TESIS DOCTORAL

MEAN CURVATURE OF A SPACELIKE HYPERSURFACE IN A SPACETIME WITH CERTAIN CAUSAL SYMMETRIES

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Contents

Resumen

A lo largo de la historia, las hipersuperficies espaciales han sido objeto de un largo y fructífero estudio en Relatividad General [79]. De manera intuitiva, cada una de ellas representa el espacio físico medible en un instante dado de tiempo. Es importante resaltar que la completitud de una hipersuperficie espacial es imprescindible siempre que queramos estudiar sus propiedades globales y, desde un punto de vista físico, esta completitud implica que estamos considerando el espacio físico en toda su extensión (no es suficiente imponer "cerrada" pues una hipersuperficie espacial cerrada no es en general completa, aún cuando se considere en espaciotiempos ambiente muy concretos [41]). Estos objetos geométricos atraen un gran interés físico debido, entre otras razones, al hecho de que el problema de valores iniciales para la ecuación de campo de Einstein en Relatividad General se formula en términos de una hipersuperficie espacial (ver, por ejemplo, [96] y referencias allí). De hecho, en Electromagnetismo, cuando usamos una hipersuperficie espacial como conjunto inicial de datos, queda unívocamente determinado tanto el futuro de un campo electromagnético que obedezca las ecuaciones de Maxwell [106, Thm. 3.11.1], como el futuro de un flujo de partículas que obedezca las ecuaciones materiales simples [106, Thm. 3.11.2]. En Teoría de la Causalidad, la existencia de cierta hipersuperficie espacial determina las propiedades causales del espaciotiempo. Concretamente, un espaciotiempo es globalmente hiperbólico [23, Def. 3.15] si y sólo si admite una hipersuperficie de Cauchy [61]. De hecho, cualquier espaciotiempo globalmente hiperbólico es difeomorfo a $\mathbb{R} \times S$, siendo S una hipersuperficie de Cauchy espacial y diferenciable [25].

Gran parte de la geometría extrínseca de una hipersuperficie espacial está codificada en la función curvatura media asociada a su operador forma. De hecho, las hipersuperficies espaciales de curvatura media constante (y en particular, las maximales) juegan un papel esencial tanto en Relatividad General como en Geometría Lorentziana, pues constituyen un conjunto inicial muy útil para el estudio del problema de Cauchy [96]. En concreto, Lichnerowicz probó que el problema de Cauchy con condiciones iniciales sobre una hipersuperficie maximal se reduce a una ecuación diferencial elíptica no lineal de segundo orden y un sistema de ecuaciones diferenciales de primer orden [76].

Además, cada hipersuperficie maximal puede describir en algunos casos relevantes el espacio físico en la transición entre una fase expansiva y otra contractiva de un universo relativista. Más aún, la existencia de hipersuperficies espaciales de curvatura media constante (y en particular maximales) es necesaria para estudiar la estructura de singularidades en el espacio de soluciones de la ecuación de Einstein [18]. Un conocimiento profundo de este tipo de hipersuperficies es también esencial para demostrar la positividad de la masa gravitatoria [111], siendo también interesantes en Relatividad Numérica, donde las hipersuperficies maximales se utilizan para integrar hacia adelante en el tiempo [66].

Desde una perspectiva matemática, el estudio de las hipersuperficies maximales de un espaciotiempo nos permite comprender su estructura. Concretamente, para algunos espaciotiempos asintóticamente llanos se demostró en [31] la existencia de una foliación por hipersuperficies maximales. De hecho, una hipersuperficie maximal es (localmente) un punto crítico del funcional área (véase por ejemplo [30]), por lo que aparecen como puntos críticos de un problema variacional natural.

Un hecho sorprendente en el estudio de las hipersuperficies maximales fue el descubrimiento de nuevos problemas elípticos no lineales asociados a estos objetos geométricos. De hecho, la función que define un grafo maximal en el espaciotiempo de Lorentz-Minkowski de dimensión $n + 1$ satisface una EDP de segundo orden conocida como ecuación de hipersuperficies maximales en $\mathbb{L}^{n+1}$. Más aún, el famoso teorema de Calabi-Bernstein establece que las únicas soluciones enteras de la ecuación de hipersuperficies maximales en $\mathbb{L}^{n+1}$ son funciones afines. Este resultado fue demostrado por Calabi [36] para $n \leq 4$ y posteriormente extendido a dimensión arbitraria por Cheng y Yau [41] gracias al principio del máximo de Omori-Yau [84], [116]. Este resultado sorprendente difiere del teorema clásico de Bernstein para hipersuperficies minimales en $\mathbb{R}^{n+1}$, que únicamente es cierto para $n \leq 7$ [110].

Este tipo de resultados de unicidad y no existencia para hipersuperficies espaciales han sido extendidos a espaciotiempos ambiente más generales. En [31], Brill y Flaherty reemplazaron el espaciotiempo de Lorentz-Minkowski por un universo espacialmente cerrado que satisface la ecuación de campo de Einstein en el vacío y probaron unicidad global. Para seguir extendiendo este tipo de resultados a una mayor variedad de espaciotiempos se asume frecuentemente la existencia de una simetría en el espaciotiempo ambiente. En Relatividad General, una simetría se basa normalmente en asumir la existencia de un grupo uniparamétrico de transformaciones generado por un campo vectorial de Killing o, de forma más general, por un campo vectorial conforme [50]. De hecho, una simplificación usual en la búsqueda de soluciones exactas de la ecuación de campo de Einstein consiste en asumir a priori la existencia de este tipo de simetría infinitesimal (véase [45], [51]). Así, una simetría en un espaciotiempo $(\overline{M}^{n+1}, \overline{g})$ se debe a la existencia de un campo vectorial conforme (no trivial) K, normalmente con cierto caráter causal prefijado. Recordemos que K es un campo vectorial conforme si la derivada de Lie de la métrica a lo largo de K satisface

$$\mathcal{L}_K \overline{g} = 2\rho\overline{g},$$

donde ρ es una función diferenciable conocida como factor conforme de K. De manera equivalente, un campo vectorial K es conforme si y sólo si sus flujos locales ϕ_t son transformaciones conformes. En el caso particular de que ρ sea idénticamente cero, K se llama campo vectorial de Killing y sus flujos locales son isometrías.

En esta tesis obtendremos nuevos resultados de unicidad para hipersuperficies espaciales completas en espaciotiempos con ciertas simetrías infinitesimales causales. En particular, nos centraremos en la familia de espaciotiempos conocidos como espaciotiempos de Robertson-Walker generalizados (GRW) (también llamados espaciotiempos de Friedmann-Lemaître-Robertson-Walker generalizados), así como en espaciotiempos estáticos estándar y espaciotiempos pp-wave.

Recordemos que por espaciotiempo GRW nos referimos a la variedad producto $\overline{M} = I \times F$ de un intervalo abierto I de la recta real $\mathbb{R}$ y una variedad Riemanniana (conexa) $n(\geq 2)$-dimensional (F, g_F), dotada con la métrica Lorentziana

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F),$$

donde π_I y π_F denotan las proyecciones sobre I y F, respectivamente. La variedad Lorentziana $(\overline{M}, \overline{g})$ es un producto alabeado (en el sentido de [85, Chap. 7]) con base $(I, -dt^2)$, fibra (F, g_F) y función de alabeo f. Estos espaciotiempos fueron introducidos en [13] e incluyen algunos espaciotiempos bien conocidos en Relatividad General como el espaciotiempo de Lorentz-Minkowski, el espaciotiempo de De Sitter, el espaciotiempo de Einstein-De Sitter, así como los espaciotiempos clásicos de Robertson-Walker (dimensión cuatro y fibra con curvatura seccional constante).

En cualquier espaciotiempo GRW hay un campo vectorial conforme temporal que define una simetría infinitesimal en el espaciotiempo. Recordemos que un espaciotiempo que admite un campo vectorial conforme temporal se denomina conformemente estacionario. En términos generales, un espaciotiempo conformemente estacionario es globalmente conforme a uno estacionario [16].

Cualquier espaciotiempo GRW $\overline{M} = I \times_f F$ posee una familia distinguida de hipersuperficies espaciales, a saber: sus slices espaciales $\{t_0\} \times F$, $t_0 \in I$. En general, un slice espacial es una hipersuperfice espacial totalmente umbilical con curvatura media constante. Además, un slice espacial es maximal (y por tanto, totalmente geodésico) siempre que t_0 sea un punto crítico de la función de alabeo.

Una vez hemos establecido nuestros espaciotiempos ambiente, en el Capítulo 3 nos centraremos en la familia de espaciotiempos GRW espacialmente cerrados, esto es, espaciotiempos GRW que admiten una hipersuperficie espacial compacta (y, por tanto, tienen fibra compacta [13, Prop. 3.2]). Nuestro objetivo es caracterizar la foliación por slices espaciales en estos modelos por su variación de volumen. En concreto, obtendremos las soluciones del siguiente problema variacional:

Sea $f : I \longrightarrow \mathbb{R}$ una función diferenciable positiva en un intervalo $I \subseteq \mathbb{R}$ y sea (F, g_F) una variedad Riemanniana compacta de dimensión n. Consideremos la clase de funciones diferenciables u sobre F tal que $u(F) \subset I$ y que la norma del gradiente de u en (F, g_F) satisfaga $|Du| < f(u)$. En esta clase consideremos el funcional

$$\mathcal{I}(u) := \int_F \left[f(u)^{n-1}\sqrt{f(u)^2 - |Du|^2} - f(u)^n \right] dV^F.$$

Los puntos estacionarios de este funcional determinan foliaciones de $\overline{M}$ por grafos espaciales $\Sigma_u = \{(u(p), p) : p \in F\}$ con la misma variación de volumen que la foliación por slices espaciales. Para puntos estacionarios de este funcional, la ecuación de Euler-Lagrange se escribe como

$$H(u) = \frac{f'(u)}{f(u)}, \tag{E.1}$$

$$|Du| < f(u), \tag{E.2}$$

siendo $H(u)$ la función diferenciable dada por

$$H(u) = \operatorname{div}\left(\frac{Du}{n f(u)\sqrt{f(u)^2 - |Du|^2}} \right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}} \left(n + \frac{|Du|^2}{f(u)^2} \right).$$

De hecho, $H(u)$ representa la curvatura media de Σ_u en $\overline{M}$ con respecto a su campo vectorial normal temporal unitario futuro. Por tanto, nuestro problema variacional es equivalente a un problema de prescripción de curvatura media donde la función de prescripción es la función de Hubble f'/f del espaciotiempo, la cual tiene un importante significado físico en Relatividad General.

Uno de los primeros trabajos donde se estudió el problema de Dirichlet de prescripción de curvatura media en el contexto Euclídeo se debe a Serrin [112], quien encontró condiciones necesarias y suficientes para la existencia de soluciones. En el espaciotiempo de Lorentz-Minkowski, un resultado crucial sobre condiciones necesarias y suficientes para la existencia de soluciones espaciales diferenciables del problema de Dirichlet para el operador curvatura media fue obtenido por Bartnik y Simon en [22]. Gerhardt [57] también estudió la existencia de hipersuperficies espaciales cerradas con curvatura media prescrita en una variedad Lorentziana globalmente hiperbólica con una hipersuperficie de Cauchy compacta. Posteriormente, Ecker y Huisken dieron una prueba más simple en [52] de algunos de los resultados obtenidos en [57]. No obstante, impusieron hipótesis más restrictivas para poder obtenerlos. Finalmente, Gerhardt demostró en [58] que el método de Ecker y Huisken podía utilizarse para demostrar sus resultados iniciales sin imponer hipótesis adicionales. Más recientemente, algunos autores han continuado estudiado problemas de prescripción de curvatura media en el espaciotiempo de Lorentz-Minkowski (véase [19], [24], [44], [59]) así como en ambientes más generales [47], [48]. Nótese que estos artículos trabajan con soluciones locales, mientras que nosotros obtendremos resultados de unicidad para ciertas soluciones globales de nuestro problema.

Nuestro principal resultado de unicidad en la Sección 3.2 para la ecuación (E) (es decir, (E.1) y (E.2)) es

> *Sea F una variedad Riemanniana compacta de dimensió n y sea $f : I \longrightarrow]0, \infty[$ una función diferenciable tal que f' tiene signo. Entonces, las únicas soluciones enteras de la ecuación* (E) *son las funciones constantes.*

Este resultado es consecuencia del Corolario 3.4. De hecho, problemas relacionados con el resuelto en el Corolario 3.4 han sido previamente estudiados en el caso particular de que la fibra sea una superficie Riemanniana no compacta (ver [35], [98] y [102]) así como para fibra Riemanniana compacta de dimensión dos en [99]. Las técnicas usadas en esos casos dependen fuertemente de la rica geometría conforme existente en el caso de dimensión dos. Aquí, abordamos el problema cuando la fibra F es una variedad Riemanniana compacta de dimensión arbitraria. Nuestro enfoque para resolver la ecuación (E) es puramente geométrico, ya que estudiamos primero una versión paramétrica del problema (Teorema 3.1) que nos permitirá resolver (E).

Más aún, también resolveremos la ecuación (E) para el caso en el que la fibra sea $F = \mathbb{S}^n$, la esfera redonda de radio unidad y dimensión n, y la función de alabeo sea $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(t) =$

$\cosh(t)$, i.e., para grafos espaciales enteros en el espaciotiempo de De Sitter $\overline{M} = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$. Para ello, abordaremos el problema más general de ver cuándo una hipersuperficie espacial compacta M en un espaciotiempo GRW Einstein cuya función curvatura media H sea igual a la restricción de la función de Hubble sobre M es un slice espacial (Teorema 3.9). Hemos obtenido este resultado gracias a una nueva aplicación de una conocida fórmula integral (ver Sección 3.3). Así, tenemos (Corolario 3.12)

Las únicas soluciones enteras en $\mathbb{S}^n$ *de*

$$H(u) = \tanh(u),$$

$$|Du| < \cosh(u)$$

son las funciones constantes.

Para concluir este capítulo, en la Sección 3.4 trataremos con espaciotiempos GRW que obedezcan la Condición de Convergencia Nula (NCC). Esta hipótesis de curvatura aparece como consecuencia algebraica de la ecuación de campo de Einstein para tensores impulso-energía razonables y es satisfecha por numerosos modelos cosmológicos relevantes, incluyendo algunos que describen escenarios inflacionarios. Además, aparece como una hipótesis natural al estudiar las singularidades del espaciotiempo. Siguiendo un procedimiento análogo al de la sección anterior, clasificamos las hipersuperficies espaciales compactas con esa prescripción de curvatura media en ciertos espaciotiempos GRW que obedecen la NCC (Teorema 3.13). Como consecuencia, obtenemos (Corolario 3.15)

Sea F *una variedad Riemanniana compacta de dimensión* n *y sea* $f : I \longrightarrow]0, \infty[$ *una función diferenciable tal que* $\log f$ *sea convexa. Si la curvatura de Ricci de* F *está estrictamente acotada inferiormente por* $(n-1)\sup_F \left(f^2(\log f)''\right)$ *o ocurre que* $\log f$ *es estrictamente convexa, entonces las únicas soluciones enteras de la ecuación* (E) *son las funciones constantes.*

Los resultados que hemos obtenido nos permiten responder a la pregunta que nos planteamos al principio del Capítulo 3. Esto se debe a que el problema de prescripción de curvatura media que hemos resuelto es equivalente a nuestro problema inicial de encontrar las foliaciones de un espaciotiempo GRW espacialmente cerrado con fibra simplemente conexa por hipersuperficies espaciales cuya variación de volumen es la misma que la que los observadores comóviles miden de sus espacios en reposo. Por tanto, nuestros resultados de unicidad para la ecuación (E) permiten caracterizar la foliación por slices espaciales por su variación de volumen en una amplia variedad de modelos con interés físico, incluyendo el espaciotiempo de De Sitter (Corolario 3.12), así como otros espaciotiempos GRW espacialmente cerrados con fibra simplemente conexa donde la función de alabeo es monótona (Corolario 3.4) o donde la NCC se satisface estrictamente (Corolario 3.15).

Tras esta caracterización de los slices espaciales en un espaciotiempo GRW espacialmente cerrado, dedicaremos el Capítulo 4 al estudio de hipersuperficies maximales en espaciotiempos GRW espacialmente abiertos. Pese a la importancia histórica de los modelos espacialmente cerrados, observaciones recientes sugieren que los modelos espacialmente abiertos son más adecuados a la hora de describir nuestro universo actual [42]. Más aún, los universos espacialmente cerrados violan el principio holográfico [20], que afirma que la entropía contenida en una región acotada del universo debe estar acotada por el área del correspondiente borde [29], lo que los convierte en modelos inapropiados para una posible teoría gravitatoria cuántica.

Así, en este capítulo obtendremos nuevos resultados de no existencia para hipersuperficies maximales completas en algunos espaciotiempos de Robertson-Walker conocidos gracias a una cota para la curvatura de Ricci de estas hipersuperficies maximales. Concretamente, demostraremos (Corolario 4.8)

No hay hipersuperficies maximales completas en el espaciotiempo steady state de dimensión $(n+1)$, $\mathbb{R} \times_{e^t} \mathbb{R}^n$.

El espaciotiempo steady state es de tipo Robertson-Walker y es isométrico a una región abierta del espaciotiempo de De Sitter. Bondi, Gold [27] y Hoyle [64] propusieron independientemente este modelo en el caso de dimensión cuatro para describir un universo que es espacialmente homogéneo e isótropo en todo instante de tiempo.

También obtendremos el siguiente resultado para hipersuperficies maximales en el espaciotiempo de Einstein-De Sitter, el cual en dimensión cuatro es una solución exacta clásica de la ecuación de campo de Einstein sin constante cosmológica que incorpora la homogeneidad espacial y la isotropía al mismo tiempo que permite la expansión. Así, en el Corolario 4.9 probamos

No hay hipersuperficies maximales completas acotadas por el infinito futuro en el espaciotiempo de Einstein-De Sitter de dimensión $(n+1)$, $\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$.

Nuestro último resultado de no existencia (Corolario 4.10) se refiere a hipersuperficies maximales completas en los modelos de Robertson-Walker de radiación, los cuales describen en dimensión cuatro universos donde la radiación predomina sobre la materia.

No hay hipersuperficies maximales completas acotadas por el infinito futuro en el modelo de Robertson-Walker de radiación de dimensión $(n+1)$*;* $\mathbb{R}^+ \times_{(2at)^{1/2}} \mathbb{R}^n$, *con* $a > 0$.

Tras estos corolarios, demostraremos el resultado principal de unicidad del Capítulo 4 sobre hipersuperficies maximales completas en una cierta familia de espaciotiempos GRW espacialmente abiertos. A saber, para espaciotiempos Robertson-Walker espacialmente abiertos con fibra llana que obedezcan la NCC obtendremos, haciendo uso del principio del máximo de Omori-Yau, lo siguiente (Corolario 4.12)

Sea $\psi : M \to \overline{M}$ *una hipersuperficie maximal de dimensión* n *en un espaciotiempo de Robertson-Walker* $\overline{M} = I \times_f F$ *con fibra llana que obedece la NCC. Si*

$$\inf_M \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} > 0$$

se cumple, entonces M *es un slice espacial* $\{t_0\} \times F$ *con* $f'(t_0) = 0$.

Notemos que este resultado mejora algunos teoremas previos de unicidad para hipersuperficies maximales completas (véase [101] y [103], por ejemplo) al no imponer hipótesis restrictivas sobre la hipersuperficie maximal como tener el ángulo hiperbólico acotado o estar contenida entre dos slices espaciales, permitiéndonos tratar el caso de hipersuperficies maximales que se aproximan a la frontera luminosa así como a su infinito pasado e infinito futuro. Además, tampoco hemos impuesto ninguna hipótesis de compacidad sobre la fibra como la parabolicidad, lo cual ha sido usado recientemente en ciertos modelos espacialmente abiertos [101].

Concluimos este capítulo mostrando en la Sección 4.4 un modelo de fluido perfecto para espaciotiempos de Robertson-Walker espacialmente abiertos con fibra llana que nos ayude a clarificar algunas de las hipótesis usadas en los resultados que hemos obtenido a lo largo de este capítulo. Este modelo nos permite expresar las condiciones de energía usuales en términos de ciertas cantidades con un claro significado físico, proporcionándonos una forma de entender mejor nuestras hipótesis matemáticas desde una perspectiva física.

El Capítulo 5 está dedicado a la obtención de un resultado que relaciona las soluciones de la ecuación de superficies maximales en un espaciotiempo estático estándar con las soluciones de la ecuación de superficies minimales en cierta variedad Riemanniana, extendiendo ampliamente un resultado bien conocido [36], [12]. Se establece una conexión entre hechos que ocurren en Geometría Lorentziana y Geometría Riemanniana representados por dos EDP elípticas distintas.

Los espaciotiempos estáticos estándar son parte de los llamados espaciotiempos estacionarios. Un espaciotiempo estacionario es una variedad Lorentziana en la que existe una simetría infinitesimal dada por un campo de Killing temporal. Esta variedad Lorentziana es temporalmente orientable y por tanto un espaciotiempo. La existencia de un campo de Killing temporal $\mathcal{K}$ permite definir alrededor de cada punto un sistema de coordenadas $(t, x^1, \ldots, x^n)$ de manera que $\mathcal{K}$ concide con el campo coordenado ∂_t en su entorno de definición y tal que las componentes del tensor métrico relativo a dichas coordenadas sean independientes de la coordenada t. Al normalizar $\mathcal{K}$ obteniendo un campo de observadores Z, éstos constantan que con el paso de su tiempo propio el tensor métrico no cambia. Más aún, si este campo de Killing temporal también es irrotacional (es decir, la distribución ortogonal $\mathcal{K}^\perp$ es involutiva), entonces aparece una estructura local de producto alabeado y el espaciotiempo se dice estático (ver [8], por ejemplo). De hecho, cuando esta estructura es global este espaciotiempo se conoce como espaciotiempo estático estándar.

Concretamente, por espaciotiempo estático estándar denotaremos a la variedad producto $\overline{M} = B \times \mathbb{R}$, siendo (B, g_B) una variedad Riemanniana (conexa) de dimesión $n(\geq 2)$, dotada con la métrica Lorentziana

$$\overline{g} = \pi_B^*(g_B) - h(\pi_B)^2 \pi_{\mathbb{R}}^*(dt^2),$$

donde π_B y $\pi_{\mathbb{R}}$ denotan las proyecciones sobre B y $\mathbb{R}$, respectivamente, y h es una función diferenciable positiva definida sobre B. La variedad Lorentziana $(\overline{M} = B \times \mathbb{R}, \overline{g})$ es un producto alabeado con base (B, g_B), fibra $(\mathbb{R}, -dt^2)$ y función de alabeo h.

Los espaciotiempos estáticos estándar incluyen algunos espaciotiempos clásicos como el espaciotiempo de Lorentz-Minkowski y el universo estático de Einstein, así como modelos que describen un universo donde únicamente hay una masa con simetría esférica que no rota, como una estrella, como el espaciotiempo exterior de Schwarzschild [106].

En cualquier espaciotiempo estático estándar $\overline{M} = B \times_h \mathbb{R}$ hay una familia distinguida de hipersuperficies espaciales, a saber, sus slices espaciales $B \times \{t_0\}$, $t_0 \in \mathbb{R}$, los cuales son las hipersuperficies de nivel de la función $\pi_{\mathbb{R}}$. Estos slices espaciales son totalmente geodésicos y constituyen una foliación en cualquier espaciotiempo estático estándar.

Nuestro objetivo será construir una correspondencia (en ambos sentidos) entre soluciones de la ecuación de superficie minimal en un cierto producto alabeado Riemanniano de dimensión tres y soluciones de la ecuación de superficie maximal en un espaciotiempo estático estándar de dimensión tres. El estudio de la dualidad entre grafos minimales y maximales se remonta a 1970, cuando Calabi descubrió una interesante correspondencia entre soluciones de la ecuación de superficie minimal en el espacio Euclídeo $\mathbb{R}^3$ y soluciones de la ecuación de superficie maximal en el espaciotiempo de Lorentz-Minkowski $\mathbb{L}^3$ [36].

Posteriormente, varios autores han estudiado esta correspondencia por medio de diferentes enfoques. En [12] se mostró cómo el teorema clásico de Bernstein para superficies minimales en el espacio Euclídeo puede verse como consecuencia del teorema de Calabi-Bernstein para superficies maximales en el espaciotiempo de Lorentz-Minkowski y viceversa (véase también [7]). Usando la representación local de Enneper-Weierstrass de superficie minimal en $\mathbb{R}^3$ y la representación local de Enneper-Weierstrass de superficie maximal en $\mathbb{L}^3$ (ver [68] y [80]), esta correspondencia puede darse también en términos de los datos locales de Enneper-Weierstrass de las superficies [17], reprobando con una técnica completamente distinta la equivalencia entre los teoremas de Bernstein y Calabi-Bernstein [55].

Recientemente, la idea de Calabi ha sido extendida a distintos tipos de superficies en diferentes variedades Riemannianas ambientes y espaciotiempos. De hecho, en [6] y [7] la correspondencia de Calabi se extiende a grafos minimales en variedades producto Riemannianas y grafos maximales en variedades producto Lorentzianas. Por otra parte, en [72] se construye una dualidad entre grafos de curvatura media constante H en el espacio de Bianchi-Cartan-Vranceanu $E^3(\kappa, \tau)$ y grafos espaciales de curvatura media constante τ en el espacio Lorentziano de Bianchi-Cartan-Vranceanu $L^3(\kappa, H)$. En [73] se da una dualidad entre superficies maximales en el grupo de Heisenberg Lorentziano y superficies de curvatura media constante en $\mathbb{R}^3$. Además, la correspondencia de Calabi también ha sido generalizada para mayor codimensión en [74].

En esta parte de la tesis, inspirados por [6] y [36], extendemos la correspondencia de Calabi a ciertos espacios producto alabeado, estableciendo una dualidad entre dos clases de problemas elípticos. Debemos remarcar que nuestra técnica no es una aplicación directa de la usada en [6]. De hecho, trabajaremos con dos ecuaciones elípticas no homogéneas en forma de divergencia y haremos uso de un truco para trabajar con estas ecuaciones tras aplicar adecuados cambios de métrica conformes.

Así, daremos un resultado de dualidad entre soluciones de la ecuación de superficie minimal en un producto alabeado Riemanniano con base una superficie de Riemann (B, g_B), fibra la recta real y función de alabeo $\frac{1}{\sqrt{\gamma}}$, dada por

$$\operatorname{div}\left(\frac{Du}{\sqrt{\gamma+|Du|^2}}\right)=\frac{1}{2\gamma}\frac{g_B(D\gamma, Du)}{\sqrt{\gamma+|Du|^2}}, \tag{R}$$

y soluciones de la ecuación de superficie maximal en un espaciotiempo estático estándar con la misma base (B, g_B), fibra la recta real dotada con la métrica usual con signo negativo y función de alabeo $\sqrt{\gamma}$, la cual se escribe como

$$\operatorname{div}\left(\frac{D\omega}{\sqrt{\frac{1}{\gamma}-|D\omega|^2}}\right)=\frac{\gamma}{2}\frac{g_B\left(D\left(\frac{1}{\gamma}\right), D\omega\right)}{\sqrt{\frac{1}{\gamma}-|D\omega|^2}}, \tag{L.1}$$

$$|D\omega|^2<\frac{1}{\gamma}. \tag{L.2}$$

Así, el Teorema 5.1 establece

Sea $\Omega \subset B$ un dominio simplemente conexo. Entonces, existe una solución no trivial (i.e., no constante) de (R) *en Ω si y sólo si existe una solución no trivial de* (L) *en Ω.*

Esta correspondencia nos permite usar un teorema conocido de tipo Bernstein para obtener un nuevo resultado de tipo Calabi-Bernstein para espaciotiempos estáticos estándar con base compacta (Corolario 5.5).

Las únicas soluciones enteras de la ecuación (L) *en la esfera de dimensión* 2, $\mathbb{S}^2$, *dotada de una métrica Riemanniana son las funciones constantes.*

Para concluir esta tesis daremos nuevos resultados de unicidad para hipersuperficies maximales completas en una familia de espaciotiempos con un tipo distinto de simetría infinitesimal.

Así, en el Capítulo 6 estudiaremos estos objetos geométricos en espaciotiempos pp-wave. Por espaciotiempo pp-wave nos referimos a una variedad Lorentziana $(\overline{M}, \overline{g})$ con un campo vectorial luminoso paralelo globalmente definido ξ; esto es, $\overline{g}(\xi, \xi) = 0$, $\xi \neq 0$ tal que $\overline{\nabla}\xi = 0$. Nótese que tal variedad Lorentziana ha de ser temporalmente orientable, y por tanto, un espaciotiempo. Por otro lado, si tenemos un campo de vectores paralelo que no se anula en un punto, entonces no se anulará en ningún punto. Además, si un campo de vectores paralelo es luminoso en un punto, entonces será luminoso en todos los puntos (i.e., un campo de vectores luminoso).

Este tipo de simetría infinitesimal se ha usado como hipótesis a priori para la obtención de soluciones exactas de la ecuación de campo de Einstein, describiendo radiación (electromagnética o gravitatoria) que se propaga a la velocidad de la luz [113]. Pese a que el estudio de las ondas gravitatorias se remonta a Einstein [54], el modelo estándar que conocemos fue introducido por Brinkmann en [32]. en los últimos años se ha estudiado la geometría de los espaciotiempos pp-wave desde diferentes puntos de vista (ver [26], [37], [38] y [56]). Recientemente, los espaciotiempos pp-wave han atraído una gran cantidad de atención debido a la detección experimental de ondas gravitatorias [1].

Así, en la Sección 6.1 estudiaremos hipersuperficies espaciales compactas de curvatura media constante en un espaciotiempo pp-wave de dimensión arbitraria que satisfaga la Condición de Convergencia Temporal (TCC) (Teorema 6.2).

> *Sea* $\psi : M \longrightarrow \overline{M}$ *una hipersuperficie espacial compacta en un espaciotiempo pp-wave de dimensión* $(n+1)$ *que satisface la TCC. Si* M *tiene curvatura media constante, entonces debe ser totalmente geodésica. Por consiguiente, no hay ninguna hipersuperficie espacial compacta con curvatura media constante* M *cuya curvatura media sea distinta de cero.*

Posteriormente, en la Sección 6.2 generalizamos el teorema paramétrico clásico de Calabi-Bernstein para superficies maximales completas en espaciotiempos pp-wave de dimensión tres, obteniendo (Teorema 6.7)

> *Sea* $\psi : S \longrightarrow \overline{M}$ *una superficie maximal completa en un espaciotiempo pp-wave de dimensión* 3 *que obedece la TCC. Entonces,* S *es totalmente geodésica y llana.*

Finalmente, presentaremos una breve discusión de las conclusiones de esta tesis, así como diversas líneas futuras de investigación que nos resultan de gran interés.

Chapter 1

Introduction

The study of spacelike hypersurfaces has a long and fruitful history in General Relativity [79]. Roughly speaking, each of them represents the physical space that can be measured in a given instant of time. We would also like to remark that the completeness of a spacelike hypersurface is required whenever we want to study its global properties and, from a physical standpoint, completeness implies that the whole physical space is taken into consideration (it is not enough to assume the spacelike hypersurface to be closed since a closed spacelike hypersurface is not complete in general, even in certain concrete spacetimes [41]). These geometric objects attract great physical interest, among other reasons, due to the fact that the initial value problem for the Einstein's field equation in General Relativity is formulated in terms of a spacelike hypersurface (see, for instance [96] and references therein). Indeed, in Electromagnetism, when we use a spacelike hypersurface as an initial data set, the future of an electromagnetic field that satisfies the Maxwell equations is univocally determined [106, Thm. 3.11.1], as well as the future of a particle flow that obeys the simple matter equations [106, Thm. 3.11.2]. In Causality Theory, the existence of a certain spacelike hypersurface can determine the causal properties of the spacetime. For example, a spacetime is globally hyperbolic [23, Def. 3.15] if and only if it admits a Cauchy hypersurface [61]. As a matter of fact, any globally hyperbolic spacetime is diffeomorphic to $\mathbb{R} \times S$, being S a smooth spacelike Cauchy hypersurface [25].

Most of the extrinsic geometry of a spacelike hypersurface is codified in the mean curvature function associated to its shape operator. In fact, constant mean curvature spacelike hypersurfaces (and in particular, maximal hypersurfaces) play a key role in General Relativity as well as in Lorentzian Geometry, since they constitute a useful initial set for the Cauchy problem [96]. In particular, Lichnerowicz proved that a Cauchy problem with initial conditions on a maximal hypersurface is reduced to a second order nonlinear elliptic differential equation and a first order linear differential system [76].

Furthermore, each maximal hypersurface can describe in some relevant cases the physical space in the transition between an expanding and a contracting phase of a relativistic universe. Moreover, the existence of constant mean curvature spacelike hypersurfaces (and in particular maximal) is necessary for the study of the structure of singularities in the space of solutions of Einstein's equations [18]. The deep understanding of this kind of hypersurfaces is also essential to prove the positivity of the gravitational mass [111]. Also, they are interesting for Numerical Relativity, where maximal hypersurfaces are used for integrating forward in time [66].

From a mathematical perspective, the study of a spacetime's maximal hypersurfaces enable us to understand its structure. Indeed, for some asymptotically flat spacetimes the existence of a foliation by maximal hypersurfaces was proved in [31]. As a matter of fact, a maximal hypersurface is (locally) a critical point of the area functional (see for instance [30]), so they appear as critical points of a natural variational problem.

A striking fact in the study of maximal hypersurfaces was the discovery of new non-linear elliptic problems associated to these geometric objects. In fact, the function defining a maximal graph in the $(n+1)$-dimensional Lorentz-Minkowski spacetime satisfies a second order PDE known as the maximal hypersurface equation in $\mathbb{L}^{n+1}$. Moreover, the famous Calabi-Bernstein theorem states that the only entire solutions to the maximal hypersurface equation in $\mathbb{L}^{n+1}$ are affine functions. This result was proved by Calabi [36] for $n \leq 4$ and later extended to arbitrary dimension by Cheng and Yau [41] by means of the Omori-Yau maximum principle [84], [116]. This surprising result differs from the classical Bernstein theorem for minimal hypersurfaces in $\mathbb{R}^{n+1}$, which only holds true for $n \leq 7$ [110].

These kind of uniqueness and non-existence results for spacelike hypersurfaces have been extended to more general ambient spacetimes. In [31], Brill and Flaherty replaced the Lorentz-Minkowski spacetime by a spatially closed universe that satisfies the Einstein vacuum equations and proved uniqueness in the large. In order to continue extending these type of results to a wider variety of spacetimes it has been frequently assumed the existence of a symmetry in the ambient spacetime. In General Relativity, symmetry is usually based on the assumption of the existence of a one-parameter group of transformations generated by a Killing or, more generally, a conformal vector field [50]. In fact, a usual simplification for the search of exact solutions to the Einstein equation is to assume a priori the existence of such an infinitesimal symmetry (see [45], [51]). Indeed, a symmetry in a spacetime $(\overline{M}^{n+1}, \overline{g})$ is due to the existence of a (non-trivial) conformal vector field K, normally assumed to have a certain causal character. Let us recall that K is a conformal vector field provided that the Lie derivative of the metric tensor along K satisfies

$$\mathcal{L}_K \overline{g} = 2\rho \overline{g},$$

where ρ is a smooth function known as the conformal factor of K. Equivalently, a vector field K is conformal if and only if the stages ϕ_t of all its local flows are conformal maps. In the particular case where ρ identically vanishes, K is called a Killing vector field and the stages of all its local flows are isometries.

In this thesis we will try to obtain new uniqueness results for complete spacelike hypersurfaces immersed in spacetimes with certain causal infinitesimal symmetries. In particular, we will focus on the family of spacetimes known as Generalized Robertson-Walker (GRW) spacetimes, as well as standard static spacetimes and pp-wave spacetimes.

Let us recall that by a GRW spacetime we mean a product manifold $\overline{M} = I \times F$ of an open interval I of the real line $\mathbb{R}$ and an $n(\geq 2)$-dimensional (connected) Riemannian manifold (F, g_F), endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F),$$

where π_I and π_F denote the projections onto I and F, respectively. The Lorentzian manifold $(\overline{M}, \overline{g})$ is a warped product (in the sense of [85, Chap. 7]) with base $(I, -dt^2)$, fiber (F, g_F) and warping function f. These spacetimes were introduced in [13] and include some well-known spacetimes such as the Lorentz-Minkowski spacetime, de Sitter spacetime, Einstein-de Sitter spacetime as well as the classical Robertson-Walker spacetimes (dimension four and fiber of constant sectional curvature).

In any GRW spacetime there is a timelike conformal vector field which introduces an infinitesimal symetry in the spacetime. Remember that a spacetime that a spacetime that admits a timelike conformal vector field is said to be conformally stationary. Roughly speaking, a conformally stationary spacetime turns out to be stationary when equipped with a conformal metric [16].

Any GRW spacetime $\overline{M} = I \times_f F$ has a distinguished family of spacelike hypersurfaces, namely its spacelike slices $\{t_0\} \times F$, $t_0 \in I$. In general, a spacelike slice is a totally umbilic spacelike hypersurface with constant mean curvature. Moreover, a spacelike slice is maximal (and hence, totally geodesic) whenever t_0 is a critical point of the warping function.

Once we have stablished our ambient spacetimes, in Chapter 3 we focus on the family of spatially closed GRW spacetimes, i.e., GRW spacetimes that admit a compact spacelike hypersurface (and therefore, they have compact fiber [13, Prop. 3.2]). Our aim is to characterize the foliation by spacelike spacelike slices in these models by their volume variation. In particular, we will find the solutions to the following variational problem:

Let $f : I \longrightarrow \mathbb{R}$ be a positive smooth function on an open interval $I \subseteq \mathbb{R}$ and let (F, g_F) be an n-dimensional compact Riemannian manifold. Consider the class of smooth real valued functions u on F such that $u(F) \subset I$ and the length of the gradient of u in (F, g_F) satisfies $|Du| < f(u)$. On this class consider the functional

$$\mathcal{I}(u) := \int_F \left[f(u)^{n-1}\sqrt{f(u)^2 - |Du|^2} - f(u)^n \right] dV^F.$$

The stationary points of this functional will determine foliations of $\overline{M}$ by spacelike graphs $\Sigma_u = \{(u(p), p) : p \in F\}$ with the same volume variation than the foliation by spacelike slices. For stationary points of this functional, the Euler-Lagrange equation is written as

$$H(u) = \frac{f'(u)}{f(u)}, \tag{E.1}$$

$$|Du| < f(u), \tag{E.2}$$

being $H(u)$ the smooth function given by

$$H(u) = \operatorname{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|^2}} \right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}} \left(n + \frac{|Du|^2}{f(u)^2} \right).$$

Indeed, $H(u)$ represents the mean curvature of Σ_u in $\overline{M}$ with respect to its unitary normal future pointing timelike vector field. Thus, our variational problem is equivalent to a mean curvature prescription problem where the prescription function will be the Hubble function f'/f of the spacetime, which has an important physical meaning in General Relativity.

One of the first works on the prescribed mean curvature Dirichlet problem in the Euclidean context is due to Serrin [112], who found necessary and sufficient conditions for its solvability. In Lorentz-Minkowski spacetime, a crucial result on necessary and sufficient conditions for the existence of smooth spacelike solutions of the Dirichlet problem for the mean curvature operator was given in [22] by Bartnik and Simon. Gerhardt [57] also studied the existence of closed spacelike hypersurfaces with prescribed mean curvature in a globally hyperbolic Lorentzian manifold with a compact Cauchy hypersurface. Later, Ecker and Huisken gave a simpler proof in [52] for some of the results in [57] using an evolutionary approach. However, they needed to impose restrictive assumptions to obtain their results. Finally, Gerdhart showed in [58] that the evolutionary method could be used to prove his earlier result without imposing additional assumptions. More recently, several authors have also studied mean curvature prescription problems in Lorentz-Minkowski spacetime (see [19], [24], [44], [59] and references therein) as well as in more general ambiences [47], [48]. Note that these papers deal with local solutions, whereas we will give uniqueness results for global solutions of our problem.

Our main uniqueness result in Section 3.2 for equation (E) (that is, (E.1) and (E.2)) is

> *Let F be an n-dimensional compact Riemannian manifold and let $f : I \longrightarrow]0, \infty[$ be a smooth function such that f' is signed. Then, the only entire solutions of equation* (E) *are the constant functions.*

This result is in fact a consequence of Corollary 3.4. Indeed, problems related to the one solved in Corollary 3.4 have been previously studied in the particular case where the fiber is a complete noncompact Riemannian surface (see [35], [98] and [102]) as well as for compact 2-dimensional Riemannian fiber in [99]. The techniques used in these cases strongly depended on the rich conformal geometry in the 2-dimensional case. Here, we deal with this problem when the fiber F is a compact Riemannian manifold of arbitrary dimension. Our approach to equation (E) is purely geometric, since we study first a parametric version of the problem (Theorem 3.1) that will enable us to solve (E).

Furthermore, we will also solve equation (E) in the case where $F = \mathbb{S}^n$, the unit round n-sphere and $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(t) = \cosh(t)$, i.e., for entire spacelike graphs in de Sitter spacetime $\overline{M} = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$. In order to do so, we will deal with the more general problem of deciding when a compact spacelike hypersurface M in an Einstein GRW spacetime whose mean curvature function H is equal to the restriction of the Hubble function to M, is a spacelike slice (Theorem 3.9). We have done this by means of a new application of a well-known integral formula (see Section 3.3). Thus, we have (Corollary 3.12)

> *The only entire solutions on $\mathbb{S}^n$ of*
>
> $$H(u) = \tanh(u),$$
>
> $$|Du| < \cosh(u)$$
>
> *are the constant functions.*

To conclude this chapter, in Section 3.4 we will deal with GRW spacetimes obeying the so-called Null Convergence Condition (NCC). This curvature assumption appears as an algebraic consequence of Einstein field equation for reasonable stress-energy tensors and is satisfied by a great deal of relevant cosmological models, including some describing inflationary scenarios. Moreover, it appears as a natural assumption when studying spacetimes' singularities. Following an analogous procedure as in the previous section, we classify the compact spacelike hypersurfaces with this mean curvature prescription in certain GRW spacetimes obeying NCC (Theorem 3.13). As a consequence we get (Corollary 3.15)

> *Let F be an n-dimensional compact Riemannian manifold and let $f : I \longrightarrow]0, \infty[$ be a smooth function such that $\log f$ is convex. If the Ricci curvature of F is strictly bounded from below by $(n-1)\sup_F \left(f^2(\log f)''\right)$ or $\log f$ is strictly convex, then the only entire solutions of equation* (E) *are the constant functions.*

The results that we have obtained enable us to answer the question at the beginning of Chapter 3. This is due to the fact that the mean curvature prescription problem that we have solved is equivalent to our initial problem of finding the foliations of a spatially closed GRW spacetime with simply connected fiber by spacelike hypersurfaces whose volume variation is the same that the one measured by the comoving observers for their restspaces. Therefore, our uniqueness results for equation (E) lead to characterize the foliation by spacelike slices by their volume variation in a wide variety of models with physical interest, including de Sitter spacetime (Corollary 3.12) as well as other spatially closed GRW spacetime with simply connected fiber where the warping function is monotone (Corollary 3.4) or where NCC is strictly satisfied (Corollary 3.15).

After this characterization of the spacelike slices in a spatially closed GRW spacetime, we will devote Chapter 4 to study maximal hypersurfaces in spatially open GRW spacetimes. Despite the historical importance of spatially closed models, recent observations suggest that spatially open models are more convenient to describe our current universe [42]. Even more, spatially closed universes violate the holographic principle [20], which states that the entropy contained in a region of the universe is bounded by the area of the region's boundary [29], making them unsuitable models for a possible quantum theory of gravity.

Therefore, in this chapter we will obtain new non-existence results for complete maximal hypersurfaces in some well-known spatially Robertson-Walker spacetimes by means of a bound on the Ricci curvature of these maximal hypersurfaces. Namely, we will prove (Corollary 4.8)

There are no complete maximal hypersurfaces in the $(n+1)$-dimensional steady state spacetime $\mathbb{R} \times_{e^t} \mathbb{R}^n$.

The steady state spacetime is a Robertson-Walker spacetime which is isometric to an open region of de Sitter spacetime. Bondi and Gold [27] and Hoyle [64] independently proposed this model in the 4-dimensional case to describe a universe that looks the same at all points and in all directions (i.e., spatially isotropic and homogeneous) as well as at all times.

Moreover, we will also be able to obtain the following result for complete maximal hypersurfaces in the Einstein-de Sitter spacetime, which in dimension four is a classical exact solution to Einstein's field equation without cosmological constant that incorporates spatial homogeneity and isotropy as well as allows expansion. Therefore, Corollary 4.9 states as follows

There are no complete maximal hypersurfaces bounded away from future infinity in the $(n+1)$-dimensional Einstein-de Sitter spacetime $\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$.

Our last non-existence result (Corollary 4.10) will deal with complete maximal hypersurfaces in Robertson-Walker Radiation models, which in dimension four describe universes where radiation dominates over matter.

There are no complete maximal hypersurfaces bounded away from future infinity in the $(n+1)$-dimensional Roberson-Walker Radiation model $\mathbb{R}^+ \times_{(2at)^{1/2}} \mathbb{R}^n$, with $a > 0$.

Furthermore, the main result in Chapter 4 will be a uniqueness result for complete maximal hypersurfaces in a certain family of spatially open GRW spacetimes. Namely, for spatially open Robertson-Walker spacetimes with flat fiber that obey the NCC we obtain, making use of the Omori-Yau maximum principle for the Laplacian, the following (Corollary 4.12)

Let $\psi : M \to \overline{M}$ be an n-dimensional maximal hypersurface in a Robertson-Walker spacetime $\overline{M} = I \times_f F$ with flat fiber that obeys the NCC. If

$$\inf_M \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} > 0$$

holds on M, then M is a spacelike slice $\{t_0\} \times F$ with $f'(t_0) = 0$.

Notice that this result improves some previous uniqueness theorems for complete maximal hypersurfaces (see [101] and [103], for instance) without making restrictive assumptions on the maximal hypersurface such as having a bounded hyperbolic angle or lying between two spacelike slices, enabling us to deal with maximal spacelike hypersurfaces that approach the null boundary as well as both the future and past infinity. Moreover, we have neither imposed any compactness assumption on the fiber such as parabolicity, which has recently been used in certain spatially open models [101].

We end this chapter by giving in Section 4.4 a perfect fluid model for spatially open Robertson-Walker spacetimes with flat fiber that helps clarifying certain assumptions that we have used in some of the results along this chapter. This model enables us to express the usual energy conditions in terms of certain quantities with a clear physical meaning, providing a way to better understand from a physical perspective our mathematical assumptions.

Chapter 5 is devoted to obtain a result that relates the solutions of the maximal surface equation in a standard static spacetime with the solutions of the minimal surface equation in a certain Riemannian manifold, widely extending a well-known result [36], [12]. We establish a correspondence between Lorentzian and Riemannian Geometry results in terms of two different elliptic PDEs.

Standard static spacetimes are part of the so called stationary spacetimes. A stationary spacetime is a Lorentzian manifold where there exists an infinitesimal symmetry given by a timelike Killing vector field. This Lorentzian manifold is time-orientable and therefore, a spacetime. The existence of a timelike Killing vector field $\mathcal{K}$ enables us to define around each point a coordinate system $(t, x^1, \ldots, x^n)$ such that $\mathcal{K}$ coincides with the coordinate vector field ∂_t on its domain of definition and such that the components of the metric tensor in these coordinates are independent of t. When we normalize $\mathcal{K}$ we obtain an observers vector field Z. These observers measure a metric tensor that does not change with time. Furthermore, if this timelike Killing vector field is also irrotational (i.e., the orthogonal distribution $\mathcal{K}^\perp$ is involutive), then a local warped product structure appears and the spacetime is called static (see, for instance, [8]). In fact, when this structure is global this spacetime is known as a standard static spacetime.

More precisely, by a standard static spacetime we denote the product manifold $\overline{M} = B \times \mathbb{R}$, being (B, g_B) an $n(\geq 2)$-dimensional (connected) Riemannian manifold, endowed with the Lorentzian metric

$$\overline{g} = \pi_B^*(g_B) - h(\pi_B)^2 \pi_{\mathbb{R}}^*(dt^2),$$

where π_B and $\pi_{\mathbb{R}}$ denote, respectively, the projections on B and $\mathbb{R}$ and h is a smooth positive function on B. The Lorentzian manifold $(\overline{M} = B \times \mathbb{R}, \overline{g})$ is a warped product with base (B, g_B), fiber $(\mathbb{R}, -dt^2)$ and warping function h.

Standard static spacetimes include some classical spacetimes, such as Lorentz-Minkowski spacetime, Einstein static universe as well as models that describe a universe where there is only a spherically symmetric non-rotating mass, such as a star, like exterior Schwarzschild spacetime [106].

In any standard static spacetime $\overline{M} = B \times_h \mathbb{R}$ there is a remarkable family of spacelike hypersurfaces, namely, it spacelike slices $B \times \{t_0\}$, $t_0 \in \mathbb{R}$, which are the level hypersurfaces of the function $\pi_{\mathbb{R}}$. These spacelike slices are totally geodesic and constitute a foliation in any standard static spacetime.

Our aim is to obtain a correspondence between the solutions of the minimal surface equation in a certain 3-dimensional Riemannian warped product and the solutions of the maximal surface equation in a 3-dimensional standard static spacetime. The study of the duality between minimal and maximal graphs goes back to 1970, when Calabi discovered a nice correspondence between solutions of the minimal surface equation in the Euclidean space $\mathbb{R}^3$ and solutions of the maximal surface equation in the Lorentz-Minkowski spacetime $\mathbb{L}^3$ [36].

Later on, several authors have studied this correspondence using different approaches. In [12] it was shown how classical Bernstein theorem on minimal surfaces in the Euclidean space can be seen as a consequence of Calabi-Bernstein theorem on maximal surfaces in the Lorentz-Minkowski spacetime and vice versa (see also [7]). Using the local Enneper-Weierstrass representation of a minimal surface in $\mathbb{R}^3$ and the local Enneper-Weierstrass representation of a maximal surface in $\mathbb{L}^3$ (see [68] and [80]), this correspondence can be given in terms of the local Enneper-Weierstrass data of the surfaces [17], reproving with a very different approach the equivalence between Bernstein and Calabi-Bernstein theorems [55].

More recently, Calabi's idea has been extended to another types of surfaces in different ambient Riemannian manifolds and spacetimes. In fact, in [6] and [7] Calabi's correspondence has been extended to minimal graphs in certain Riemannian product manifolds and maximal graphs in Lorentzian product manifolds. On the other hand, in [72] a duality between graphs of constant mean curvature H in Bianchi-Cartan-Vranceanu space $E^3(\kappa, \tau)$ and spacelike graphs of constant mean curvature τ in Lorentzian Bianchi-Cartan-Vranceanu space $L^3(\kappa, H)$ is constructed. In [73] a duality between maximal surfaces in the Lorentzian Heisenberg group and constant mean curvature surfaces in $\mathbb{R}^3$ is given. Moreover, Calabi's correspondence has been generalized to higher codimension in [74].

In this part of the thesis, inspired in [6] and [36], we extend Calabi's correspondence to certain warped product spaces, establishing a duality between two wide classes of elliptic problems. It should be noted that our procedure is not a direct application of the one in [6]. In fact, we will deal with two non-homogeneous elliptic divergent form equations and will make use of a trick to deal with these equations performing suitable conformal changes of metric.

Thus, we give a duality result between solutions of the minimal surface equation in a Riemannian warped product with base a 2-dimensional Riemannian surface (B, g_B), fiber the real line and warping function $\frac{1}{\sqrt{\gamma}}$, given by

$$\operatorname{div}\left(\frac{Du}{\sqrt{\gamma + |Du|^2}}\right) = \frac{1}{2\gamma}\frac{g_B(D\gamma, Du)}{\sqrt{\gamma + |Du|^2}}, \tag{R}$$

and solutions of the maximal surface equation in a standard static spacetime with the same base (B, g_B), fiber the real line endowed with the negative of its standard metric and warping function $\sqrt{\gamma}$, which is written as

$$\operatorname{div}\left(\frac{D\omega}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}\right) = \frac{\gamma}{2}\frac{g_B\left(D\left(\frac{1}{\gamma}\right), D\omega\right)}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}, \tag{L.1}$$

$$|D\omega|^2 < \frac{1}{\gamma}. \tag{L.2}$$

Hence, Theorem 5.1 states

> *Let $\Omega \subset B$ be a simply connected domain. Then, there exists a non-trivial solution (i.e., non-constant) of* (R) *on Ω if and only if there exists a non-trivial solution of* (L) *on Ω.*

This correspondence enables us to use a known Bernstein type theorem to give a new Calabi-Bernstein result for standard static spacetimes with compact base (Corollary 5.5).

The only entire solutions of equation (L) *on the* 2*-dimensional sphere* $\mathbb{S}^2$ *endowed with a Riemannian metric are the constants.*

To conclude this thesis, we will give new uniqueness results for complete maximal hypersurfaces in a family of spacetimes with a different kind of infinitesimal symmetry. Hence, in Chapter 6 we will study these geometric objects in pp-wave spacetimes. By a pp-wave spacetime we mean a Lorentzian manifold $(\overline{M}, \overline{g})$ with a globally defined parallel lightlike vector field ξ; i.e., $\overline{g}(\xi, \xi) = 0$, $\xi \neq 0$ such that $\overline{\nabla}\xi = 0$. Note that such a Lorentzian manifold is time-orientable and, therefore, a spacetime. On the other hand, given a parallel vector field that does not vanish at one point, then it does not vanish anywhere. Moreover, if a parallel vector field is lightlike at one point, it is lightlike everywhere (i.e., a parallel lightlike vector field).

This kind of infinitesimal symmetry has been assumed a priori to obtain exact solutions to the Einstein's field equation, describing radiation (electromagnetic or gravitational) propagating at the speed of light [113]. Despite the fact that the study of gravitational waves goes back to Einstein [54], the standard exact model was introduced by Brinkmann in order to determine Einstein spaces which can be improperly mapped conformally on some Einstein one (see [32]). More recently, the geometry of pp-wave spacetimes has been studied (see [26], [37], [38] and [56]). Lately, pp-wave spacetimes have attracted a great deal of attention due to the experimental detection of gravitational waves [1].

Thus, in Section 6.1 we will study compact spacelike hypersurfaces of constant mean curvature in pp-wave spacetimes of arbitrary dimension that obey the Timelike Convergence Condition (TCC) (Theorem 6.2).

Let $\psi : M \longrightarrow \overline{M}$ *be a compact spacelike hypersurface in an* $(n+1)$*-dimensional pp-wave spacetime that satisfies the TCC. If* M *has constant mean curvature, then it must be totally geodesic. As a direct consequence, there is no compact constant mean curvature spacelike hypersuface in such* $\overline{M}$ *whose mean curvature is different from zero.*

Moreover, in Section 6.2 we generalize the classical parametric Calabi-Bernstein theorem for complete maximal surfaces in 3-dimensional pp-wave spacetimes, obtaining (Theorem 6.7)

Let $\psi : S \longrightarrow \overline{M}$ be a complete maximal surface in a 3-dimensional pp-wave spacetime that obeys the TCC. Then, S is totally geodesic and flat.

Finally, we present a brief discussion of the conclusions of this thesis as well as provide several interesting future research lines.

Chapter 2

Preliminaries

2.1 Spacelike hypersurfaces

Let $(\overline{M}, \overline{g})$ be an $(n+1)$-dimensional spacetime, i.e., a time-oriented Lorentzian manifold. Given an n-dimensional manifold M, an immersion $\psi : M \to \overline{M}$ is said to be spacelike if the Lorentzian metric $\overline{g}$ induces, via ψ, a Riemannian metric g on M. In this case, M is called a spacelike hypersurface in $\overline{M}$.

If $\mathcal{T}$ is a unitary timelike vector field on $\overline{M}$ defining its time-orientation [85, Lemma 5.32], a unique unitary timelike normal vector field $N \in \mathfrak{X}^{\perp}(M)$ can be globally defined on M in the same time-orientation of $\mathcal{T}$, i.e., such that $\overline{g}(N, \mathcal{T}) \leq -1$ and $\overline{g}(N, \mathcal{T}) = -1$ at a point $p \in M$ if and only if $N = \mathcal{T}$ at p [85, Prop. 5.30]. In order to do so, at each point $p \in M$ we can decompose $\mathcal{T}$ along ψ into its tangent and normal parts to M as follows,

$$\mathcal{T} = \mathcal{T}^T + \mathcal{T}^N. \tag{2.1}$$

Note that $\mathcal{T}^N \in \mathfrak{X}^{\perp}(M)$ has no zeros. Now, the explicit form of N is given by

$$N := \frac{1}{\sqrt{1 + |\mathcal{T}^T|^2}} \mathcal{T}^N. \tag{2.2}$$

We can also relate the Levi-Civita connections of M and $\overline{M}$ by the Weingarten and Gauss formulas for isometric immersions, which for a spacelike hypersurface are, respectively,

$$AX = -\overline{\nabla}_X N, \tag{2.3}$$

and

$$\overline{\nabla}_X Y = \nabla_X Y + \sigma(X, Y) = \nabla_X Y - g(AX, Y)N, \tag{2.4}$$

for all $X, Y \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the Levi-Civita connection of $\overline{g}$, ∇ the Levi-Civita connection of the induced metric g on M, σ is the second fundamental form and A denotes the shape operator associated to N.

In this thesis we will also use the Riemann curvature tensors of the Levi-Civita connections ∇ of M and $\overline{\nabla}$ of $\overline{M}$, which are defined, respectively, by

$$\mathrm{R}(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$ and

$$\overline{\mathrm{R}}(X, Y)Z = \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(\overline{M})$.

Thanks to (2.3) and (2.4) we can deduce the Gauss and Codazzi equations for a spacelike hypersurface M in a spacetime $\overline{M}$, which are, respectively,

$$g(\mathrm{R}(X, Y)Z, W) = \overline{g}(\overline{\mathrm{R}}(X, Y)Z, W) - g(AY, Z)g(AX, W) + g(AX, Z)g(AY, W), \tag{2.5}$$

and

$$\overline{g}(\overline{\mathrm{R}}(X, Y)N, Z) = \overline{g}((\nabla_Y A)X, Z) - \overline{g}((\nabla_X A)Y, Z), \tag{2.6}$$

for $X, Y, Z, W \in \mathfrak{X}(M)$. In fact, we will also use that the derivative of the shape operator is given by

$$(\nabla_X A)Y = \nabla_X (AY) - A(\nabla_X Y), \tag{2.7}$$

for all $X, Y \in \mathfrak{X}(M)$.

Moreover, most of the extrinsic geometry of a spacelike hypersurface M is codified in the mean curvature function associated to N, which is given by

$$H := -\frac{1}{n}\mathrm{trace}(A). \tag{2.8}$$

Note that the minus sign is introduced in order to write the mean curvature vector field of ψ as HN. We can give a physical interpretation of the sign of H in terms of the expanding/contracting behaviour of the spacetime [105]. To do so, we know that for each $p \in M$ there exists a neighborhood U of p in the spacetime $\overline{M}$ and a unitary timelike vector field $\widetilde{N} \in \mathfrak{X}(U)$ such that $\widetilde{N} = N$ at $U \cap M$, i.e., $\widetilde{N}$ locally extends N. Since on every point $p \in U \cap M$ we have

$$\overline{\nabla}_v \widetilde{N} = \overline{\nabla}_v N, \tag{2.9}$$

for all $v \in T_pM$, we can compute the divergence in $\overline{M}$ of this vector field at $p \in U \cap M$ and obtain

$$\overline{\operatorname{div}}(\widetilde{N})_p = nH(p). \tag{2.10}$$

Since the integral curves of this vector field $\widetilde{N}$ are known as the normal observers, if the mean curvature function is positive (resp. negative) at some point $p \in M$, these normal observers will measure that they are spreading out (resp. coming together).

To conclude this section, we recall that for a spacelike hypersurface $\psi : M \to \overline{M}$ we can define the hyperbolic angle φ at any point of M between the unit timelike vectors N and $\mathcal{T}$ by

$$\cosh \varphi = -\overline{g}(N, \mathcal{T}). \tag{2.11}$$

Indeed, we can give a phyisical interpretation of the hyperbolic angle. In any point p of a spacelike hypersurface M in a spacetime $\overline{M}$ we can define two families of instantaneous observers [106, pg. 73], namely, $\mathcal{T}_p$ and N_p at $p \in M$. From the orthogonal decomposition

$$N(p) = \mathcal{E}(p)\mathcal{T}(p) + N_S(p),$$

being N_S orthogonal to $\mathcal{T}$, we obtain that $\cosh \varphi(p)$ coincides with the energy $\mathcal{E}(p)$ that $\mathcal{T}(p)$ measures for $N(p)$. Moreover, the velocity measured by $\mathcal{T}(p)$ for $N(p)$ is (see [71])

$$v(p) := \frac{1}{\cosh \varphi(p)} N_S(p).$$

2.1.1 Higher order mean curvatures

Besides the mean curvature there are another extrinsic invariants of higher order on any spacelike hypersurface in a spacetime (see for instance [9] and [10]). Given the shape operator A associated to the future pointing normal vector field of a spacelike hypersurface M in a spacetime, we can define at any point $p \in M$ the following algebraic invariants

$$S_k(p) = \sum_{i_1 < \cdots < i_k} \kappa_{i_1}(p) \dots \kappa_{i_k}(p),\ 1 \leq k \leq n,$$

where $\kappa_1(p), \dots, \kappa_n(p)$ are the principal curvatures of the hypersurface (i.e., the eigenvalues of the shape operator) and $S_0 = 1$ by definition. The kth-mean curvature of the hypersurface is then defined by

$$\binom{n}{k} H_k = (-1)^k S_k,$$

for every $0 \leq k \leq n$. In particular, for $k = 1$ we have

$$H_1 = -\frac{1}{n} \sum_{i=1}^{n} \kappa_i = -\frac{1}{n} \text{trace}(A) = H.$$

Taking this into account, we can define the classical Newton transformations $P_k : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ by

$$P_k = \sum_{i=0}^{k} \binom{k}{i} H_i A^{k-i}.$$

Note that $P_0 = \mathbb{I}$ and by Cayley-Hamilton theorem we have $P_n = 0$. Let us recall that each $P_k(p)$ is a self-adjoint linear operator on each tangent space T_pM that commutes with $A(p)$. In fact, $P_k(p)$ and $A(p)$ can be simultaneously diagonalized, allowing us to obtain

$$\text{trace}(P_k) = c_k H_k \tag{2.12}$$

and

$$\text{trace}(A \circ P_k) = -c_k H_{k+1}, \tag{2.13}$$

where

$$c_k = (n-k)\binom{n}{k}$$

and

$$H_k = 0 \ \text{ if } \ k > n.$$

Associated to each Newton transformation P_k, we define the second order linear differential $L_k : C^\infty(M) \longrightarrow C^\infty(M)$ by

$$L_k(u) = \mathrm{trace}(P_k \circ \mathrm{Hess}(u)),$$

being Hess(f) the self-adjoint linear operator metrically equivalent to the Hessian of f, which is given by

$$g(\mathrm{Hess}(u)(X), Y) = g(\nabla_X(\nabla u), Y),$$

for all $X, Y \in \mathfrak{X}(M)$.

In an analogous way to what we have done in (2.9), we can locally propagate the geometric information contained in each S_k to the future and the past of M and give a physical interpretation of it (more details about this procedure can be found in [49, Rem. 2.1]).

2.2 GRW spacetimes

Let (F, g_F) be an $n(\geq 2)$-dimensional (connected) Riemannian manifold, I an open interval in $\mathbb{R}$ and f a positive smooth function defined on I. Now, consider the product manifold $\overline{M} = I \times F$ endowed with the Lorentzian metric

$$\overline{g} = -\pi_I^*(dt^2) + f(\pi_I)^2\, \pi_F^*(g_F), \tag{2.14}$$

where π_I and π_F denote the projections onto I and F, respectively. The Lorentzian manifold $(\overline{M}, \overline{g})$ is a warped product (in the sense of [85, Chap. 7]) with base $(I, -dt^2)$, fiber (F, g_F) and warping function f. If we endow $(\overline{M}, \overline{g})$ with the time orientation induced by $\partial_t := \partial/\partial t$ we can call it , agreeing with the terminology introduced in [13], an $(n+1)$-dimensional Generalized Robertson-Walker (GRW) spacetime. This family of spacetimes extends the classical notion of Robertson-Walker spacetime to the case where the fiber does not necessarily have constant sectional curvature and has arbitrary dimension. To be fair, these spacetimes should be called Generalized Friedmann-Lemaître-Robertson-Walker (GFLRW) spacetimes. However, we will follow the most common terminology in the literature.

Notice that a GRW spacetime is not necessarily spatially homogeneous. We recall that spatial homogeneity seems appropriate as a rough approach when we consider the universe on a large scale (see [81, Chap. 30], for instance). In fact, Robertson-Walker models are very useful to describe a universe that satisfies the cosmological principle, i.e., when viewed on a sufficiently large scale, the universe is spatially homogeneous and isotropic [63, Chap. 5]. However, this assumption might not be physically realistic if we want to model the universe in a more accurate scale. Therefore, GRW spacetimes can be suitable models of universes with inhomogeneous spacelike geometry [93].

Furthermore, deformations of the fiber's metric of a classical Robertson-Walker spacetime as well as GRW spacetimes with a time-dependent conformal change of metric also fit into the class of GRW spacetimes. This suggest the use of GRW spacetimes to analyze the stability of the properties of a Robertson-Walker spacetime [60].

In any GRW spacetime there exists a distinguished vector field $K := f(\pi_I)\partial_t$, which is timelike and future pointing. From the relation between the Levi-Civita connection of $\overline{M}$ and those of the base and the fiber [85, Cor. 7.35], it follows that

$$\overline{\nabla}_X K = f'(\pi_I)\, X \tag{2.15}$$

for any $X \in \mathfrak{X}(\overline{M})$, where $\overline{\nabla}$ is the Levi-Civita connection of the Lorentzian metric (2.14). Hence, K is conformal with $\mathcal{L}_K \overline{g} = 2f'(\pi_I)\overline{g}$ and its metrically equivalent 1-form is closed. This timelike conformal vector field makes GRW spacetimes conformally stationary. Roughly speaking, conformally stationary spacetimes are those that can be turned into a stationary one by means of a conformal change of metric [16]. Furthermore, there exist several criteria to decide wether a given Lorentzian manifold is (locally or globally) a GRW spacetime (see [34], [39], [78] and [108]).

According to Causality Theory, every GRW spacetime is stably causal, since the time coordinate t constitutes a global time function [23, pg. 64]. Furthermore, it is globally hyperbolic if and only if its fiber is complete [23, Thm. 3.66]. In this case, every spacelike slice is a Cauchy hypersurface.

There are a wide variety of examples of GRW spacetimes $\overline{M} = I \times_f F$ among which are included the following classical Robertson-Walker spacetimes that will appear along this thesis.

- When F is the Euclidean space $\mathbb{R}^n$, $I = \mathbb{R}$ and $f(t) = 1$, then $\overline{M}$ is the Lorentz-Minkowski spacetime $\mathbb{L}^{n+1}$. $\mathbb{L}^4$ models the trivial gravitational field: no gravity at all, and it is the basic ground for special relativity.

- If F is the unitary sphere $\mathbb{S}^n$, $I = \mathbb{R}$ and $f(t) = \cosh(t)$, then $\overline{M}$ is de Sitter spacetime $\mathbb{S}_1^{n+1}$ of constant setional curvature equal to one. $\mathbb{S}_1^4$ is an exact solution of the Einstein's field equations for an empty space with positive cosmological constant or for a certain type of perfect fluid (see, for instance, [63, Sec. 5.2]).

- There is a relevant open region of de Sitter spacetime that can be seen as $\overline{M}$ for $F = \mathbb{R}^n$, $I = \mathbb{R}$ and $f(t) = e^t$. This spacetime is called the $(n+1)$-dimensional steady state spacetime. In the 4-dimensional case it is a model of the universe proposed by Bondi and Gold [27] and Hoyle [64]. This is a model of the universe that looks the same at all points and in all directions (i.e., spatially isotropic and homogeneous) as well as at all times (see [63, Sec. 5.2] and [115, Sec. 14.8]).

- When $F = \mathbb{R}^n$, $I = \mathbb{R}^+$ and $f(t) = t^{2/3}$, then $\overline{M}$ is called the $(n+1)$-dimensional Einstein-de Sitter spacetime. In dimension four, this spacetime is a classical exact solution to Einstein's field equation without cosmological constant that incorporates spatial homogeneity and isotropy (i.e., cosmological principle) and allows expansion (see [105, Ex. 7.3.6]). Furthermore, this model shows to fit reasonably well with recent observations [114].

- If $F = \mathbb{R}^n$, $I = \mathbb{R}^+$ and $f(t) = (2at)^{1/2}$ for a fixed $a > 0$, then $\overline{M}$ is the $(n+1)$-dimensional Robertson-Walker Radiation model. These models of the universe describe the earliest era of the universe as well as the final one (if it exists), where radiation dominates over matter [85, Ch. 12].

2.2.1 Spacelike hypersurfaces in a GRW spacetime

In any GRW spacetime $\overline{M} = I \times_f F$ there is a remarkable family of spacelike hypersurfaces, namely its spacelike slices $\{t_0\} \times F$, $t_0 \in I$. It can be easily seen that a spacelike hypersurface in $\overline{M}$ is a (piece of) spacelike slice if and only if the function $\tau := \pi_I \circ \psi$ is constant. Furthermore, a spacelike hypersurface M in $\overline{M}$ is a (piece of) spacelike slice if and only if M is orthogonal to ∂_t. The shape operator of the spacelike slice $t = t_0$, with respect to $N = \partial_t|_{\{t_0\}\times F}$, is given by

$$A = -\frac{f'(t_0)}{f(t_0)}\,\mathbb{I},$$

where $\mathbb{I}$ denotes the identity transformation. Therefore, a spacelike slice is totally umbilic and its constant mean curvature is equal to the Hubble function at t_0, i.e.,

$$H = \frac{f'(t_0)}{f(t_0)}.$$

Moreover, a spacelike slice is maximal if and only if $f'(t_0) = 0$ (and hence, totally geodesic). In addition, these spacelike slices are the restspaces of the so called comoving observers [106, pg. 160]. Since the divergence in $\overline{M}$ of this vector field is

$$\overline{\text{div}}(\partial_t) = n\frac{f'(t)}{f(t)}, \tag{2.16}$$

we can see how the comoving observers will measure that the universe is (locally) expanding or contracting depending on the sign of $f'(t)$. Thus, if $f'(t) > 0$ the observers in ∂_t are on average spreading apart. Similarly, if $f'(t) < 0$ they are coming together [106, pg. 121].

Let $\psi : M \to \overline{M}$ be an n-dimensional spacelike hypersurface in a GRW spacetime $\overline{M}$. If we denote by

$$\partial_t^T := \partial_t + \overline{g}(N, \partial_t)N$$

the tangential component of ∂_t along ψ, then it is easy to check that the gradient of τ is

$$\nabla\tau = -\partial_t^T. \tag{2.17}$$

In fact, we can relate the hyperbolic angle and the norm of the gradient of τ using (2.11) and (2.17), getting

$$\sinh^2\varphi = |\nabla\tau|^2. \tag{2.18}$$

Moreover, since the tangential component of K along ψ is given by $K^T = K + \overline{g}(K, N)N$, a direct computation from (2.15) gives

$$\nabla\overline{g}(K, N) = -AK^T, \tag{2.19}$$

being A the shape operator with respect to N chosen as in (2.2) with $\mathcal{T} = \partial_t$. Taking the tangential component in (2.15) and using (2.4) and (2.3), we get

$$\nabla_X K^T = -f(\tau)\overline{g}(N, \partial_t)AX + f'(\tau)X, \tag{2.20}$$

where $X \in \mathfrak{X}(M)$ and $f'(\tau) := f' \circ \tau$. Since $K^T = f(\tau)\partial_t^T$, it follows from (2.17) and (2.20) that the Laplacian of τ on M is

$$\Delta\tau = -\frac{f'(\tau)}{f(\tau)}\{n + |\nabla\tau|^2\} - nH\overline{g}(N, \partial_t). \tag{2.21}$$

From (2.21) we have

$$\begin{aligned}\Delta f(\tau) &= f'(\tau)\Delta\tau + f''(\tau)|\nabla\tau|^2 \\ &= -n\,\frac{f'(\tau)^2}{f(\tau)} + |\nabla\tau|^2 f(\tau)(\log f)''(\tau) + nHf'(\tau)\cosh\varphi. \qquad (2.22)\end{aligned}$$

Taking now into account the Codazzi equation (2.6) and choosing around a point $p \in M$ a local orthonormal reference frame $\{E_1, \ldots, E_n\}$ such that $\nabla_{E_i}E_j = 0$ at p, for all $i, j = 1, \ldots, n$ we get

$$\overline{\mathrm{Ric}}(K^T, N) = \sum_{i=1}^{n} g((\nabla_{K^T}A)E_i, E_i) - \sum_{i=1}^{n} g((\nabla_{E_i}A)K^T, E_i). \qquad (2.23)$$

Taking (2.7) into account, (2.23) yields at p

$$\begin{aligned}\overline{\mathrm{Ric}}(K^T, N) &= \sum_{i=1}^{n} g(\nabla_{K^T}(AE_i), E_i) - \sum_{i=1}^{n} g(\nabla_{E_i}(AK^T), E_i) \\ &\quad + \sum_{i=1}^{n} g(\nabla_{E_i}K^T, AE_i). \qquad (2.24)\end{aligned}$$

Using (2.19), (2.20) and the fact that derivations commute with contractions, from (2.24) we obtain

$$\overline{\mathrm{Ric}}(K^T, N) = -ng(K^T, \nabla H) + \Delta\overline{g}(K, N) - \overline{g}(K, N)\mathrm{trace}(A^2) - nf'(\tau)H. \qquad (2.25)$$

On the other hand, from (2.20) we also obtain

$$\mathrm{div}(K^T) = nf(\tau)H\overline{g}(N, \partial_t) + nf'(\tau).$$

Hence, we get

$$\mathrm{div}(HK^T) = nf(\tau)H^2\overline{g}(N, \partial_t) + nf'(\tau)H + f(\tau)\overline{g}(\nabla H, \partial_t). \qquad (2.26)$$

2.2.2 The mean curvature equation for a spacelike graph in a GRW spacetime

Among the family of spacelike hypersurfaces in a GRW spacetime we should highlight the subfamily of spacelike graphs. Recall that any spacelike hypersurface in a GRW spacetime is locally the graph of a smooth function on a domain of its fiber. Moreover, under certain assumptions a spacelike hypersurface must be a spacelike graph [13].

Let (F, g_F) be an $n(\geq 2)$-dimensional Riemannian manifold and let $f : I \longrightarrow \mathbb{R}^+$ be a smooth function. Consider now in the GRW spacetime $\overline{M} = I \times_f F$ the graph

$$\Sigma_u = \{(u(p), p) : p \in \Omega\},$$

where Ω is a subdomain of F, $u \in C^\infty(\Omega)$ and $u(\Omega) \subset I$. The induced metric on Ω from the Lorentzian metric on $\overline{M}$, via the graph Σ_u is given by

$$g_u = -du^2 + f(u)^2 g_F. \tag{2.27}$$

The metric g_u is positive definite (i.e., Σ_u is spacelike) if and only if u satisfies

$$|Du| < f(u). \tag{2.28}$$

In this case,

$$N = \frac{1}{f(u)\sqrt{f(u)^2 - |Du|^2}} \left(f(u)^2 \partial_t + Du\right) \tag{2.29}$$

is a future pointing unit normal vector field on Σ_u. In the particular case where $\Omega = F$, the spacelike graph is said to be entire. A long but easy computation, taking into acount [85, Prop. 7.35], allows us to obtain the shape operator corresponding to N,

$$\begin{aligned} A(X) \quad = \quad & \frac{-f(u)}{\sqrt{f(u)^2 - |Du|^2}} \left(\frac{f'(u)}{f(u)} X - \frac{f'(u)\; g_F(Du, X)}{f(u)\,(f(u)^2 - |Du|^2)} Du \right. \\ & \left. + \frac{g_F(D_X Du, Du)}{f(u)^2\,(f(u)^2 - |Du|^2)} Du + \frac{1}{f(u)^2} D_X Du \right) \end{aligned} \tag{2.30}$$

for all $X \in \mathfrak{X}(\Omega)$. Taking the trace in (2.30), we obtain that the mean curvature function of a spacelike graph associated to N is

$$H = \operatorname{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|^2}}\right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}}\left(n + \frac{|Du|^2}{f(u)^2}\right), \tag{2.31}$$

where div represents the divergence operator in (F, g_F).

2.3 Standard static spacetimes

Let us consider the product manifold $\overline{M} = B \times \mathbb{R}$, where (B, g_B) is an $n(\geq 2)$-dimensional (connected) Riemannian manifold, endowed with the Lorentzian metric

$$\overline{g} = \pi_B^*(g_B) - h(\pi_B)^2 \pi_{\mathbb{R}}^*(dt^2), \tag{2.32}$$

where π_B and $\pi_{\mathbb{R}}$ denote, respectively, the projections on B and $\mathbb{R}$ and h is a smooth positive function on B. The Lorentzian manifold $(\overline{M} = B \times \mathbb{R}, \overline{g})$ is a warped product in the sense of [85, Ch. 7] with base (B, g_B), fiber $(\mathbb{R}, -dt^2)$ and warping function h. If we endow $(\overline{M}, \overline{g})$ with the time orientation induced by $\partial_t := \partial/\partial t$, we can call it an $(n+1)$-dimensional standard static spacetime, since we can easily see that the timelike vector field ∂_t is Killing.

The importance of standard static spacetimes also comes from the fact that they include some classical spacetimes, such as Lorentz-Minkowski spacetime, Einstein static universe as well as models that describe a universe where there is only a spherically symmetric non-rotating mass, as a star or a black hole, like exterior Schwarzschild spacetime [106]. In fact, in [85, Prop. 12.38] it is proved that any static spacetime is locally isometric to a standard static one (see also [8] and [109] for sufficient conditions for a static spacetime to be standard).

In any standard static spacetime $\overline{M} = B \times_h \mathbb{R}$ there is a remarkable family of spacelike hypersurfaces, namely, it spacelike slices $B \times \{t_0\}$, $t_0 \in \mathbb{R}$, which are the level hypersurfaces of the function $\pi_{\mathbb{R}}$. It can be easily seen that the shape operator of any spacelike slice vanishes identically, i.e., they are totally geodesic. Indeed, these totally geodesic spacelike slices constitute a foliation in any standard static spacetime.

Furthermore, these spacelike slices are the restspaces of a distinguished family of observers given by the integral curves of the unitary timelike vector field $\frac{1}{h}\partial_t$. Since the vector field ∂_t is Killing, its local flows are isometries in $\overline{M}$ that preserve the restspaces of the observers in $\frac{1}{h}\partial_t$. Physically, this means that the spatial universe measured by each observer in $\frac{1}{h}\partial_t$ always looks the same.

2.3.1 The mean curvature equation for a spacelike graph in a standard static spacetime

Among the family of spacelike hypersurfaces in a standard static spacetime we will focus on the subfamily of spacelike graphs. Let (B, g_B) be a Riemannian manifold and let $h : B \longrightarrow \mathbb{R}$

be a positive smooth function. For each $u \in C^\infty(B)$ we can consider its graph

$$\Sigma_u = \{(p, u(p)) : p \in B\}$$

in the standard static spacetime $\overline{M} = B \times_h \mathbb{R}$. The induced metric on B from the Lorentzian metric on $\overline{M}$ via the graph Σ_u is given by

$$g_u = g_B - h^2 du^2, \tag{2.33}$$

which is positive definite (i.e., Σ_u is spacelike) if and only if u satisfies

$$|Du| < \frac{1}{h}, \tag{2.34}$$

where $|Du|$ denotes the norm of the gradient of u in (B, g_B). In this case, the future pointing unit normal vector field on Σ_u is

$$N = \frac{h}{\sqrt{1 - h^2|Du|^2}} \left(Du + \frac{1}{h^2}\, \partial_t \right) \tag{2.35}$$

On the other hand, standard computations show that the mean curvature function H of Σ_u in $\overline{M}$ with respect to N is given by

$$nH = \operatorname{div} \left(\frac{h\ Du}{\sqrt{1 - h^2|Du|^2}} \right) + \frac{g_B(Dh, Du)}{\sqrt{1 - h^2|Du|^2}}, \tag{2.36}$$

being div the divergence operator in (B, g_B).

The differential equation (2.36) for $H = 0$ with the constraint (2.34) is called the maximal hypersurface equation in $\overline{M}$. The solutions of this equation are maximal graphs in $\overline{M}$. Note that the constraint (2.34) is nothing but the ellipticity condition of this equation.

2.4 pp-wave spacetimes

Now, we will consider a Lorentzian manifold $(\overline{M}, \overline{g})$ that admits a globally defined parallel lightlike vector field ξ; i.e., $\overline{g}(\xi, \xi) = 0$, $\xi \neq 0$ such that $\overline{\nabla}\xi = 0$. Such a Lorentzian manifold is time-orientable. In fact, we can endow $\overline{M}$ with a time-orientation choosing as the future cone the one that contains this lightlike vector field in its boundary. Hence, if $\overline{M}$ is also connected, it becomes a spacetime. Following the terminology used in [113] we will call $\overline{M}$ a pp-wave spacetime. However, this notation is not universally accepted, since several authors call $\overline{M}$ a Brinkmann spacetime [26].

This family of spacetimes contains exact solutions to the Einstein's field equation that model radiation (electromagnetic or gravitational) moving at the speed of light [113]. Indeed, although the study of gravitational waves goes back to Einstein [54], the standard exact model was introduced by Brinkmann in order to determine Einstein spaces which can be mapped conformally, in the sense of [32], on some Einstein one. Indeed, it was proved in [32] that any point of an $n(\geq 3)$-dimensional pp-wave spacetime admits a so called Brinkmann chart where the Lorentzian metric $\overline{g}$ can be written as

$$\mathcal{H}(u,x_1,\ldots,x_{n-2})\,du^2+2\,du\,dv+2\sum_{i=1}^{n-2}W_i(u,x_1,\ldots,x_{n-2})du\,dx_i+\sum_{i,j=1}^{n-2}\overline{g}_{ij}(u,x_1,\ldots,x_{n-2})dx_i\,dx_j,$$

where $\mathcal{H}$ is a smooth function and the parallel vector field ξ coincides with $-\partial_v$ on the corresponding coordinate neighborhood (see also [26]). More recently, these spacetimes have attracted a great deal of attention due to the experimental detection of gravitational waves [1].

Although the symmetry imposed in the spacetime may seem too restrictive, from the following result we see that more general assumptions also lead to our model.

Proposition 2.1. *Every closed conformal lightlike vector field in a Lorentzian manifold is parallel.*

Proof. Let ζ be a conformal lightlike vector field in a Lorentzian manifold $(\overline{M},\overline{g})$, that is, $\overline{g}(\zeta,\zeta)=0$, $\zeta\neq 0$ and its Lie derivative $\mathcal{L}$ satisfies

$$\mathcal{L}_\zeta\overline{g}=2\;\lambda\;\overline{g}, \tag{2.37}$$

where λ is a smooth function on $\overline{M}$. If ζ is locally a gradient vector field (closed), (2.37) is equivalent to

$$\overline{\nabla}_X\zeta=\lambda\;X, \tag{2.38}$$

for all $X\in\mathfrak{X}(\overline{M})$. If ζ is lightlike, using (2.38) we get

$$0=X\overline{g}(\zeta,\zeta)=2\;\overline{g}(\overline{\nabla}_X\zeta,\zeta)=2\;\lambda\;\overline{g}(X,\zeta), \tag{2.39}$$

for all $X\in\mathfrak{X}(\overline{M})$. Therefore, since λ must be identically zero, ζ is parallel. □

2.4.1 Spacelike hypersurfaces in a pp-wave spacetime

Let $\psi : M \longrightarrow \overline{M}$ be a spacelike hypersurface in an $(n+1)$-dimensional pp-wave spacetime. We can decompose the globally defined parallel lightlike vector field $\xi \in \mathfrak{X}(\overline{M})$ along ψ into its tangent and normal part, having

$$\xi = \xi^T + \xi^N. \tag{2.40}$$

Clearly, ξ^N is timelike everywhere on M and therefore, a unitary future pointing timelike normal vector field N can be defined from ξ^N by

$$N := \frac{1}{\sqrt{\overline{g}(\xi^T, \xi^T)}} \xi^N.$$

Observe that ξ^T never vanishes on M. Recall that the existence of a vector field which never vanishes yields to some topological obstructions when the hypersurface M is compact. In the following example we find a pp-wave spacetime that admits a compact spacelike hypersurface.

Example 2.2. Consider the Lorentzian manifold $(\overline{M}, \overline{g})$ where $\overline{M}$ is given by the product $\Sigma \times \mathbb{S}^1 \times \mathbb{R}$, endowed with the Lorentzian metric

$$\overline{g} = g_{\Sigma} + 2d\alpha dv + \mathcal{H}(x, \alpha) d\alpha^2,$$

where $x \in \Sigma$, (Σ, g_{Σ}) is a compact Riemannian manifold, $\mathbb{S}^1$ denotes the unitary sphere, α is an angular coordinate on $\mathbb{S}^1$ and $v \in \mathbb{R}$. It is clear that the vector field $\frac{\partial}{\partial v}$ on $\overline{M}$ is a parallel lightlike vector field and $\overline{M}$ admits a compact spacelike hypersurface as long as $\mathcal{H}$ is positive.

Moreover, taking the covariant derivative in (2.40) with respect to any vector field $X \in \mathfrak{X}(M)$ and using the Gauss and Weingarten formulas (2.4) and (2.3), we get

$$0 = \overline{\nabla}_X \xi = \nabla_X \xi^T + \sigma(X, \xi^T) + \overline{g}(N, \xi) AX + \overline{g}(AX, \xi) N, \tag{2.41}$$

Decomposing (2.41) into its tangent part, we obtain

$$\nabla_X \xi^T = -\overline{g}(N, \xi) AX \tag{2.42}$$

In particular, from (2.42) and using the fact that the function $\overline{g}(N, \xi)$ never vanishes because N is timelike and ξ is lightlike, we see that M is totally geodesic if and only if ξ^T is parallel.

2.5 Energy conditions

Along this work we will make use of certain energy assumptions in our ambient spacetimes. Thus, we will clarify them in this section. A spacetime $\overline{M}$ obeys the Timelike Convergence Condition (TCC) if its Ricci tensor satisfies

$$\overline{\mathrm{Ric}}(X,X) \geq 0,$$

for all timelike vectors X. It is usually argued that the TCC is the mathematical way to express that gravity, on average, attracts (see [85, pg. 340]). Furthermore, if the spacetime satisfies the Einstein equation (without cosmological constant) with a physically reasonable stress-energy tensor, then it must obey the TCC [106, Ex. 4.3.7]. A weaker energy condition is the Null Convergence Condition (NCC), which reads

$$\overline{\mathrm{Ric}}(Z,Z) \geq 0,$$

for all lightlike vector Z. A continuity argument shows that the TCC implies the NCC.

In particular, in an $(n+1)$-dimensional GRW spacetime $\overline{M} = I \times_f F$, from [85, Cor. 7.43] we obtain that the Ricci tensor $\overline{\mathrm{Ric}}$ of $\overline{M}$ is given by

$$\begin{aligned}\overline{\mathrm{Ric}}(X,Y) &= \mathrm{Ric}^F(X_F,Y_F) + \left(\frac{f''(t)}{f(t)} + (n-1)\frac{f'(t)^2}{f(t)^2}\right)\overline{g}(X_F,Y_F) \\ &\quad -n\frac{f''(t)}{f(t)}\overline{g}(X,\partial_t)\overline{g}(Y,\partial_t),\end{aligned} \tag{2.43}$$

for all $X, Y \in T\overline{M}$, where $X_F := X + \overline{g}(X,\partial_t)\partial_t$ and $Y_F := Y + \overline{g}(Y,\partial_t)\partial_t$ are the projections of the vectors on the fiber F and Ric^F denotes the Ricci tensor of F.

From (2.43) we can easily see that a GRW spacetime obeys the NCC if and only if

$$\mathrm{Ric}^F(X_F,X_F) - (n-1)f(t)^2(\log f)''(t)g_F(X_F,X_F) \geq 0, \tag{2.44}$$

for all X_F tangent to the fiber F. Note that we can define G_F as the symmetric tensor whose associated quadratic form on each T_pF is given by the left hand side of (2.44) then the NCC means that G_F is positive semidefinite everywhere. If G_F is positive semidefinite everywhere, we will say that the NCC is strictly satisfied.

Furthermore, it obeys the TCC if and olny if the NCC holds and $f''(t) \leq 0$. Note that any Einstein spacetime (in particular, an Einstein GRW spacetime [14]) trivially satisfies the NCC.

It is also possible to give a characterization of the TCC for different subfamilies of pp-wave spacetimes in a similar way as we have done for GRW spacetimes. A particular class of pp-wave spacetimes are the so called plane fronted waves (PFW), which are given by the product manifold $\overline{M} = \mathbb{R}^2 \times \mathcal{M}$, where $(\mathcal{M}, g_{\mathcal{M}})$ is a connected n-dimensional Riemannian manifold endowed with the Lorentzian metric

$$\overline{g} = \mathcal{H}(u,x)\ du^2 + 2\ du\ dv + \pi^*_{\mathcal{M}} g_{\mathcal{M}},$$

where $x \in \mathcal{M}$. It was proved in [37, Prop. 2.2] that a PFW obeys the TCC if and only if the Ricci tensor of $(\mathcal{M}, g_{\mathcal{M}})$ is positive semidefinite and $\mathcal{H}(u,\cdot)$ is superharmonic on $(\mathcal{M}, g_{\mathcal{M}})$.

2.6 The Omori-Yau maximum principle

In this section we will recall this important technique to study the global behaviour of a smooth function on a non-compact Riemannian manifold. Later we will use this technique for studying maximal hypersurfaces in some spatially open spacetimes. Following the terminology introduced in [92], we have the following definition

Definition 2.3. *Let (M, g) be a (not necessarily complete) Riemannian manifold. The Omori-Yau maximum principle for the Laplacian is said to hold on M if for any function $u \in C^2(M)$ with $u^* = \sup_M u < +\infty$ there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset M$ satisfying*

$$\textit{(i) } u(x_k) > u^* - \tfrac{1}{k}, \quad \textit{(ii) } |\nabla u(x_k)| < \tfrac{1}{k}, \quad \textit{(iii) } \Delta u(x_k) < \tfrac{1}{k}$$

for each $k \in \mathbb{N}$. Equivalently, for any function $u \in C^2(M)$ with $u_* = \inf_M u > -\infty$ there exists a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$\textit{(i) } u(x_k) < u_* + \tfrac{1}{k}, \quad \textit{(ii) } |\nabla u(x_k)| < \tfrac{1}{k}, \quad \textit{(iii) } \Delta u(x_k) > -\tfrac{1}{k}$$

for each $k \in \mathbb{N}$.

The classical result given by Omori [84] and Yau [116] can be stated in this terminology in the following way

Theorem 2.4. *The Omori-Yau maximum principle for the Laplacian holds on every complete Riemannian manifold with Ricci curvature bounded from below.*

More generally, it has been shown that a controlled radial decay of the Ricci curvature suffices to guarantee the validity of the Omori-Yau maximum principle on a Riemannian manifold. In particular, [11, Thm. 5.2] states

Theorem 2.5. *Let (M, g) be a complete, noncompact, Riemannian manifold of dimension n; let $o \in M$ be a reference point and denote by $r(x)$ the Riemannian distance function from o. Assume that the Ricci curvature of M satisfies*

$$\mathrm{Ric} \leq -(n-1)G^2(r)g,$$

where $G \in C^1(\mathbb{R}_0^+)$ satisfies

$$\text{(i) } G(0) > 0, \quad \text{(ii) } G'(t) \geq 0, \quad \text{(iii) } G^{-1}(t) \notin L^1(+\infty).$$

Then the Omori-Yau maximum principle for the Laplacian holds on M.

This result generalizes [40, Thm. 2.2], where it was proved that this maximum principle holds on every complete Riemannian manifold whose Ricci curvature has a strong quadratic decay. Indeed, as it was observed in [92], the validity of the Omori-Yau maximum principle does not depend on curvature bounds as much as it may be expected. Actually, we have the next theorem obtained in [4], which improves [92, Thm. 1.9] and expresses in a function theoretic form a condition to guarantee the validity of this principle.

Theorem 2.6. *The Omori-Yau maximum principle for the Laplacian holds on every Riemannian manifold (M, g) admitting a C^2 function $\gamma : \mathbb{M} \longrightarrow \mathbb{R}$ satisfying the following requirements:*

(i) $\gamma(x) \to +\infty$ *as* $x \to \infty$;

(ii) $|\nabla \gamma| \leq G(\gamma)$ *outside a compact subset of M*;

(iii) $\Delta \gamma \leq G(\gamma)$ *outside a compact subset of M*,

with $G \in C^1(\mathbb{R}^+)$, positive near infinity and such that

$$G^{-1}(t) \notin L^1(+\infty) \quad \text{and} \quad G'(t) \geq -A(\log t + 1),$$

for $t >> 1$ and $A \geq 0$.

More sufficient conditions to guarantee the validity of this maximum principle can be found in [11]. An important consequence of the Omori-Yau maximum principle for the Laplacian is the next useful result obtained in [41] and [83].

Lemma 2.7. *Let M be a Riemannian manifold where the Omori-Yau maximum principle for the Laplacian holds and let $u : M \longrightarrow \mathbb{R}$ be a non-negative smooth function on M. If there exists a constant $c > 0$ such that*

$$\Delta u \geq cu^2,$$

then u identically vanishes on M.

Proof. Let us consider the positive function $F \in C^\infty(M)$ given by

$$F = \frac{1}{\sqrt{1+u}}.$$

If we compute the gradient and the Laplacian of this function we obtain

$$\Delta u = 6\frac{|\nabla F|^2}{F^4} - 2\frac{\Delta F}{F^3}. \tag{2.45}$$

Since there exists a positive constant c such that $\Delta u \geq cu^2$, we have from (2.45)

$$0 \leq c\frac{u^2}{(1+u^2)^2} \leq 6|\nabla F|^2 - 2F\Delta F. \tag{2.46}$$

Applying Omori-Yau maximum principle to the function F we can choose a sequence of points $\{x_k\}_{k\in\mathbb{N}} \subset M$ satisfying

$$\text{(i) } F(x_k) < \inf_M F + \tfrac{1}{k}, \quad \text{(ii) } |\nabla F(x_k)| < \tfrac{1}{k}, \quad \text{(iii) } \Delta F(x_k) > -\tfrac{1}{k}$$

for each $k \in \mathbb{N}$. Notice that F will tend to its infimum if and only if u tends to its supremum. Therefore, we get from (2.46)

$$0 \leq c\frac{u^2(x_k)}{(1+u^2(x_k))^2} \leq 6|\nabla F(x_k)|^2 - 2F(x_k)\Delta F(x_k) \leq \frac{6}{k} + \frac{2}{k}F(x_k). \tag{2.47}$$

Taking limits as k tends to infinity, from (2.47) we obtain

$$\sup_M u = \lim_{k\to+\infty} u(x_k) = 0,$$

which finishes the proof. □

2.7 Parabolicity

To introduce the notion of parabolicity we need to define first the concepts of superharmonic and subharmonic functions. A function u defined in a subdomain Ω of a Riemannian manifold

M is said to be superharmonic if it is continuous and if, for any relatively compact region $U \subset\subset \Omega$ and any harmonic function $v \in C^2(U) \cap C^0(\overline{U})$, $u \geq v$ on ∂U implies $u \geq v$ on U. If $u \in C^2(\Omega)$ this definition is equivalent to

$$\Delta u \leq 0,$$

where Δ denotes the Laplacian on M. A function u is said to be subharmonic provided that $-u$ is superharmonic. In this thesis we will assume that the functions that we will consider are differentiable enough to take as definition of subharmonicity the second one.

According to the well known "Liouville-type" property we have the following definition of parabolicity (see, for instance, [67])

Definition 2.8. *A complete (non-compact) Riemannian manifold M is parabolic if the only bounded from below, superharmonic C^2-functions on M are the constants.*

There are different approaches to study parabolicity. From a physical viewpoint, parabolicity is closely related to Brownian motion. This phenomenon describes the irregular motion of microscopic particles suspended in a still liquid. It was first observed in 1828, when the botanist Robert Brown noticed a ceaseless erratic motion of pollen grains in water. It was not until 1905 when Albert Einstein explained this effect as the result of the physical collisions between particles in suspension and molecules [53]. This stochastic process was proved to satisfy a diffusion equation and the experimental confirmation of the predictions on this phenomenon gave strong arguments in favor of the molecular-kinetic theory of heat.

In a Riemannian manifold, parabolicity is equivalent to the recurrence of the Brownian motion [62]. Roughly speaking, this means that any particle will pass through any open set at an arbitrarily large time.

From a mathematical perspective, parabolicity of Riemannian surfaces is closely related to their Gaussian curvature. Indeed, a key result by Ahlfors and Blanc-Fiala-Huber [65] avers that

Theorem 2.9. *Every complete Riemannian surface with non-negative Gaussian curvature is parabolic.*

In higher dimension, parabolicity of Riemannian manifolds has no clear relation with the sectional curvature. In fact, the Euclidean space $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$. Nevertheless, there are sufficient conditions for parabolicity of a Riemannian manifold of arbitrary dimension based on the volume growth of its geodesic balls (see [11], [75], [107] and references therein).

Chapter 3

Mean curvature of a spacelike hypersurface in a spatially closed GRW spacetime

Once we have described the ambient spacetimes where we are going to work, we will focus in this chapter on the family of spatially closed GRW spacetimes. A spacetime is said to be spatially closed if it admits any compact spacelike hypersurface. From a historical point of view, spatially closed models have a great importance. For instance, classical Robertson-Walker spacetimes include some whose fiber is $\mathbb{S}^3$. Physically, spatially closed models describe a spacetime where the physical space is finite (at least for a family of observers) although the whole spacetime is not (recall that a compact spacetime is not physically admissible [85, Lemma 14.10]).

For GRW spacetimes it was proved in [13] that any spatially closed one has compact fiber. Conversely, the compactness of the universal covering of the fiber is inherited by every complete spacelike hypersurface where $f(\pi_I)$ is bounded [13, Prop. 3.2]. Throughout this chapter we will characterize the spacelike slices of several spatially closed GRW spacetimes by their volume variation. We refer to [89] in this part of the thesis.

3.1 A new problem in spatially closed GRW spacetimes

We would like to focus now on the physical space measured by the comoving observers of a GRW spacetime. The volume of the physical space measured by any comoving observer γ at an instant t of its proper time is given by

$$\mathrm{Vol}_\gamma(t) = f(t)^n \mathrm{Vol}(F), \tag{3.1}$$

where $\mathrm{Vol}(F)$ is the volume of the fiber (F, g_F). Indeed, the comoving observers determine a foliation of the GRW spacetime by spacelike slices. From (3.1), we see that the volume change that the comoving observers measure for the spacelike slices is given by the function $nf(t)^{n-1}f'(t)$. In general, we can consider the volume variation of the leaves of a foliation of the spacetime by closed spacelike hypersurfaces and ask ourselves the following question:

Does there exist another (local) foliation of a spatially closed GRW spacetime by closed spacelike hypersurfaces with the same volume variation that the comoving observers measure for their restspaces?

Since a compact spacelike hypersurface in a spatially closed GRW spacetime with simply connected fiber is a graph [13, Prop. 3.3], to obtain the volume variation of the leaves of a foliation by spacelike hypersurfaces we will study the volume variation of entire spacelike graphs. Thus, let us consider in $\overline{M}$ the spacelike graph

$$\Sigma_u = \{(u(p), p) : p \in F\},$$

where F is the fiber, $u \in C^\infty(F)$, $u(F) \subset I$ and satisfies (2.28). It is well known that the volume of this spacelike graph is

$$\mathrm{Vol}(\Sigma_u) = \int_F f(u)^{n-1}\sqrt{f(u)^2 - |Du|^2}\, dV^F, \tag{3.2}$$

where dV^F is the canonical measure on (F, g_F). Taking (3.1) and (3.2) into account, we can easily see that the spacelike graphs that answer our question will be the ones defined by a function that solves the following variational problem:

Let $f : I \longrightarrow \mathbb{R}$ be a positive smooth function on an open interval $I \subseteq \mathbb{R}$ and let (F, g_F) be an n-dimensional compact Riemannian manifold. Consider the class of smooth real valued functions u on F such that $u(F) \subset I$ and the length of the gradient of u in (F, g_F) satisfies $|Du| < f(u)$. On this class consider the functional

$$\mathcal{I}(u) := \int_F \left[f(u)^{n-1}\sqrt{f(u)^2 - |Du|^2} - f(u)^n \right] dV^F. \tag{3.3}$$

Note that the stationary points of this functional will determine foliations of $\overline{M}$ by spacelike graphs with the same volume variation than the foliation by spacelike slices. For stationary points of this functional, the Euler-Lagrange equation is written as

$$H(u) = \frac{f'(u)}{f(u)}, \tag{E.1}$$

$$|Du| < f(u), \tag{E.2}$$

being $H(u)$ the smooth function given by

$$H(u) = \operatorname{div}\left(\frac{Du}{nf(u)\sqrt{f(u)^2 - |Du|^2}}\right) + \frac{f'(u)}{n\sqrt{f(u)^2 - |Du|^2}}\left(n + \frac{|Du|^2}{f(u)^2}\right). \tag{3.4}$$

Indeed, $H(u)$ represents the mean curvature of Σ_u in $\overline{M}$ with respect to N. Thus, the variational problem that answers our question is equivalent to a mean curvature prescription problem.

Hence, in this chapter we will prove some uniqueness results for entire solutions of a prescribed mean curvature problem in spatially closed GRW spacetimes. In fact, the prescription function is the Hubble function of the spacetime f'/f, which has an important physical meaning in General Relativity, since we have seen in (2.16) that its sign is related to the expansion of the universe. Therefore, we will study the problem of determining the compact spacelike hypersurfaces in this class of spacetimes whose mean curvature function at each point is the same than the mean curvature of the spacelike slice that passes through that point.

We can make some comments on the meaning of (E): **1)** This equation represents a mean curvature prescription problem, i.e., we are dealing with a second order nonlinear elliptic problem. **2)** The right member of (E.1) is, at each point $(u(p), p) \in \overline{M}$, the mean curvature of the spacelike slice $t = u(p)$. **3)** However, this is not a comparison assumption between extrinsic quantities of two spacelike hypersurfaces in $\overline{M}$, since the right member of (E.1) corresponds to a spacelike slice that changes at each point. **4)** When the warping function is constant, (E) turns into the well-known maximal hypersurface equation. **5)** Problems related to this one have been previously studied in the particular case where the fiber is a complete noncompact Riemannian surface (see [35], [98] and [102]) as well as for compact 2-dimensional Riemannian fiber in [99]. The techniques used in these cases strongly depended on the rich conformal geometry in the 2-dimensional case.

It is clear that any constant function $u = t_0$ is a solution of (E). In this section we will state several characterizations results of these trivial entire solutions.

3.2 Uniqueness results when the warping function is monotone

First of all, we will aproach our problem from a parametric viewpoint. Therefore, we want to find if there are other compact spacelike hypersurfaces appart from the spacelike slices satisfying

$$H = \frac{f'(\tau)}{f(\tau)}.$$

Indeed, we will go one step further proving the next result.

Theorem 3.1. *The only compact spacelike hypersurfaces M in a GRW spacetime $\overline{M}$ whose mean curvature function satisfies*

$$H \leq \frac{f'(\tau)}{f(\tau)} \qquad \left(\textit{resp. } H \geq \frac{f'(\tau)}{f(\tau)} \right),$$

with $f'(\tau) \leq 0$ (resp. with $f'(\tau) \geq 0$) are the spacelike slices.

Proof. Let us consider a primitive function $\mathcal{F}$ of f and write $\mathcal{F}(\tau)$ for the restriction of $\mathcal{F} \circ \pi_I$ on M. Note that

$$\nabla \mathcal{F}(\tau) = f(\tau) \nabla \tau.$$

Using (2.21) we can compute this function's Laplacian as

$$\Delta \mathcal{F}(\tau) = f(\tau)\Delta\tau + f'(\tau)|\nabla\tau|^2 = -nf'(\tau) - nf(\tau)H\overline{g}(N, \partial_t). \tag{3.5}$$

From our assumptions and (3.5) we have

$$\Delta \mathcal{F}(\tau) \leq -nf'(\tau)(1 + \overline{g}(N, \partial_t)) \leq 0.$$

Hence, the compactness of M implies that $\mathcal{F}$ must be constant. Consequently,

$$\nabla \mathcal{F}(\tau) = f(\tau)\nabla\tau = 0$$

and M is a spacelike slice. We can prove analogously the other statement. □

In particular, we have obtained

The only compact spacelike hypersurfaces in a GRW spacetime such that $f'(\tau)$ is signed and whose mean curvature function H satisfies

$$H = \frac{f'(\tau)}{f(\tau)}$$

are the spacelike slices.

Theorem 3.1 also yields to

The only compact spacelike hypersurfaces in a non-expanding (resp. non-contracting) GRW spacetime whose mean curvature function satisfies

$$H \leq \frac{f'(\tau)}{f(\tau)} \qquad \left(\textit{resp. } H \geq \frac{f'(\tau)}{f(\tau)} \right)$$

are the spacelike slices.

Remark 3.2. Notice that in Theorem 3.1: **a)** There is no curvature assumption on the spacetime. In fact, if we wanted to obtain a similar result using the technique suggested in [99, Rem. 2.3], we would need to impose a certain kind of curvature restriction. **b)** The compactness assumption on the spacelike hypersurface cannot be weakened to completeness. Indeed, a trivial obstruction is given by any spacelike hyperplane in the $(n+1)$-dimensional Lorentz-Minkowski spacetime. Even more, there exist complete spacelike hypersurfaces in the $(n+1)$-dimensional steady state spacetime satisfying $H = 1$ different from the spacelike slices [5]. **c)** Despite $f'(t)$ not being signed, we can add some assumption to the spacelike hypersurface in order to use Theorem 3.1 (see Proposition 3.6). **d)** If the inequality satisfied by the mean curvature function is strict the result turns into a nonexistence one.

Remark 3.3. We will now give an example to show how the compactness assumption in Theorem 3.1 can not weakened to completeness in general. To do so, we will consider the map

$$\psi : \mathbb{R}^n \longrightarrow \mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$$

given by

$$x \mapsto \left(\operatorname{arcsinh}\left(\frac{|x|^2}{2}\right), \frac{1}{\sqrt{1+\left(\frac{|x|^2}{2}\right)^2}} \left(\frac{|x|^2}{2} - 1\right), \frac{1}{\sqrt{1+\left(\frac{|x|^2}{2}\right)^2}}\, x \right).$$

Note that ψ is an isometric immersion with $H = 1$ with our choice of N. In fact, this hypersurface has been previously studied in dimension $n = 2$ in [46] (see also [94, Rem. 9] and [82, Ex. 1(b)]).

We can see how this map defines a complete (non compact) spacelike hypersurface in de Sitter spacetime which is not a spacelike slice and whose mean curvature function verifies

$$H = 1 \geq \frac{f'(\tau)}{f(\tau)} = \tanh(\tau),$$

with

$$f'(\tau)(x) = \frac{|x|^2}{2} \geq 0.$$

As a consequence of Theorem 3.1, we have the following uniqueness result

Corollary 3.4. *Let F be an n-dimensional compact Riemannian manifold and let $f : I \longrightarrow]0,\infty[$ be a smooth function such that $f' \leq 0$ (resp. $f' \geq 0$). Then, the only entire solutions of*

$$H(u) \leq \frac{f'(u)}{f(u)} \left(\textit{resp. } H(u) \geq \frac{f'(u)}{f(u)} \right),$$
$$|Du| < f(u)$$

are the constant functions.

The previous Corollary contains the following Calabi-Bernstein type result for (E).

> *Let F be an n-dimensional compact Riemannian manifold and let $f : I \longrightarrow]0,\infty[$ be a smooth function such that f' is signed. Then, the only entire solutions of equation* (E) *are the constant functions.*

We end this section highlighting that in Theorem 3.1 H and $f'(\tau)$ are equally signed. However, if they have different sign we get

Proposition 3.5. *The only compact spacelike hypersurfaces in a non-contracting (resp. non-expanding) GRW spacetime whose mean curvature function satisfies $H \leq 0$ (resp. $H \geq 0$) are the totally geodesic spacelike slices.*

Proof. From our assumptions we have $Hf'(\tau) \leq 0$. Now, using (2.21) we obtain that $\Delta\tau$ is signed. Since the hypersurface is compact, it must be a spacelike slice. Moreover, the only spacelike slices that satisfy this assumption are the totally geodesic ones. □

We can point out two immediate consequences of the previous results in the relevant case of de Sitter spacetime $\overline{M} = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$ (see Section 2.2). Hence, directly from Theorem 3.1 and Proposition 3.5 we get, respectively

Corollary 3.6. *The only compact spacelike hypersurfaces with $\tau \leq 0$ (resp., $\tau \geq 0$) in de Sitter spacetime $\overline{M} = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$ whose mean curvature function satisfies*

$$H \leq \tanh(\tau) \qquad (\textit{resp. } H \geq \tanh(\tau))$$

are the spacelike slices.

Corollary 3.7. *The only compact spacelike hypersurface with $\tau \geq 0$ (resp. $\tau \leq 0$) and non-positive (resp., non-negative) mean curvature in de Sitter spacetime $\overline{M} = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$ is the totally geodesic equator $\{0\} \times \mathbb{S}^n$.*

It should be noticed that the technique used to prove Theorem 3.1 may be extended to deal with other elliptic operators on the spacelike hypersurface M appart from the Laplacian. In particular, Theorem 3.1 and its applications admit an extension to study higher order mean curvatures (see Section 2.1.1). In order to obtain this extension, we can consider again a primitive function $\mathcal{F}$ of the warping function f. From (2.20), (2.12) and (2.13) we get

$$\begin{aligned} L_k(\mathcal{F}(\tau)) &= -f'(\tau)\mathrm{trace}(P_k) + f(\tau)\overline{g}(N,\partial_t)\mathrm{trace}(A \circ P_k) \\ &= -c_k\left(f'(\tau)H_k + f(\tau)H_{k+1}\overline{g}(N,\partial_t)\right). \end{aligned} \tag{3.6}$$

As it is shown in [10, Lemma 3.3], if there exists an elliptic point (for a suitable choice of N) and the $(k+1)$th-mean curvature H_{k+1} satisfies $H_{k+1} > 0$ for some $2 \leq k \leq n-1$, then L_j is elliptic for $1 \leq j \leq k$. Concretely, we can state

Theorem 3.8. *Let $\psi : M \to \overline{M}$ be a compact spacelike hypersurface in a GRW spacetime such that $H_{k+1} > 0$ for some $2 \leq k \leq n-1$. If any H_j vanishes nowhere and satisfies*

$$\frac{H_{j+1}}{H_j} \geq \frac{f'(\tau)}{f(\tau)} \qquad \left(resp. \ \frac{H_{j+1}}{H_j} \leq \frac{f'(\tau)}{f(\tau)} \right),$$

with $f'(\tau) > 0$ *(resp.* $f'(\tau) < 0$*) for* $1 \leq j \leq k$*, then* M *is a spacelike slice.*

Proof. Firstly, we have from [10, Lemma 5.3] that there exists an elliptic point of M with respect to an appropiate choice of N. Therefore, from this fact and $H_{k+1} > 0$ on M for some $2 \leq k \leq n-1$, [10, Lemma 3.3] asserts that the operator L_i is elliptic for all $1 \leq i \leq k$ (using that choice of N if i is odd). Moreover, from (3.6) we get

$$L_j(\mathcal{F}(\tau)) = -c_j f(\tau) H_j \left(\frac{f'(\tau)}{f(\tau)} + \frac{H_{j+1}}{H_j} \overline{g}(N, \partial_t) \right).$$

Hence, taking our assumptions into account we obtain that $L_j(\mathcal{F}(\tau))$ is signed. Since M is compact and L_j is an elliptic operator, the classical maximum principle can be called to conclude that $\mathcal{F}(\tau)$ is constant and M is a spacelike slice. □

3.3 Uniqueness results when the GRW spacetime is Einstein

If the GRW spacetime is Einstein we will deal directly with equation (E). Note that a GRW spacetime is Einstein with $\overline{\mathrm{Ric}} = \overline{c}\,\overline{g}$ if and only if its fiber (F, g_F) has constant Ricci curvature c and f satisfies the equations

$$\frac{f''}{f} = \frac{c}{n} \quad \text{and} \quad \frac{\overline{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2}. \tag{3.7}$$

The positive solutions of (3.7) were given in [14] and are collected in the following table (in each case, the interval of definition I of f is the maximal one where f is positive).

TABLE

1	$\bar{c} > 0$	$c > 0$	$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)} e^{-bt}, \quad a > 0, \quad b = \sqrt{\bar{c}/n}$
2	$\bar{c} > 0$	$c = 0$	$f(t) = ae^{\varepsilon bt}, \quad a > 0, \quad \varepsilon = \pm 1, \quad b = \sqrt{\bar{c}/n}$
3	$\bar{c} > 0$	$c < 0$	$f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)} e^{-bt} \quad a \neq 0, \quad b = \sqrt{\bar{c}/n}$
4	$\bar{c} = 0$	$c = 0$	$f(t) = a, \quad a > 0$
5	$\bar{c} = 0$	$c < 0$	$f(t) = \varepsilon \sqrt{\frac{-c}{(n-1)}} t + a, \quad \varepsilon = \pm 1$
6	$\bar{c} < 0$	$c < 0$	$f(t) = a_1 \cos(bt) + a_2 \sin(bt), \quad a_1^2 + a_2^2 = cn/\bar{c}(n-1), \quad b = \sqrt{-\frac{\bar{c}}{n}}$

Furthermore, from (3.7) we also have

$$(n-1)(\log f)'' = \frac{c}{f^2}. \tag{3.8}$$

In order to obtain our results, for a compact spacelike hypersurface M in a GRW spacetime $\overline{M}$ we have from (2.25) and (2.26) the following integral formula (previously obtained in [13])

$$\int_M \left\{(n-1)f(\tau)\,\overline{g}(\nabla H, \partial_t) + f(\tau)\,\overline{\mathrm{Ric}}(\partial_t^T, N) + f(\tau)\,\overline{g}(\partial_t, N)\Big(\mathrm{trace}(A^2) - nH^2\Big)\right\} dV = 0, \tag{3.9}$$

where ∇H represents the gradient of the mean curvature function of M. As far as we know formula (3.9) has been previously used only for the constant mean curvature case (see for instance [15] and references therein). Taking into account (3.9) and (3.8), we can state the following result

Theorem 3.9. *Let M be a compact spacelike hypersurface in an Einstein GRW spacetime whose mean curvature function verifies*

$$H = \frac{f'(\tau)}{f(\tau)}.$$

If the constant Ricci curvature of the fiber satisfies $c \geq 0$, then M is totally umbilic. Moreover, if $c > 0$, M must be a spacelike slice.

Proof. From our assumptions and (3.8), the gradient of $H = \frac{f'(\tau)}{f(\tau)}$ in M is

$$\nabla H = (\log f)''(\tau)\,\nabla\tau = \frac{c}{(n-1)f^2(\tau)}\,\nabla\tau. \tag{3.10}$$

Thus, using (3.10) and (2.17) we can write (3.9) as

$$-c\int_M \frac{|\nabla\tau|^2}{f(\tau)} dV + \int_M f(\tau)\,\overline{g}(\partial_t, N)\Big(\mathrm{trace}(A^2) - nH^2\Big) dV = 0. \tag{3.11}$$

The result follows using that the Schwarz inequality gives $\mathrm{trace}(A^2) - nH^2 \geq 0$, with equality holding if and only if M is totally umbilic. When $c > 0$ we can go one step further, since in this case τ must be constant. □

Particularly, in the case of de Sitter spacetime, the previous result yields to

Corollary 3.10. *The only compact spacelike hypersurfaces in de Sitter spacetime* $\overline{M} = \mathbb{R} \times_{\cosh(t)} \mathbb{S}^n$ *whose mean curvature function satisfies*

$$H = \tanh(\tau)$$

are the spacelike slices.

Remark 3.11. Note that despite this result's assumption on H is stronger than the one in Corollary 3.6, we are not requiring our spacelike hypersurfaces to have signed τ.

Taking into account (3.4) and Corollary 3.10 we are able to get the following nonparametric uniqueness result

Corollary 3.12. *The only entire solutions on* $\mathbb{S}^n$ *of*

$$\operatorname{div}\left(\frac{Du}{n\cosh(u)\sqrt{\cosh^2(u) - |Du|^2}}\right) + \frac{\sinh(u)}{n\sqrt{\cosh^2(u) - |Du|^2}}\left(n + \frac{|Du|^2}{\cosh^2(u)}\right) = \tanh(u),$$

$$|Du| < \cosh(u)$$

are the constant functions.

3.4 Uniqueness results when the GRW spacetime obeys the NCC

As we said in Section 2.5, an Einstein spacetime automatically obeys the NCC. However, the family of spacetimes that satisfy this energy condition is much wider. In this section we will consider as our ambient spacetime a GRW spacetime that obeys the NCC. We will now make use again of formula (3.9) to state

Theorem 3.13. *Let* M *be a compact spacelike hypersurface in a GRW spacetime whose mean curvature function verifies*

$$H = \frac{f'(\tau)}{f(\tau)}.$$

If the spacetime obeys the NCC and its warping function satisfies $(\log f)'' \geq 0$*, then* M *is totally umbilic. Moreover, if the NCC is strictly satisfied or* $(\log f)'' > 0$ *on* M*, then* M *must be a spacelike slice.*

Proof. From (3.9), we have

$$-(n-1)\int_M f(\tau)(\log f)''(\tau)|\nabla\tau|^2 dV + \int_M f(\tau)\,\overline{\mathrm{Ric}}(\partial_t^T, N)dV +$$

$$+\int_M f(\tau)\,\overline{g}(\partial_t, N)\Big(\mathrm{trace}(A^2) - nH^2\Big)dV = 0. \qquad (3.12)$$

Our next step is to show following the argument in [13] and [15] that $\overline{\mathrm{Ric}}(\partial_t^T, N) \leq 0$. To do so, writing

$$N = -\overline{g}(\partial_t, N)\partial_t + N_F \qquad (3.13)$$

where N_F denotes the projection of N on the fiber F, we have

$$\partial_t^T = \overline{g}(\partial_t, N)N_F + \left(1 - \overline{g}(\partial_t, N)^2\right)\partial_t. \qquad (3.14)$$

Now we can obtain

$$\overline{\mathrm{Ric}}(\partial_t^T, N) = \overline{g}(\partial_t, N)\left\{\overline{\mathrm{Ric}}(N_F, N_F) - \left(1 - \overline{g}(\partial_t, N)^2\right)\overline{\mathrm{Ric}}(\partial_t, \partial_t)\right\}. \qquad (3.15)$$

Besides, from [85, Cor. 7.43] we know that

$$\overline{\mathrm{Ric}}(\partial_t, \partial_t) = -n\,\frac{f''(\tau)}{f(\tau)} \qquad (3.16)$$

and

$$\overline{\mathrm{Ric}}(N_F, N_F) = \mathrm{Ric}^F(N_F, N_F) - \left(1 - \overline{g}(\partial_t, N)^2\right)\left(\frac{f''(\tau)}{f(\tau)} + (n-1)\frac{f'(\tau)^2}{f(\tau)^2}\right). \qquad (3.17)$$

Hence, using (3.16) and (3.17) in (3.15) as well as the fact that

$$\overline{g}(N_F, N_F) = -1 + \overline{g}(\partial_t, N)^2,$$

we obtain

$$\overline{\mathrm{Ric}}(\partial_t^T, N) = \overline{g}(\partial_t, N)\left\{\mathrm{Ric}^F(N_F, N_F) - (n-1)f(t)^2(\log f)''(t)g_F(N_F, N_F)\right\}. \tag{3.18}$$

Thus, using (2.44) in (3.18) we see that the second term in (3.12) is non-positive under the NCC.

On the other hand, the first term of (3.12) is non-positive when $(\log f)'' \geq 0$. Therefore, M is totally umbilic. Moreover, we also obtain $\overline{\mathrm{Ric}}(\partial_t^T, N) = 0$ on M. Therefore, the symmetric tensor G_F introduced in Section 2.5 satisfies

$$G_F(N_F, N_F) = 0.$$

If the NCC is strictly satisfied, then G_F is positive definite and we conclude $N_F = 0$, which means that M is a spacelike slice. On the other hand, if $(\log f)'' > 0$ on M from the firs term in (3.12) we also obtain that M is a spacelike slice. □

Notice that Theorem 3.13 can be extended to a more abstract scenario.

Theorem 3.14. *Let M be a compact spacelike hypersurface in a GRW spacetime whose mean curvature function verifies*

$$H = \varphi(\tau),$$

being $\varphi : I \longrightarrow \mathbb{R}$ smooth and increasing. If the spacetime obeys the NCC, then M is totally umbilic. Moreover, if the NCC is strictly satisfied on M or $\varphi'(\tau) > 0$, then M must be a spacelike slice.

As a consequence of the previous theorem, we obtain the following Calabi-Bernstein type result

Corollary 3.15. *Let F be an n-dimensional compact Riemannian manifold and let $f : I \longrightarrow]0, \infty[$ be a smooth function such that $\log f$ is convex. If the Ricci curvature of F is strictly bounded from below by $(n-1)\sup_F \left(f^2(\log f)''\right)$ or $\log f$ is strictly convex, then the only entire solutions of equation* (E) *are the constant functions.*

Chapter 4

Maximal hypersurfaces in a spatially open GRW spacetime

After dealing with spatially closed models, we will now study complete maximal hypersurfaces in spatially open GRW spacetimes. In particular, we will be able to characterize them in spatially open GRW spacetimes with flat fiber, which fit with experimental observations of our current universe [42]. Our main tool here will be the Omori-Yau maximum principle for the Laplacian. This chapter's contents can be found in [86] and [88].

4.1 Set up

The aim of this section is to obtain the Laplacian of the function $\sinh^2\varphi$, where φ denotes the hyperbolic angle defined in (2.11). Using the same notation as in Section 2.2.1, let $\psi : M \to \overline{M}$ be an n-dimensional spacelike hypersurface in a GRW spacetime $\overline{M} = I \times_f F$. We can use (2.19) to compute the gradient on M of $\cosh\varphi$ to get

$$\nabla \cosh\varphi = -A\nabla\tau - \frac{f'(\tau)}{f(\tau)}\, \cosh\varphi\, \nabla\tau. \tag{4.1}$$

Note that an immediate consequence of this formula is the fact that the hyperbolic angle is constant if and only if the right hand side of (4.1) is equal to zero. Now, using (2.22) and (4.1) we get

$$\begin{aligned}\Delta\left(f(\tau)\cosh\varphi\right) &= \cosh\varphi\ \Delta f(\tau) + f(\tau)\ \Delta\cosh\varphi + 2g(\nabla f(\tau), \nabla\cosh\varphi)\\ &= -n\,\frac{f'(\tau)^2}{f(\tau)}\ \cosh\varphi + f(\tau)\cosh\varphi\ \sinh^2\varphi\,(\log f)''(\tau)\\ &\quad +nHf'(\tau)\cosh^2\varphi + f(\tau)\ \Delta\cosh\varphi\\ &\quad -2f'(\tau)g(A\nabla\tau,\nabla\tau) - 2\,\frac{f'(\tau)^2}{f(\tau)}\ \cosh\varphi\ \sinh^2\varphi,\end{aligned} \tag{4.2}$$

where we have used (2.17)-(2.19). On the other hand, in the maximal case, (2.25) reduces to

$$\Delta\left(f(\tau)\cosh\varphi\right) = -\Delta\overline{g}(K,N) = -\overline{\mathrm{Ric}}(K^T,N) + f(\tau)\cosh\varphi\ \mathrm{trace}(A^2). \tag{4.3}$$

Therefore, combining (4.2) and (4.3) we have

$$\begin{aligned}\overline{\mathrm{Ric}}(K^T,N) &= f(\tau)\cosh\varphi\ \mathrm{trace}(A^2) + n\,\frac{f'(\tau)^2}{f(\tau)}\ \cosh\varphi\\ &\quad -f(\tau)\cosh\varphi\ \sinh^2\varphi\,(\log f)''(\tau) - f(\tau)\ \Delta\cosh\varphi\\ &\quad +2f'(\tau)g(A\partial_t^T,\partial_t^T) + 2\,\frac{f'(\tau)^2}{f(\tau)}\ \cosh\varphi\ \sinh^2\varphi.\end{aligned} \tag{4.4}$$

If we use (3.13) to obtain the projection of N on the fiber, it is easy to get from (2.18) that

$$\sinh^2\varphi = f(\tau)^2\ g_F(N_F,N_F). \tag{4.5}$$

Besides, using (4.5) in (3.17) we have

$$\overline{\mathrm{Ric}}(N_F,N_F) = \mathrm{Ric}^F(N_F,N_F) + \sinh^2\varphi\left(\frac{f''(\tau)}{f(\tau)} + (n-1)\frac{f'(\tau)^2}{f(\tau)^2}\right). \tag{4.6}$$

Now, if we assume that the fiber F is Ricci-flat, from (3.16) and (4.6) we obtain

$$\begin{aligned}\overline{\mathrm{Ric}}(K^T,N) &= -f(\tau)\cosh\varphi\ \overline{\mathrm{Ric}}(N_F,N_F) - f(\tau)\cosh\varphi\ \sinh^2\varphi\ \overline{\mathrm{Ric}}(\partial_t,\partial_t)\\ &= (n-1)f(\tau)\cosh\varphi\ \sinh^2\varphi\ (\log f)''(\tau).\end{aligned} \tag{4.7}$$

Finally, from (4.4) and (4.7) we get

$$\begin{aligned}\Delta \cosh\varphi &= -n\cosh\varphi\sinh^2\varphi\ (\log f)''(\tau) + \frac{f'(\tau)^2}{f(\tau)^2}\ \cosh\varphi\left(n+2\sinh^2\varphi\right)\\ &\quad +\cosh\varphi\ \mathrm{trace}(A^2) + 2\ \frac{f'(\tau)}{f(\tau)}\ g(A\partial_t^T,\partial_t^T). \end{aligned} \tag{4.8}$$

In order to analyze the last term in the previous equation we will consider the square algebraic trace-norm of the Hessian tensor of τ

$$|\mathrm{Hess}(\tau)|^2 = \mathrm{trace}(H_\tau \circ H_\tau),$$

where H_τ denotes the operator defined by $g(H_\tau(X), Y) := \mathrm{Hess}(\tau)(X,Y)$ for all $X, Y \in \mathfrak{X}(M)$. We can compute this (in the general case) taking the tangential component in (2.15) and using (2.17) to get

$$\begin{aligned}|\mathrm{Hess}(\tau)|^2 &= \sum_{i=1}^n g(\nabla_{E_i}\partial_t^T, \nabla_{E_i}\partial_t^T) = \frac{f'(\tau)^2}{f(\tau)^2}\sinh^4\varphi + \cosh^2\varphi\ \mathrm{trace}(A^2)\\ &\quad +n\ \frac{f'(\tau)^2}{f(\tau)^2} + 2\ \frac{f'(\tau)}{f(\tau)}\cosh\varphi\ g(A\partial_t^T,\partial_t^T) + 2\ \frac{f'(\tau)^2}{f(\tau)^2}\sinh^2\varphi\\ &\quad -2\ n\ \frac{f'(\tau)}{f(\tau)} H\cosh\varphi. \end{aligned} \tag{4.9}$$

Since M is assumed to be maximal, (4.9) leads to

$$\begin{aligned}|\mathrm{Hess}(\tau)|^2 &= \frac{f'(\tau)^2}{f(\tau)^2}\left(n-1+\cosh^4\varphi\right) + \cosh^2\varphi\,\mathrm{trace}(A^2)\\ &\quad +2\ \frac{f'(\tau)}{f(\tau)}\cosh\varphi\, g(A\partial_t^T,\partial_t^T). \end{aligned} \tag{4.10}$$

Since $|\mathrm{Hess}(\tau)|^2 \geq 0$, it is a straightforward computation to obtain, making use of (4.8) and (4.10), that

$$\begin{aligned}\cosh\varphi\ \Delta\cosh\varphi &\geq -n\cosh^2\varphi\sinh^2\varphi\ (\log f)''(\tau) + n\ \frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\\ &\quad +2\ \frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\sinh^2\varphi - \frac{f'(\tau)^2}{f(\tau)^2}\left(n-1+\cosh^4\varphi\right). \end{aligned} \tag{4.11}$$

Now, we can manipulate the right hand side of inequality (4.11) to get

$$\begin{aligned}
\cosh\varphi\,\Delta\cosh\varphi \;\geq\; & -n\,\frac{f''(\tau)}{f(\tau)}\cosh^2\varphi\sinh^2\varphi + n\,\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\sinh^2\varphi \\
& +n\,\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi + 2\,\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\sinh^2\varphi \\
& -(n-1)\frac{f'(\tau)^2}{f(\tau)^2} - \frac{f'(\tau)^2}{f(\tau)^2}\Big(\sinh^2\varphi+1\Big)\cosh^2\varphi \\
& = -n\,\frac{f''(\tau)}{f(\tau)}\cosh^2\varphi\sinh^2\varphi + (n-1)\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\sinh^2\varphi \\
& +2\,\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\sinh^2\varphi + (n-1)\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi \\
& -(n-1)\frac{f'(\tau)^2}{f(\tau)^2}. \qquad (4.12)
\end{aligned}$$

Since $\cosh^2\varphi \geq 1$, we obtain from (4.12)

$$\cosh\varphi\,\Delta\cosh\varphi \geq -n\frac{f''(\tau)}{f(\tau)}\cosh^2\varphi\sinh^2\varphi + (n+1)\frac{f'(\tau)^2}{f(\tau)^2}\cosh^2\varphi\sinh^2\varphi. \qquad (4.13)$$

Now, we will impose the NCC in order to obtain our results. From (2.44) we see that in a GRW spacetime with Ricci-flat fiber, this energy condition is satisfied if and only if $(\log f)''(t) \leq 0$. Using this assumption, in (4.13) we have

$$\cosh\varphi\,\Delta\cosh\varphi \geq \left((n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\,\frac{f''(\tau)}{f(\tau)}\right)\sinh^4\varphi. \qquad (4.14)$$

Moreover, we know that

$$\frac{1}{2}\Delta\sinh^2\varphi = \cosh\varphi\Delta\cosh\varphi + |\nabla\cosh\varphi|^2 \geq \cosh\varphi\Delta\cosh\varphi. \qquad (4.15)$$

To conclude this calculations, we combine (4.14) and (4.15) to obtain the following result

Lemma 4.1. *Let $\psi : M \to \overline{M}$ be an n-dimensional maximal hypersurface in a GRW spacetime $\overline{M} = I \times_f F$ with Ricci-flat fiber that obeys the NCC, then*

$$\frac{1}{2}\Delta\sinh^2\varphi \geq \left((n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\,\frac{f''(\tau)}{f(\tau)}\right)\sinh^4\varphi. \qquad (4.16)$$

Remark 4.2. For the case of a constant mean curvature spacelike hypersurface M in a GRW spacetime $\overline{M}$ that obeys the NCC (and whose fiber is not necessarily Ricci-flat), the same procedure as above gives that the hyperbolic angle of M verifies

$$\frac{1}{2}\Delta \sinh^2 \varphi \geq -n\ \frac{f'(\tau)}{f(\tau)} H \cosh \varphi \sinh^2 \varphi + \frac{f'(\tau)^2}{f(\tau)^2} \sinh^2 \varphi (n + \sinh^2 \varphi).$$

When $H = 0$, this inequality reduces to

$$\frac{1}{2}\Delta \sinh^2 \varphi \geq \frac{f'(\tau)^2}{f(\tau)^2} \sinh^4 \varphi,$$

which can also be obtained from (4.16) taking into account that the NCC in a spatially open GRW spacetime with Ricci-flat fiber is equivalent to

$$-(\log f)''(\tau) \geq 0.$$

However, inequality (4.16) is stronger than the previous one (which is not useful for characterizing the spacelike slices).

Remark 4.3. Note that in [71, Prop. 3.1], under more general assumptions than in Lemma 4.1, the authors obtain another inequality of this type by means of Bochner-Lichnerowicz's formula. In [71, Thm. 4.2] they use that inequality when the hyperbolic angle function attains a local maximum on the spacelike hypersurface.

We will later use inequality (4.16) to obtain our uniqueness results in these ambient spacetimes. Indeed, we will also need the following bound for the Ricci curvature for maximal hypersurfaces, which will enable us to obtain some non-existence results.

Lemma 4.4. *Let $\psi : M \to \overline{M}$ be an n-dimensional maximal hypersurface in a Robertson-Walker spacetime $\overline{M} = I \times_f F$ with flat fiber that obeys the NCC. Then, the Ricci curvature of M must be non-negative.*

Proof. Given $p \in M$, let us take a local orthonormal frame $\{U_1, \ldots, U_n\}$ around p. From the Gauss equation (2.5) we get that the Ricci curvature of M, Ric, satisfies

$$\mathrm{Ric}(Y,Y) = \sum_k \overline{g}(\overline{\mathrm{R}}(Y,U_k)U_k,Y) + g(A^2Y,Y) \geq \sum_k \overline{g}(\overline{\mathrm{R}}(Y,U_k)U_k,Y), \qquad (4.17)$$

for all $Y \in \mathfrak{X}(M)$. Now, from [85, Prop. 7.42] (see also [69, Lemma 4.4.14]) and using the fact that F is flat, we have

$$\sum_k \overline{g}(\overline{\mathrm{R}}(Y,U_k)U_k,Y) = (n-1)\frac{f'(\tau)^2}{f(\tau)^2}|Y|^2 - (n-2)(\log f)''(\tau)\, g(Y,\nabla\tau)^2 - (\log f)''(\tau)|\nabla\tau|^2|Y|^2. \tag{4.18}$$

From this equation, taking into account the NCC, we have the Ricci curvature of M to be non-negative. □

Using the classical result given by Omori [84] and Yau [116] (Theorem 2.4) and the previous lemma we have

Lemma 4.5. *Let $\psi : M \to \overline{M}$ be an n-dimensional complete maximal hypersurface in a Robertson-Walker spacetime $\overline{M} = I \times_f F$ with flat fiber that obeys the NCC. Then, the Omori-Yau maximum principle for the Laplacian holds on M.*

4.2 Non-existence results

Thanks to the lower bound that we have found for the Ricci curvature of any maximal hypersurface in a Robertson-Walker spacetime with flat fiber, we are able to obtain the next non-existence results in these ambiences.

Proposition 4.6. *There is no complete maximal hypersurface M in a spatially open Robertson-Walker spacetime $\overline{M} = I \times_f F$ with flat fiber that obeys the NCC such that the restriction of the function $\overline{\mathrm{div}}(\partial_t)$ to M satisfies*

$$\inf_M |\overline{\mathrm{div}}(\partial_t)| > 0. \tag{4.19}$$

Proof. From (4.17) and (4.18), considering (2.16), we get for any maximal hypersurface M in $\overline{M}$

$$\mathrm{Ric}(Y,Y) \geq \frac{n-1}{n^2}\, \overline{\mathrm{div}}(\partial_t)\Big|_M^2 \, |Y|^2, \tag{4.20}$$

for all $Y \in \mathfrak{X}(M)$. Now, using (4.19) in (4.20) we obtain that the Ricci curvature of M is bounded from below by a positive constant. Thus, if M is complete the classical Bonnet-Myers theorem ensures its compactness. However, this contradicts the fact that in a spatially open spacetime there are no compact spacelike hypersurfaces. □

Remark 4.7. According to (2.16), assumption (4.19) in Proposition 4.6 means that the comoving observers measure that the universe is strictly expanding or contracting near M.

Proposition 4.6 enables us to obtain the following non-existence results in some Robertson-Walker spacetimes that we introduced in Section 2.2. First of all, we get

Corollary 4.8. *There are no complete maximal hypersurfaces in the* $(n+1)$*-dimensional steady state spacetime* $\mathbb{R} \times_{e^t} \mathbb{R}^n$.

Now, taking into account that being bounded away from future infinity analytically means that $\sup_M \tau < +\infty$ we obtain

Corollary 4.9. *There are no complete maximal hypersurfaces bounded away from future infinity in the* $(n+1)$*-dimensional Einstein-de Sitter spacetime* $\mathbb{R}^+ \times_{t^{2/3}} \mathbb{R}^n$.

Corollary 4.9 generalizes and improves the result in [104, Thm. 4.2] to the case of arbitrary dimension, since there the maximal hypersurface was assumed to lie between two spacelike slices. Analogously, for another relevant relativistic model we have

Corollary 4.10. *There are no complete maximal hypersurfaces bounded away from future infinity in the* $(n+1)$*-dimensional Roberson-Walker Radiation model* $\mathbb{R}^+ \times_{(2at)^{1/2}} \mathbb{R}^n$*, with* $a > 0$.

4.3 Uniqueness results

In order to obtain our uniqueness results we will make use of the well-known Omori-Yau maximum principle, which has proved to be a powerful tool to study spacelike hypersurfaces in spatially open spacetimes. For instance, this maximum principle has been also used in [33] to study complete constant mean curvature spacelike surfaces in 3-dimensional GRW spacetimes as well as in [77] for complete maximal surfaces in certain Lorentzian product spacetimes. Thus, combining this principle with our previous results we are now in a position to prove our main uniqueness result for complete maximal hypersurfaces in spatially open GRW spacetimes with Ricci-flat fiber.

Theorem 4.11. *Let $\psi : M \to \overline{M}$ be an n-dimensional maximal hypersurface in a GRW spacetime $\overline{M} = I \times_f F$ with Ricci-flat fiber that obeys the NCC. If the Omori-Yau maximum principle for the Laplacian holds on M and*

$$\inf_M \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} > 0. \tag{4.21}$$

Then, M is a spacelike slice $\{t_0\} \times F$ with $f'(t_0) = 0$.

Proof. Using Lemma 2.7 in Lemma 4.1 we get that $\sinh^2 \varphi$ identically vanishes on M. □

As an immediate consequence of Theorem 4.11 and Lemma 4.5 we get

Corollary 4.12. *Let $\psi : M \to \overline{M}$ be a complete n-dimensional maximal hypersurface in a Robertson-Walker spacetime $\overline{M} = I \times_f F$ with flat fiber that obeys the NCC. If (4.21) holds on M, then M is a spacelike slice $\{t_0\} \times F$ with $f'(t_0) = 0$.*

Remark 4.13. Under the assumptions of Theorem 4.11, the existence of such a maximal hypersurface implies the existence of a critical point of the warping function. Even more, since in this ambient spacetimes the NCC is equivalent to $(\log f)'' \leq 0$, if f is not locally constant, this critical point is a global maximum of f. We can prove this statement following [70].

Remark 4.14. Observe that assumption (4.21) on the function

$$(n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)},$$

defined on the maximal hypersurface is scarcely restrictive, even if combined with the NCC. In fact, if we consider its extension

$$(n+1)\frac{f'(t)^2}{f(t)^2} - n\frac{f''(t)}{f(t)},$$

defined on the spacetime, we have from the NCC that

$$(n+1)\frac{f'(t)^2}{f(t)^2} - n\frac{f''(t)}{f(t)} = \frac{f'(t)^2}{f(t)^2} - n(\log f)''(t) \geq 0.$$

However, if we assume that the warping function is defined on the largest possible domain, i.e., it is inextendible, we can find two cases where (4.21) does not hold:

1. When both f' and f'' vanish simultaneously at some point in $I =]a, b[$. This obviously happens in the Lorentz-Minkowski spacetime, where an analogous uniqueness result does not hold. Nevertheless, note that $\mathbb{L}^{n+1}$ is a vacuum solution. Indeed, if there is real presence of matter in the spacetime we can discard this case.

2. If the warping function f verifies

$$\lim_{t \to b} \frac{f'(t)^2}{f(t)^2} = \lim_{t \to b} (\log f)''(t) = 0.$$

This is the case in the Einstein-de Sitter spacetime. Even more, the inequality will not hold either in the less realistic case where

$$\lim_{t \to a} \frac{f'(t)^2}{f(t)^2} = \lim_{t \to a} (\log f)''(t) = 0.$$

Furthermore, this theorem improves some previous uniqueness results for complete maximal hypersurfaces in GRW spacetimes with Ricci-flat fiber (see [101] and [103], for instance) without making restrictive assumptions on the maximal hypersurface such as having a bounded hyperbolic angle or lying between two spacelike slices. This allows us to study maximal hypersurfaces that approach both the future and past infinity as well as the null boundary.

Indeed, a previous attempt to characterize maximal hypersurfaces in these models was based on assuming the parabolicity of the fiber (see [101], [107] and references therein). However, by avoiding this compactness assumption in this result we can study maximal hypersurfaces in Robertson-Walker spacetimes with fiber $\mathbb{R}^n$ of arbitrary dimension, since $\mathbb{R}^n$ for $n \geq 3$ is no longer parabolic. These spacetimes constitute a relevant family of spatially open cosmological models both from a historic as well as from a physical perspective.

We will give now two examples where Theorem 4.11 holds.

Example 4.15. Let us consider the Robertson-Walker spacetime $\overline{M} = \mathbb{R} \times_f \mathbb{R}^n$ with warping function $f(t) = e^{-t^2}$. This spacetime obeys the NCC, since $(\log f)''(t) = -2$. Moreover, any maximal hypersurface in $\overline{M}$ satisfies

$$\inf_M \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} = \inf_M \{2n + 4\tau^2\} \geq 2n > 0.$$

Therefore, the only complete maximal hypersurface in $\overline{M}$ is the spacelike slice $\{0\} \times \mathbb{R}^n$.

This spacetime models a relativistic universe without singularities (in the sense of [85, Def. 12.16]) that goes from an expanding phase to a contracting one. The physical space in this transition of phase is represented by the spacelike slice $\{0\} \times \mathbb{R}^n$.

Example 4.16. We obtain another example of a Robertson-Walker spacetime satisfying the corresponding assumptions in Theorem 4.11 by considering $\overline{M} = I \times_f \mathbb{R}^n$. Where $I =]-a, a[$ and the warping function is $f(t) = \sqrt{a^2 - t^2}$, being a a positive constant. Let us remark that this spacetime behaves like the Robertson-Walker model proposed by Friedmann with constant sectional curvature of the fiber equal to one (see [85, Chap. 12]), since it has a big bang singularity at $t = -a$ as well as a big crunch one at $t = a$ [85, Def. 12.16].

For this spacetime,

$$(\log f)''(t) = -\frac{a^2 + t^2}{(a^2 - t^2)^2} \leq 0,$$

so it satisfies the NCC. Furthermore, for every maximal hypersurface in $\overline{M}$

$$\inf_M \left\{ (n+1)\frac{f'(\tau)^2}{f(\tau)^2} - n\frac{f''(\tau)}{f(\tau)} \right\} = \inf_M \left\{ \frac{n(a^2 + \tau^2) + \tau^2}{(a^2 - \tau^2)^2} \right\} > 0.$$

Hence, the only complete maximal hypersurface in this spacetime is the spacelike slice $\{0\} \times \mathbb{R}^n$, which represents the physical space in the transition from an expanding phase of the spacetime to a contracting one.

4.4 A physical model to better understand our mathematical assumptions

We can build a simple cosmological model that will help us to understand the physical meaning of some mathematical assumptions that we have made in this chapter. Following the ideas in [85, Chap. 12], consider in the spacetime $\overline{M} = I \times_f F$ for $p \in F$ the integral curve γ_p of the galactic flow ∂_t through p, i.e., each γ_p represents the evolution of a galaxy in $\overline{M}$. The time function t may be seen as the common proper time of every γ_p. On the spacelike slice $t = t_0$ the distance with respect to the induced Riemannian metric between two galaxies at the same time (t_0, p), (t_0, q) is $f(t_0)d_F(p, q)$, where d_F is the distance associated to (F, g_F). Thus, when f has positive derivative (resp. negative derivative) the galaxies will be spreading out (resp. coming together). Furthermore, the sign of f'' measures the acceleration of this expansion (resp. contraction). Therefore, if $f' > 0$ and $f'' > 0$ our GRW spacetime will model a universe in accelerated expansion; whereas if $f' < 0$ and $f'' < 0$, it will model a universe in accelerated contraction.

We consider now the Einstein field equation

$$\overline{\mathrm{Ric}} - \frac{1}{2}\overline{\mathrm{S}}\,\overline{g} = 8\pi T, \tag{4.22}$$

where $\overline{\mathrm{S}}$ is the scalar curvature of $\overline{g}$ and T is the stress-energy momentum tensor which describes the distribution of matter and energy in the spacetime. For an instantaneous observer v, the quantity $T(v,v)$ is interpreted as the energy density, i.e., the mass-energy per unit of volume, measured by this observer. Usually, this quantity must be non-negative, i.e., the tensor T must obey the Weak Energy Condition [63, Chap. 4.3]. It is easy to see that an exact solution of (4.22) for a stress-energy tensor that obeys the Weak Energy Condition must satisfy the NCC [63, pg. 95]. Indeed, directly from (4.22) we have

$$\overline{\mathrm{Ric}}(z,z) = 8\pi T(z,z),$$

for all null tangent vector z.

In order to find exact solutions to the Einstein field equation several families of stress-energy momentum tensor have been considered. In particular, the case where T describes a continuous distribution of energy and matter as a perfect fluid is specially relevant. Recall that a perfect fluid (see, for example, [85, Def. 12.4]) in a spacetime $\overline{M}$ is a triple $(U, \rho, \mathfrak{p})$ where

1. U is a timelike future-pointing unit vector field on $\overline{M}$ called the flow vector field.

2. $\rho, \mathfrak{p} \in C^\infty(\overline{M})$ are, respectively, the energy density and the pressure functions.

3. The stress-energy momentum tensor is

$$T = (\rho + \mathfrak{p})\, U^{\bar{\flat}} \otimes U^{\bar{\flat}} + \mathfrak{p}\,\overline{g}.$$

Classically, the dominant contribution to the energy density of the galactic fluid is the mass of the galaxies, with a smaller pressure due mostly to radiation. Indeed, for a perfect fluid model the usual energy conditions [63, Sec. 4.3] impose certain restrictions to its energy density and pressure functions. Thus, we have:

- The Weak Energy Condition implies that

$$\rho \geq 0 \;\text{ and }\; \rho + \mathfrak{p} \geq 0.$$

- The Strong Energy Condition stipulates that

$$\rho + \mathfrak{p} \geq 0 \;\text{ and }\; \rho + n\mathfrak{p} \geq 0.$$

- The Dominant Energy Condition gives

$$\rho \geq |\mathfrak{p}|.$$

Note that (4.22) is the Einstein field equation with zero cosmological constant. If we introduce in (4.22) the term given by the cosmological constant, the functions ρ and $\mathfrak{p}$ may behave in a different way. Indeed, a positive vacuum energy density resulting from a cosmological constant implies a negative pressure and vice versa. In [91] and [95] we can find evidences of the existence of a cosmological constant that is driving the current acceleration of the universe. Hence, perfect fluids can also be used to model universes at the dark energy dominated stage (see [21] and [43]).

In the case of a Robertson-Walker spacetime with flat fiber and flow vector field $U = \partial_t$, the density and pressure functions are given by

$$8\pi\rho = \frac{n(n-1)}{2}\frac{f'^2}{f^2}$$

and

$$8\pi\mathfrak{p} = -(n-1)\frac{f''}{f} - \frac{(n-1)(n-2)}{2}\frac{f'^2}{f^2}.$$

In the classical situation with positive density and non-negative pressure ($\rho > 0$, $\mathfrak{p} \geq 0$), every maximal hypersurface included in a region of the spacetime in which the stress-energy momentum tensor is far from zero satisfies (4.21). In fact, if we consider the extension of (4.21) to the whole spacetime, this condition is equivalent in our perfect fluid model to

$$\inf \left\{ \frac{8\pi}{n-1} \left(\frac{n^2+2}{n}\rho + n\mathfrak{p} \right) \right\} > 0.$$

This happens, for instance near a physical singularity. On the other hand, as the example of the steady state spacetime shows, (4.21) also holds in certain models with negative pressure.

Finally, we may wonder whether our assumptions in Corollary 4.12 are compatible with the previous energy conditions. In order to better understand these conditions, we will express them using the warping function f. Hence, for a Robertson-Walker spacetime with flat fiber endowed with its perfect fluid structure we have:

- The Weak Energy Condition is equivalent to

  $$(\log f)'' \leq 0,$$

  since in our perfect fluid model the energy density is always non-negative. Thus, it is equivalent to the NCC as expected.

- The Strong Energy Condition implies that

  $$(\log f)'' \leq 0 \ \text{ and } \ \frac{n-3}{2}\frac{f'^2}{f^2} + \frac{f''}{f} \leq 0.$$

  For $n = 3$, this last equation leads to $f'' \leq 0$, which is equivalent to the TCC in this case.

- The Dominant Energy Condition for positive pressure can be written as

  $$(n-1)\frac{f'^2}{f^2} + \frac{f''}{f} \geq 0$$

  and for negative pressure implies

  $$(\log f)'' \leq 0.$$

Therefore, we see how our assumptions (in particular, the NCC and (4.21)) are compatible in many cases with these energy conditions, which are satisfied by most of the models that are used to describe our universe.

Chapter 5

Maximal surfaces in a standard static spacetime

The aim of this chapter is to obtain new non-parametric uniqueness results for complete maximal graphs in standard static spacetimes. Most of this chapter's content can be found in [90].

5.1 An extension of Calabi's correspondence between minimal and maximal graphs

Our uniqueness results in this chapter will be based on a duality between solutions to the minimal surface equation in certain Riemannian warped products and the maximal surface equation in a standard static spacetime. In order to obtain this correspondence, let us consider a two dimensional Riemannian manifold (B, g_B) and a smooth positive function h on B. On the product manifold $B \times I$ consider the Riemannian metric

$$g_R = \pi_B^*(g_B) + h(\pi_B)^2 \pi_{\mathbb{R}}^*(dt^2), \tag{5.1}$$

where π_B and $\pi_{\mathbb{R}}$ denote the projections onto B and $\mathbb{R}$, respectively. Denote by $B \times_h \mathbb{R}$ this three dimensional Riemannian manifold. In the terminology of [85, Def. 7.33], $B \times_h \mathbb{R}$ is the warped product with base (B, g_B), fiber $(\mathbb{R}, dt^2)$ and warping function h.

The graph of a smooth function u defined on a (connected) domain $\Omega \subset B$ is given by

$$\Sigma_u = \{(p, u(p)) : p \in \Omega\}.$$

This graph is minimal in $B \times_h \mathbb{R}$ (i.e., it has zero mean curvature) if and only if u satisfies the equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{\gamma + |Du|^2}}\right) = \frac{1}{2\gamma}\frac{g_B(D\gamma, Du)}{\sqrt{\gamma + |Du|^2}}, \tag{R}$$

where div is the divergence operator in (B, g_B), $|Du|$ denotes the norm of the gradient Du of u and $\gamma := 1/h^2$.

Analogously, for the same smooth positive function h on B we can endow the product manifold $B \times \mathbb{R}$ with the Lorentzian metric

$$g_L = \pi_B^*(g_B) - \frac{1}{h(\pi_B)^2}\, \pi_{\mathbb{R}}^*(dt^2). \tag{5.2}$$

We will denote the Lorentzian warped product obtained in this way by $B \times_{\frac{1}{h}} (-\mathbb{R})$ to clearly distinguish it from the Riemannian case and will call it a standard static spacetime with base (B, g_B), fiber $(\mathbb{R}, -dt^2)$ and warping function $\frac{1}{h}$ (see Section 2.3).

Given a smooth function ω defined on a domain $\Omega \subset B$, its graph

$$\Sigma_\omega = \{(p, \omega(p)) : p \in \Omega\}$$

in $B \times_{\frac{1}{h}} (-\mathbb{R})$ is spacelike, i.e., it inherits a Riemannian metric from (5.2), if and only if ω satisfies

$$|D\omega| < h.$$

From (2.36), we obtain that the graph of ω is maximal, i.e., spacelike with zero mean curvature, if and only if ω satisfies the equation

$$\operatorname{div}\left(\frac{D\omega}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}\right) = \frac{\gamma}{2}\frac{g_B\left(D\left(\frac{1}{\gamma}\right), D\omega\right)}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}, \tag{L.1}$$

$$|D\omega|^2 < \frac{1}{\gamma}, \tag{L.2}$$

being $\gamma = 1/h^2$ as before.

Equations (R) and (L) are clearly elliptic, in fact, (L.2) is the ellipticity condition for the second equation as we have seen in Section 2.3.1. Observe that the constant functions are trivially solutions of both equations. Any other solution is called non-trivial.

Let us remark that for $h = 1$, (R) and (L) are, respectively, the minimal surface equation in the Riemannian product $B \times \mathbb{R}$ and the maximal surface equation in the Lorentzian product $B \times (-\mathbb{R})$. Obviously, when $B = \mathbb{R}^2$, they specialize to the minimal surface equation in Euclidean space $\mathbb{R}^3$ and the maximal surface equation in Lorentz-Minkowski spacetime $\mathbb{L}^3$, respectively.

In order to obtain the desired duality, note that equations (R) and (L) are of divergence form but in both of them the right member is not identically zero, in general. We can overcome this difficulty considering two conformal changes of metric in B. Recall that for a conformal change of metric

$$g' = e^{2u}\, g_B,$$

with $u \in C^\infty(M)$, the relation between the corresponding gradient operators is

$$D'\varphi = e^{-2u}\, D\varphi, \tag{5.3}$$

for all $\varphi \in C^\infty(M)$. On the other hand, for the corresponding divergence operators and taking into account that M is two dimensional, we have

$$\mathrm{div}'(X) = \mathrm{div}(X) + 2\, g_B(X, Du), \tag{5.4}$$

for all $X \in \mathfrak{X}(M)$.

Now, we consider the first conformal change

$$\widetilde{g} = \frac{1}{\sqrt{\gamma}}\, g_B\,. \tag{5.5}$$

Making use of (5.4), we can rewrite (R) as follows

$$\widetilde{\mathrm{div}} \left(\frac{Du}{\sqrt{\gamma + |Du|^2}} \right) = 0, \tag{$\widetilde{\mathrm{R}}$}$$

where now $\widetilde{\mathrm{div}}$ is the divergence operator with respect to $\widetilde{g}$. It should be pointed out that in ($\widetilde{\mathrm{R}}$) the gradient of u and its norm are taken with respect to the original metric g_B on B.

In a similar way, with the conformal metric

$$\widehat{g} = \sqrt{\gamma}\, g_B\,, \tag{5.6}$$

equation (L) can be rewritten with the help of (5.4) as

$$\widehat{\operatorname{div}}\left(\frac{D\omega}{\sqrt{\frac{1}{\gamma} - |D\omega|^2}}\right) = 0, \tag{$\widehat{\text{L}}$.1}$$

$$|D\omega|^2 < \frac{1}{\gamma}. \tag{$\widehat{\text{L}}$.2}$$

We are now in a position to prove the following correspondence between solutions of the previous equations.

Theorem 5.1. *Let $\Omega \subset B$ be a simply connected domain. Then, there exists a non-trivial (i.e., non-constant) solution of* (R) *on Ω if and only if there exists a non-trivial solution of* (L) *on Ω.*

Proof. Recall that $\widetilde{\operatorname{div}}(X) = *d*\left(X^{\widetilde{\flat}}\right)$ for all vector field X, where $*$ denotes the Hodge operator on B associated to the conformal class of g_B, d is the exterior differential and $X^{\widetilde{\flat}}$ denotes the 1-form $\widetilde{g}$-equivalent to X, i.e., $X^{\widetilde{\flat}}(Y) = \widetilde{g}(X, Y)$, for all vector field Y.

If u is a non-trivial solution of (R), or equivalently, of ($\widetilde{\text{R}}$). Then, we have

$$d*\left(U^{\widetilde{\flat}}\right) = 0, \tag{5.7}$$

where $U := \dfrac{Du}{\sqrt{\gamma + |Du|^2}}$, i.e., the 1-form $*\left(U^{\widetilde{\flat}}\right)$ is closed. Now, classical Poincaré's lemma asserts that there exists $\omega \in C^\infty(\Omega)$ such that

$$d\,\omega = *\left(U^{\widetilde{\flat}}\right). \tag{5.8}$$

Equivalently, (5.8) is written, using (5.3), in terms of vector fields as

$$\left(*\left(U^{\widetilde{\flat}}\right)\right)^{\widetilde{\sharp}} = \widetilde{D}\omega = \sqrt{\gamma}\; D\omega, \tag{5.9}$$

where $\widetilde{D}$ denotes the gradient with respect to the metric $\widetilde{g}$ given in (5.5) and $\alpha^{\widetilde{\sharp}}$ represents the vector field $\widetilde{g}$-equivalent to a 1-form α, i.e., $\widetilde{g}(\alpha^{\widetilde{\sharp}}, Y) = \alpha(Y)$ for all vector fields Y.

Moreover, since $*$ preserves the length of a 1-form, we obtain using (5.9)

$$\frac{|Du|^2}{\gamma+|Du|^2}=\gamma|D\omega|^2. \tag{5.10}$$

From here, we have

$$|D\omega|^2<\frac{1}{\gamma}. \tag{5.11}$$

Moreover, using (5.10) we obtain

$$1-\frac{|Du|^2}{\gamma+|Du|^2}=1-\gamma|D\omega|^2,$$

which yields to

$$\gamma+|Du|^2=\frac{1}{\frac{1}{\gamma}-|D\omega|^2}\ . \tag{5.12}$$

Therefore, from (5.8) and (5.12), using (5.3), we get

$$*\left(\frac{d\omega}{\sqrt{\frac{1}{\gamma}-|D\omega|^2}}\right)=(\widehat{D}(-u))^\flat, \tag{5.13}$$

where $\widehat{D}$ represents the gradient respect to the conformal metric $\widehat{g}$ defined in (5.6). Using (5.13) we obtain

$$*\left(W^{\widehat{\flat}}\right)=d(-u), \tag{5.14}$$

being W the vector field on Ω given by

$$W:=\frac{D\omega}{\sqrt{\frac{1}{\gamma}-|D\omega|^2}}.$$

Hence, it follows from (5.11) and (5.14) that ω is a non-trivial solution of $(\widehat{L})$ on Ω. The converse follows in an analogous way. □

In particular, when B is assumed to be simply connected, we can take $\Omega=B$ and obtain

Corollary 5.2. *If B is simply connected, there exists a non-trivial entire solution of equation* (R) *if and only if there exists a non-trivial entire solution of equation* (L).

Remark 5.3. Once we know the conformal changes of metric (5.5) and (5.6), we can also see that we can reprove Theorem 5.1 in the original metric ambiences using an analogous procedure by modifying equations (R) and (L). Indeed, it is enough to observe that (R) is equivalent to $\operatorname{div}\left(\gamma^{-\frac{1}{2}}\dfrac{Du}{\sqrt{\gamma+|Du|^2}}\right)=0$ and (L.1) can be written as $\operatorname{div}\left(\gamma^{\frac{1}{2}}\dfrac{D\omega}{\sqrt{\frac{1}{\gamma}-|D\omega|^2}}\right)=0$.

Remark 5.4. Note that this correspondence enables us to relate Bernstein type results in $B\times_h\mathbb{R}$ to Calabi-Bernstein type results in $B\times_{1/h}(-\mathbb{R})$. This is due to the fact that if the only entire solutions of (R) are the constant functions, then making use of (5.12) we get that the only entire solutions of (L) must also be the constants and vice versa. Thus, [100, Th. 5] yields to

Corollary 5.5. *The only entire solutions of equation* (L) *on the* 2*-dimensional sphere* $\mathbb{S}^2$ *endowed with a Riemannian metric are the constants.*

Chapter 6

Maximal hypersurfaces in a pp-wave spacetime

In this chapter we will continue obtaining uniqueness results for maximal hypersurfaces. In particular, we will study them in the spacetimes known as pp-wave spacetimes that we have already described in Section 2.4. We refer to [87] in this part of the thesis.

6.1 Uniqueness of constant mean curvature spacelike hypersurfaces

Let $\psi : M \longrightarrow \overline{M}$ be a spacelike hypersurface in an $(n+1)$-dimensional pp-wave spacetime. Let us now consider the product of the normal vector field N to M and the parallel lightlike vector field ξ, which we will denote by

$$\eta = \overline{g}(N, \xi),$$

and choose N such that $\overline{g}(N, \xi) < 0$, since a timelike and a lightlike vector field are never orthogonal at any point. It is not difficult to see that

$$\nabla \eta = -A\xi^T, \tag{6.1}$$

where ∇ denotes the gradient operator on M with respect to the induced metric g and A the shape operator of ψ corresponding to N. Hence, from (6.1) we obtain that η is constant if and only if the nowhere zero tangent vector field ξ^T on M is geodesic (i.e., $\nabla_{\xi^T}\xi^T = 0$).

Take now a local orthonormal reference frame $\{E_1, \ldots, E_n\}$ for (M, g). From (2.42) we obtain for all $i = 1, \dots, n$,

$$\nabla_{E_i} \xi^T = -\overline{g}(N, \xi) A E_i. \tag{6.2}$$

We compute now $\mathrm{div}(A\xi^T)$ as follows

$$\mathrm{div}(A\xi^T) = \sum_{i=1}^{n} g\left(\nabla_{E_i}(A\xi^T), E_i\right) = \sum_{i=1}^{n} g\left((\nabla_{E_i} A)\xi^T, E_i\right) + \sum_{i=1}^{n} g\left(A(\nabla_{E_i}\xi^T), E_i\right). \tag{6.3}$$

Using (6.2) in (6.3), we have

$$\begin{aligned} \mathrm{div}(A\xi^T) &= \sum_{i=1}^{n} g\left((\nabla_{E_i} A)\xi^T, E_i\right) + \sum_{i=1}^{n} g\Big(A(-\overline{g}(N, \xi) A E_i), E_i\Big) \\ &= \sum_{i=1}^{n} g\left((\nabla_{E_i} A)\xi^T, E_i\right) - \overline{g}(N, \xi) \sum_{i=1}^{n} g(A^2 E_i, E_i) \\ &= \sum_{i=1}^{n} g\left((\nabla_{E_i} A)\xi^T, E_i\right) - \overline{g}(N, \xi)\mathrm{trace}(A^2). \end{aligned} \tag{6.4}$$

The Codazzi equation (2.6) allows us to obtain from (6.4)

$$\begin{aligned} \mathrm{div}(A\xi^T) &= \sum_{i=1}^{n} g\left((\nabla_{\xi^T} A)E_i, E_i\right) + \sum_{i=1}^{n} \overline{g}(\overline{\mathrm{R}}(\xi^T, E_i)N, E_i) - \overline{g}(N, \xi)\mathrm{trace}(\mathrm{A}^2) \\ &= \sum_{i=1}^{n} g\left((\nabla_{\xi^T} A)E_i, E_i\right) - \overline{\mathrm{Ric}}(\xi^T, N) - \overline{g}(N, \xi)\mathrm{trace}(A^2). \end{aligned} \tag{6.5}$$

Finally, taking into account that tensor derivations commute with contractions we get

$$\mathrm{div}(A\xi^T) = -n\overline{g}(\nabla H, \xi) - \overline{\mathrm{Ric}}(\xi^T, N) - \overline{g}(N, \xi)\mathrm{trace}(A^2), \tag{6.6}$$

where H is the mean curvature function of M defined in (2.8). From (6.1) and (6.6), the Laplacian of the distinguished function η is given by

$$\Delta\eta = n\overline{g}(\nabla H, \xi) + \overline{\mathrm{Ric}}(\xi^T, N) + \eta \ \mathrm{trace}(A^2). \tag{6.7}$$

Moreover, since ξ is parallel, the Ricci tensor can be expressed as

$$\overline{\mathrm{Ric}}(\xi^T, N) = \overline{\mathrm{Ric}}(\xi, N) + \overline{g}(N, \xi)\overline{\mathrm{Ric}}(N, N) = \eta \ \overline{\mathrm{Ric}}(N, N). \tag{6.8}$$

We see how (6.7) and (6.8) lead to

$$\Delta\eta = n\ \overline{g}(\nabla H, \xi) + \eta\{\overline{\mathrm{Ric}}(N,N) + \mathrm{trace}(A^2)\}. \tag{6.9}$$

These computations enable us to obtain the following result for spacelike hypersurfaces with constant mean curvature in pp-wave spacetimes.

Proposition 6.1. *Let $\psi : M \longrightarrow \overline{M}$ be a spacelike hypersurface with constant mean curvature in an $(n+1)$-dimensional pp-wave spacetime that satisfies the TCC. Then, the function $\eta = \overline{g}(N, \xi)$ is constant if and only if M is totally geodesic.*

Proof. If $\overline{M}$ satisfies the TCC and M has constant mean curvature, provided that η is constant, (6.9) yields

$$0 = \eta\{\overline{\mathrm{Ric}}(N,N) + \mathrm{trace}(A^2)\} \leq 0.$$

Since $\eta < 0$ we obtain

$$\overline{\mathrm{Ric}}(N,N) = \mathrm{trace}(A^2) = 0.$$

The converse follows as an immediate consequence of (6.1) and it is true for every spacelike hypersurface in any pp-wave spacetime. □

We can also obtain the following result for compact spacelike hypersurfaces with constant mean curvature in pp-wave spacetimes.

Theorem 6.2. *Let $\psi : M \longrightarrow \overline{M}$ be a compact spacelike hypersurface in an $(n+1)$-dimensional pp-wave spacetime that satisfies the TCC. If M has constant mean curvature, then it must be totally geodesic. As a direct consequence, there is no compact constant mean curvature spacelike hypersuface in such $\overline{M}$ whose mean curvature is different from zero.*

Proof. From (6.9) we have that the function η is negative and superharmonic on a compact manifold. Thus, η is constant. From Proposition 6.1 we obtain that $A = 0$. □

6.2 A new extension of the parametric Calabi-Bernstein theorem

In this section we will deal with 3-dimensional pp-wave spacetimes. Several examples of such spacetimes can be found in [28] and [38]. Moreover, we can give the following example of a maximal hypersurface in such a spacetime.

Example 6.3. Consider the differentiable manifold $\mathbb{R}^3 = \{(u, v, x) : u, v, x \in \mathbb{R}\}$ endowed with the family of Lorentzian metrics given by

$$\overline{g} = \mathcal{H}(u, x)\, du^2 + 2\, du\, dv + dx^2. \tag{6.10}$$

When the function $\mathcal{H}$ is constant we have a realization of the Lorentz-Minkowski spacetime $\mathbb{L}^3$. Nonetheless, taking into account that the Ricci tensor (see [56, Sec. 2.2]) of this family of metrics is given by

$$\overline{\mathrm{Ric}} = -\frac{\partial^2 \mathcal{H}}{\partial x^2} du \otimes du,$$

it is clear that this family admits spacetimes which are not isometric to $\mathbb{L}^3$ and admit a global parallel lightlike vector field. So, it is enough to consider a function $\mathcal{H}$ such that

$$\frac{\partial^2 \mathcal{H}}{\partial x^2}(p) \neq 0,$$

for some point $p \in \mathbb{R}^3$.

Futhermore, we can now give examples of closed maximal hypersurfaces in spacetimes of the previous family, which are not isometric to $\mathbb{L}^3$. Indeed, let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function and let $c \in \mathbb{R}$ be a regular value of F. It is well-known that the level set $\Sigma = F^{-1}(c)$ is a closed embedded hypersurface in $\mathbb{R}^3$. On the other hand, this hypersurface is spacelike with the induced metric, if and only if the vector field $\overline{\nabla} F$ is timelike on Σ. Taking into account the Christoffel symbols of the metric (see [37, Sec. 2]), it is easy to obtain

$$\overline{\nabla} F = \frac{\partial F}{\partial v}\frac{\partial}{\partial u} + \left(\frac{\partial F}{\partial u} - \mathcal{H}\frac{\partial F}{\partial v}\right)\frac{\partial}{\partial v} + \frac{\partial F}{\partial x}\frac{\partial}{\partial x}.$$

Therefore, Σ is spacelike if and only if the inequality

$$2\frac{\partial F}{\partial u}\frac{\partial F}{\partial v} - \mathcal{H}\left(\frac{\partial F}{\partial v}\right)^2 + \left(\frac{\partial F}{\partial x}\right)^2 < 0, \tag{6.11}$$

holds on Σ.

Now, if we denote by A the shape operator on Σ associated to the unitary normal vector field $\frac{1}{|\overline{\nabla} F|}\overline{\nabla} F$, we have that

$$\overline{g}(AW_1, W_2) = -\frac{\overline{\mathrm{Hess}}(F)(W_1, W_2)}{|\,\overline{\nabla} F\,|}, \tag{6.12}$$

where $\overline{\mathrm{Hess}}(F)$ denotes the Hessian of the function F in $(\mathbb{R}^3, \overline{g})$ and $W_1, W_2 \in \mathfrak{X}(\Sigma)$. Considering $W_1, W_2 \in \mathfrak{X}(\Sigma)$ given by

$$W_1 = w_1^u \frac{\partial}{\partial u} + w_1^v \frac{\partial}{\partial v} + w_1^x \frac{\partial}{\partial x}, \quad W_2 = w_2^u \frac{\partial}{\partial u} + w_2^v \frac{\partial}{\partial v} + w_2^x \frac{\partial}{\partial x},$$

then

$$\begin{aligned}
\overline{\mathrm{Hess}}(F)(W_1, W_2) &= \overline{g}(\overline{\nabla}_{W_1}(\overline{\nabla} F), W_2) = W_1\left(\frac{\partial F}{\partial v}\right)(\mathcal{H} w_2^u + w_2^v) \\
&+ \frac{\partial F}{\partial v}\left(\frac{1}{2} w_1^u w_2^u \frac{\partial \mathcal{H}}{\partial u} - \frac{1}{2} w_1^u w_2^x \frac{\partial \mathcal{H}}{\partial x} + \frac{1}{2} w_1^x w_2^u \frac{\partial \mathcal{H}}{\partial x}\right) \\
&+ W_1\left(\frac{\partial F}{\partial u} - \mathcal{H}\frac{\partial F}{\partial v}\right) w_2^u + W_1\left(\frac{\partial F}{\partial x}\right) w_2^x \\
&+ \frac{\partial F}{\partial x}\left(\frac{1}{2} w_1^u w_2^u \frac{\partial \mathcal{H}}{\partial x}\right).
\end{aligned} \tag{6.13}$$

Now, consider the pp-wave spacetime $(\mathbb{R}^3, \overline{g})$ whose metric is given in (6.10), taking $\mathcal{H}$ a positive function such that

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \qquad \text{and} \qquad \frac{\partial^2 \mathcal{H}}{\partial x^2}(p) \neq 0,$$

at some point $p \in \mathbb{R}^3$. For $\lambda \in \mathbb{R}^+$, consider

$$F(u, v, x) = \lambda v.$$

Taking into account (6.11) it is not difficult to see that the hypersurface $F^{-1}(c)$ is spacelike and from (6.12) and (6.13) it is also maximal, since $\mathrm{trace}(A) = 0$. Furthermore, if $\inf_\Sigma \mathcal{H} > 0$, Σ is complete.

Since we have given explicit examples of these type of hypersurfaces in pp-wave spacetimes, we will consider now a maximal surface S in a 3-dimensional pp-wave spacetime. Taking into account that $H = 0$, we have from (6.9),

$$\Delta \eta = \eta\{\overline{\mathrm{Ric}}(N, N) + \mathrm{trace}(A^2)\}. \tag{6.14}$$

Moreover, in this case, using Cayley-Hamilton theorem we get

$$|\nabla\eta|^2 = \overline{g}(A^2\xi^T, \xi^T) = \frac{1}{2}\text{trace}(A^2)|\xi^T|^2 = \frac{1}{2}\text{trace}(A^2)\eta^2. \tag{6.15}$$

Now, we can characterize the totally geodesic spacelike surfaces in a 3-dimensional pp-wave spacetime by means of the following result.

Proposition 6.4. *Every Riemannian surface that admits a globally defined parallel vector field different from zero is flat.*

Proof. Let (S, g) be a Riemannian surface and let $\zeta \in \mathfrak{X}(S)$ be such that $\nabla\zeta = 0$, with $\zeta \neq 0$. From the definition of the Riemann curvature tensor we have

$$\mathrm{R}(X,Y)\zeta = \nabla_X\nabla_Y\zeta - \nabla_Y\nabla_X\zeta - \nabla_{[X,Y]}\zeta = 0, \tag{6.16}$$

for all $X, Y \in \mathfrak{X}(S)$. Contracting in (6.16) we obtain

$$\mathrm{Ric}(Y,\zeta) = \mathsf{K}\ g(Y,\zeta) = 0, \tag{6.17}$$

for all $Y \in \mathfrak{X}(S)$. Choosing $Y = \zeta$ we obtain that the Gaussian curvature K of S identically vanishes. □

As a consequence of this result and taking (2.42) into account we have the following intrinsic characterization of totally geodesic surfaces in 3-dimensional pp-wave spacetimes.

Corollary 6.5. *Every totally geodesic spacelike surface in a 3-dimensional pp-wave spacetime is flat.*

On the other hand, considering the Gaussian curvature of a spacelike surface in these ambient spacetimes, we can enunciate the next theorem.

Theorem 6.6. *Let $\psi : S \longrightarrow \overline{M}$ be a maximal surface in a 3-dimensional pp-wave spacetime. Then,*

(i) the Gaussian curvature of S satisfies $\mathsf{K} \geq \overline{\mathrm{Ric}}(N,N)$, being N the timelike unitary normal vector field to S. Moreover, equality holds if and only if S is totally geodesic (and therefore, flat).

(ii) if the spacetime satisfies the TCC and the surface is complete, then S is parabolic.

Proof. Since S is maximal, from the Gauss equation (2.5) we obtain,

$$\mathrm{Ric}(\xi^T, \xi^T) = \overline{\mathrm{Ric}}(\xi^T, \xi^T) + \overline{g}(\overline{\mathrm{R}}(\xi^T, N)N, \xi^T) + \overline{g}(A^2\xi^T, \xi^T). \tag{6.18}$$

Moreover, being $\dim(S) = 2$ we have

$$\mathrm{Ric}(\xi^T, \xi^T) = \mathsf{K}\ |\xi^T|^2 = K\ \eta^2 \tag{6.19}$$

and as $\overline{\nabla}\xi = 0$, then

$$\overline{\mathrm{Ric}}(\xi^T, \xi^T) = \eta^2\ \overline{\mathrm{Ric}}(N, N), \tag{6.20}$$

and

$$\overline{g}(\overline{\mathrm{R}}(\xi^T, N)N, \xi^T) = \overline{g}(\overline{\mathrm{R}}(\xi, N)N, \xi) + 2\ \eta\ \overline{g}(\overline{\mathrm{R}}(\xi, N)N, N) + \eta^2\ \overline{g}(\overline{\mathrm{R}}(N, N)N, N) = 0. \tag{6.21}$$

Therefore, substituting (6.19), (6.20), (6.21) and (6.15) in (6.18) we have

$$\mathsf{K} = \overline{\mathrm{Ric}}(N, N) + \frac{1}{2}\mathrm{trace}(A^2) \tag{6.22}$$

and as a direct consequence of (6.22) and Corollary 6.5, *(i)* holds.

Finally, the TCC guarantees $\mathsf{K} \geq 0$. Thus, if S is a complete Riemannian surface, Theorem 2.9 ensures its parabolicity. □

We can give the following rigidity's result by means of a procedure used in [97].

Theorem 6.7. *Let $\psi : S \longrightarrow \overline{M}$ be a complete maximal surface in a 3-dimensional pp-wave spacetime that obeys the TCC. Then, S is totally geodesic and flat.*

Proof. Making use of (6.14) and (6.15),

$$\Delta\left(\frac{1}{\eta}\right) = -\frac{1}{\eta^2}\Delta\eta + \frac{2}{\eta^3}|\nabla\eta|^2 = -\frac{1}{\eta}\left\{\overline{\mathrm{Ric}}(N, N) + \mathrm{trace}(A^2)\right\} + \frac{1}{\eta}\mathrm{trace}(A^2)$$

Thus,

$$\Delta\left(\frac{1}{\eta}\right) = -\frac{1}{\eta}\overline{\mathrm{Ric}}(N, N). \tag{6.23}$$

Since S is parabolic due to Theorem 6.6, it is enough to obseve that the function η must be constant, and so, Proposition 6.1 applies to obtain that S is totally geodesic. Moreover, its flatness is due to Corollary 6.5. □

As a special case of the previous theorem we obtain the following well-known result.

Corollary 6.8. *(Classical Calabi-Bernstein theorem) The only complete maximal surfaces in the Lorentz-Minkowski spacetime* $\mathbb{L}^3$ *are the spacelike affine planes.*

Conclusions and future research

In this thesis we have studied complete spacelike hypersurfaces in several spacetimes from different perspectives. These spacetimes include both spatially closed and open GRW spacetimes as well as standard static spacetimes and pp-wave spacetimes.

In Chapter 3 we have introduced a new way of regarding the spacelike slices in a spatially closed GRW spacetime as the critical points of a variational problem with a clear physical interpretation. We believe that an interesting future line of research can be based on extending some of the results in this chapter for the solutions of equation (E) to the spatially open n-dimensional case. In order to do this, we may need to rely on certain compactness assumptions such as parabolicity.

Moreover, after dealing with the previous problem in spatially closed GRW spacetimes, we have devoted Chapter 4 to study complete maximal hypersurfaces in the spatially open case. Therefore, by means of our technique based on the Omori-Yau maximum principle for the Laplacian we have been able to obtain new uniqueness results for these hypersurfaces in these models that are in agreement with the observations of our actual universe and include some of the most famous Robertson-Walker cosmological models.

Indeed, we are now working on extending some of the results given in Chapter 4 to constant mean curvature spacelike hypersurfaces (not necesarily maximal) in spatially open GRW spacetimes. In particular, using the formula obtained in Remark 4.2 we will probably be able to study constant mean curvature spacelike hypersurfaces in spatially open GRW spacetimes that obey the NCC.

Furthermore, in Chapter 5 we give a new correspondence between the solutions of the minimal surface equation in a certain 3-dimensional Riemannian warped product and the solutions of the maximal surface equation in a 3-dimensional standard static spacetime. This widely extends the classical duality between minimal graphs in 3-dimensional Euclidean space and maximal graphs in 3-dimensional Lorentz-Minkowski spacetime ([36], [12]). This correspondence can be restricted to the respective classes of entire solutions, obtaining a Calabi-Bernstein type result for certain static standard spacetimes. Moreover, this result can be used to construct solutions of the maximal surface equation in this ambiences from known solutions of the corresponding minimal surface equation and vice versa, extending [6, Sect. 5] and [7, Sect. 6]. We believe that it might be interesting to extend this duality to higher dimension, where we will have to make use of a different technique than the one used here.

To conclude, we have studied in Chapter 6 maximal hypersurfaces in the family of pp-wave spacetimes. These models have attracted a great deal of attention due to the recent experimental detection of gravitational waves [1]. Therefore, the theoretical study of these hypersurfaces in these models will lead to a better understanding of our universe and the development of more suitable models to describe it. We have obtained a uniqueness result for compact maximal hypersurfaces in pp-wave spacetimes of arbitrary dimension as well as for complete (not necessarily compact) maximal surfaces in 3-dimensional pp-wave spacetimes. Thus, we think that a future research line can consist in extending these uniqueness results for complete, non-compact, maximal hypersurfaces to the case of arbitrary dimension.

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