Versión revisada y corregida del 14/06/2012 sobre la versión original de la tesis, defendida el 13/06/2012 y depositada en la Comisión de Doctorado de la Universidad de Granada el 02/05/2012, que incluye, con respecto a la depositada:

- Las proposiciones 2.1.4 y 2.1.5 de la sección 2.1.4.
- La demostración de la proposición 2.3.8 en la sección 2.3.
- El lema 5.1.5 de la sección 5.1.
- La demostración de la existencia del vector paralelo luminoso en la sección 6.2.
- El apéndice B.
- Una reestructuración de las secciones 5.1 y 6.1 de la tesis depositada.
- Las sugerencias de los miembros del Tribunal.

A Senovilla y Miguel,
sin los cuales esta tesis
hubiera sido imposible.

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Eskerrik beroenak danoi.

Contents

Introducción

La *Geometría euclídea*, basada en el libro *Los Elementos* de Euclides escrito en el año 300 a.C., perduró como la base de la geometría hasta el siglo XIX. En 1829 el matemático ruso Lobachevski publicó un trabajo en el que el quinto postulado de Euclides, *el postulado de las paralelas*, no se satisfacía, dando lugar a una nueva geometría a la que se le conoce hoy en día como *geometría hiperbólica*. De esta manera, se iniciaba el estudio de las *Geometrías no euclídeas*. Tres años después el matemático húngaro Bolyai también llegó a la conclusión de que el quinto postulado de Euclides era independiente de los demás axiomas del libro, de forma que nuevos tipos de geometrías podían construirse basándose en la negación de aquél. En su trabajo por tanto aparecen otras geometrías aparte de la hiperbólica. Se sabe incluso que Gauss también había llegado a las mismas conclusiones, aunque nunca publicó sus resultados. Fue Riemann, matemático alemán, quien hizo la contribución más radical a la geometría en este siglo XIX, presentando un trabajo general en el que la geometría del espacio podía variar de punto a punto. Con esta obra inauguró nuevas áreas de investigación que combinaban análisis y geometría, originándose así desde este nuevo punto de vista la *Geometría riemanniana*.

Un poco después, en el siglo XX, se marcó otro gran hito en el área de la geometría. Desde la observación de la naturaleza, la física trata de describir las propiedades del universo mediante el uso de las matemáticas. Un ejemplo claro de ello es la aplicación de la Geometría riemanniana en la Teoría Gravitatoria de la Relatividad General. Haciendo gala de una intuición sin igual, en 1905 Einstein en su trabajo [28] presentó ideas radicalmente novedosas sobre el comportamiento de la luz en el vacío. Se dio cuenta de que todos los observadores inerciales debían ser físicamente equivalentes (primer postulado de la *Relatividad Especial*), y de que la velocidad de la luz en el vacío debía ser igual para todos ellos (segundo postulado). A su vez, rescató las ecuaciones de transformación de Lorentz entre estos sistemas, que Lorentz ideó para tratar de entender el resultado del experimento de Michelson-Morley sobre la velocidad de la luz. Tales transformaciones contenían dos hipótesis: la contracción de cuerpos rígidos y la dilatación del tiempo a través del llamado *éter*. Aunque la esencia de la Relatividad Especial está contenida en estas transformadas, Einstein las dedujo exclusivamente de los dos postulados mencionados arriba.

En 1907 el físico y matemático Hermann Minkowski observó que la teoría de la Relatividad Especial podía entenderse mejor en un espaciotiempo de cuatro dimensiones, conocido hoy en día como el *espaciotiempo de Lorentz-Minkowski*, en el cual el tiempo y el espacio no son entes separados sino que se entremezclan en un espaciotiempo de cuatro dimensiones y las transformadas de Lorentz de la Relatividad Especial pueden representarse fácilmente. Como bien puntualizó en su discurso de inauguración de la 80ª reunión de la Asamblea general alemana de científicos naturalistas y físicos el 21 de septiembre de 1908:

> *"Las ideas sobre el espacio y el tiempo que deseo mostrarles hoy descansan en el suelo firme de la física experimental, en la cual yace su fuerza. Son ideas radicales. Por lo tanto, el espacio y el tiempo por separado están destinados a desvanecerse entre las sombras y tan sólo una unión de ambos puede representar la realidad."*

Su muerte en el año 1909 a la edad de 44 años le impidió comprobar lo mucho que revolucionaría este nuevo punto de vista tanto la geometría como la teoría de la gravitación a partir del año 1915. Precisamente en ese año, cuando Einstein introdujo la gravedad en su teoría (desarrollando así la *Teoría de la Relatividad General*) en su trabajo [29], éste dedujo que el campo gravitatorio (concepto físico) es el efecto natural de la existencia de una curvatura en el espaciotiempo (concepto geométrico). Por ello y para poder formular las nuevas ecuaciones del campo gravitatorio adecuadamente, Einstein se dedicó a estudiar esta nueva geometría que Riemann había ideado, además del cálculo tensorial. Dichas ecuaciones vienen dadas por la siguiente fórmula tensorial:

$$\mathrm{Ric} - \frac{1}{2}Sg + \Lambda g = \frac{8\pi G}{c^4}T.$$

donde g es el tensor métrico, Ric su tensor de Ricci, $S = \mathrm{tr}_g(\mathrm{Ric})$ su curvatura escalar, Λ es la constante cosmológica de Einstein, G la constante de gravitación newtoniana, c la velocidad de la luz en el vacío y T el tensor energía-momento o impulso-energía. Estas ecuaciones relacionan el tensor de Ricci (que es parte de la curvatura del espaciotiempo cuadri-dimensional) con T, que representa la concentración de materia y energía en el universo.

De esta forma, combinando las ideas de Riemann para la geometría con las ideas de Lorentz y Minkowski, se dio lugar a una nueva rama de las matemáticas, la *Geometría lorentziana o de Lorentz*. Desde entonces y hasta nuestros días ambas ramas, la Geometría Lorentziana y la Relatividad General, han sido muy fructíferas. Sin duda, hay una continua retroalimentación entre ellas, que ha dado lugar a una nueva rama como intersección de ambas, a la que se le podría denominar *relatividad matemática*.

Una de las virtudes del enfoque matemático de la Geometría lorentziana es la contribución de ésta a la resolución de muchos problemas a los que se enfrentan los físicos en su investigación sobre la teoría del campo gravitatorio. Sin embargo, en alguna ocasión pudiera ser que los geómetras no hubiesen considerado algún problema en particular o planteado alguna pregunta que en cambio sí surge de manera natural en la teoría gravitatoria. En esos casos la colaboración entre ambas ramas, esto es, entre matemáticos y físicos, es fundamental y muchas veces imprescindible. Existen a su vez casos en que problemas geométricos han sido resueltos o planteados por físicos, al igual que existen ideas que surgen desde el enfoque puramente matemático y rápidamente se extienden a aplicaciones físicas.

Esta memoria trata de aportar un minúsculo grano de arena en esta interacción, clasificando los espacios lorentzianos *segundo-simétricos*, esto es, las variedades lorentzianas que satisfacen que $\nabla\nabla R = 0$, donde R es el tensor de curvatura de la variedad. Como apunta en [79] José M.M. Senovilla, introductor de tales espacios, el interés físico que suscitan estos espacios es múltiple. Entre otras cosas, podrían ser soluciones de teorías físicas en formulación lagrangiana de orden superior que incluyan la gravedad, y en particular podrían ser soluciones exactas en teoría de cuerdas bajo un contenido de materia específico (veáse por ejemplo [70, 47, 23]); también podrían proporcionar soluciones exactas de fondo para supergravedad de dimensión 11. En [79] también se aporta una primera aproximación a estos espacios, ya que se demuestra que si $\nabla R \neq 0$, necesariamente el espacio posee un campo vectorial paralelo luminoso. De esta forma, la familia constituida por estos espacios queda contenida en la de los espacios de Brinkmann. En esta tesis, mediante un meticuloso estudio preliminar puramente geométrico de estos últimos espacios, se consigue dar una clasificación completa de los espacios lorentzianos segundo-simétricos. Los resultados de este trabajo están recogidos en [10, 11, 12]. En los Encuentros Relativistas Españoles celebrados en Bilbao en Septiembre del 2009 se presentó el caso cuadri-dimensional (véase [10]), y en los mismos encuentros celebrados el siguiente año en Granada se comunicó el resultado para el caso más general de dimensión arbitraria (véase [11]).

El interés que suscitan estas variedades lorentzianas desde el punto de vista más puramente matemático es el siguiente. En Geometría Diferencial existe un venerable resultado (Nomizu y Ozeki [64], Tanno [87]) enunciado de la siguiente manera: para una variedad riemanniana (M, g), la anulación de la r-ésima derivada covariante del tensor curvatura R,

$$\nabla^r R \,(= \nabla \overset{(r)}{\ldots} \nabla R) \equiv 0, \qquad r \geq 2, \tag{1}$$

implica la anulación de la primera, esto es, (M, g) es *localmente simétrica*. Pero cuando (M, g) es una variedad lorentziana, la igualdad (1) no necesariamente implica $\nabla R = 0$. Por tanto, la ecuación (1) conduce a una escalera jerárquica de generalizaciones naturales de los espacios semi-riemannianos localmente simétricos. Y es con $r = 2$ cuando aparecen los arriba mencionados espacios segundo-simétricos. Llamaremos a estas nuevas variedades semi-riemannianas *espacios simétricos de r-ésimo orden* (o más abreviadamente *espacios r-ésimo-simétricos*). En contraposición, la generalización típica de los espacios riemannianos localmente simétricos son los espacios semi-simétricos, introducidos por Cartan [20] y definidos mediante la conmutatividad de las derivadas covariantes aplicada a R:

$$R(X, Y)R := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})R = 0,$$

para cualesquiera campos vectoriales X, Y. Szabó determinó su estructura local en [84] y más tarde la global en [85].

La clasificación de los espacios riemannianos localmente simétricos es, por tanto, conocida desde el trabajo de Cartan [19] (véase también [46, 8]). La clasificación de los espacios lorentzianos simétricos simplemente conexos fue llevada a cabo por Cahen y Wallach en [17]. Para ampliaciones a otras signaturas y para el caso no simplemente conexo, consúltense los trabajos de Cahen y Parker [16], Neukirchner [63] y especialmente Kath y Olbrich [49, 50]. Los espacios lorentzianos semi-simétricos también han sido estudiados, veáse por ejemplo su clasificación en cuatro dimensiones [31, 40] y las referencias que ahí aparecen. No obstante, con anterioridad al trabajo de Senovilla [79] no existía estudio alguno de los espacios segundo-simétricos. Como se subraya en esta referencia, es posible construir dentro de la clase de ondas planas n-dimensionales ejemplos simples de espacios lorentzianos *propios r-ésimo-simétricos* —esto es, r-ésimo-simétricos pero no $(r-1)$ésimo-simétricos— (véase la Sección 5.4 de esta memoria). Esta familia constituye una clara generalización de los espacios localmente simétricos que aquí denominamos *espacios de Cahen-Wallach* [17] y, como demostraremos, representan de manera esencial toda la clase de espacios lorentzianos propios segundo-simétricos.

Vale la pena señalar que los espacios segundo-simétricos son tratables desde el punto de vista del grupo de simetrías local de la variedad, ya que es posible expresar la condición $\nabla^2 R = 0$ en términos del algebra infinitesimal de holonomía (puesto que esta álgebra se puede generar mediante la imagen de las dos-formas de curvatura y sus derivadas primeras, véase por ejemplo [52, Th. 9.2 Ch. III]); de hecho, el resultado principal en [79] es una propiedad del grupo de holonomía. Con la ayuda de esta propiedad y del conocimiento previo de los espacios localmente simétricos debido a Cahen y Wallach, nuestra demostración será totalmente autocontenida, resolviendo las ecuaciones de segundo-simetría de un modo directo. Específicamente, este es el resultado principal que vamos a demostrar:

Teorema 1. *Un espacio lorentziano segundo-simétrico no localmente simétrico (M, g) de dimensión n es localmente isométrico al producto directo $(M_1 \times M_2, g_1 \oplus g_2)$ donde (M_2, g_2) es un espacio riemanniano simétrico no llano y (M_1, g_1) es un espacio propio de Cahen-Wallach generalizado de orden 2, definido como $M_1 = \mathbb{R}^{d+2}$ ($d \geq 0$) dotado de la métrica*

$$g_1 = -2du \left(dv + du \sum_{i,j=2}^{d+1} p_{ij}(u) x^i x^j \right) + \sum_{i=2}^{d+1} (dx^i)^2,$$

donde $(u, v, x^2, \ldots, x^{d+1})$ son las coordenadas naturales de $\mathbb{R}^{d+2}$ y cada función p_{ij} es afín: $p_{ij}(u) = \alpha_{ij} u + \beta_{ij}$ para algunos $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$ con al menos uno de los α_{ij} no nulo, y para todo $i, j = 2, \ldots, d+1$.

Es más, si (M, g) es también geodésicamente completo y simplemente conexo, entonces (M, g) es globalmente isómetrico a uno de tales productos.

En líneas muy generales, la idea de la demostración es la siguiente. Como ya se ha comentado, el punto de partida es el resultado significativo obtenido por Senovilla ([79, Teorema 4.2]): *cualquier espacio lorentziano simplemente conexo y propio segundo-simétrico*

(M, g) *admite un campo de vectores paralelo luminoso* K. Brinkmann en [15] obtuvo los espacios lorentzianos con el mencionado K y serán estudiados en la Sección 4.2.1, donde se introducirán bases locales asociadas a lo que llamaremos *cartas de Brinkmann* $\{u, v, x^i\}$. El hecho de que, para cualquiera de dichas cartas, los subconjuntos con u y v constantes resultan ser localmente simétricos sugiere una reducción de las ecuaciones de segundo-simetría. Esta reducción se lleva a cabo explotando íntegramente las ecuaciones de integrabilidad, y aplicando algunas propiedades algebraicas puramente técnicas. Después, usando descomposiciones de tipo Eisenhart para transformar las ecuaciones originales de (M, g) en ecuaciones de segundo-simetría asociadas a dos espaciotiempos más simples, surgen localmente los enunciados espacios (M_1, g_1) y (M_2, g_2). El resultado global proviene de ciertas elaboraciones geométricas. De hecho, a lo largo de la prueba el hecho de que (M, g) admite una dirección paralela luminosa globalmente definida se hará evidente. A partir de ahí, los requerimientos globales completan fácilmente la demostración.

Merece la pena recalcar que, en paralelo, Alekseevsky y Galaev en [2, 35] han desarrollado una aproximación local diferente, completamente basada en grupos de holonomía (que incluyen resultados clave como la clasificación de los grupos de holonomía en signatura lorentziana [55]). Mediante su uso, se recupera el resultado crucial en [79, Theorem 4.2] y se proporciona el Teorema 1. Por tanto, hay un sustancial conjunto de herramientas a día de hoy para este excitante campo de investigación. Al final de esta memoria se presentarán algunos problemas abiertos que podrían ser de interés.

Resumen de contenidos

Esta tesis está organizada de la siguiente manera. En el Capítulo 2 y para complemento de esta memoria, se dan algunos resultados matemáticos de base, junto con ciertos resultados algebraicos necesarios y una breve explicación de la notación con índices abstractos. Somos especialmente meticulosos en esto último, ya que usaremos una combinación potente de expresiones en forma intrínseca y cálculo en componentes tensoriales. En el Capítulo 3 revisamos algunos resultados en espacios localmente simétricos y simétricos. En particular, se presentan los espacios de Cahen-Wallach en la Sección 3.2.4 como aquellos localmente simétricos de signatura lorentziana. El Capítulo 4 está dedicado a un estudio local de los espacios de Brinkmann, que podría tener interés por derecho propio aún a pesar de su naturaleza técnica. En la Sección 4.1 recordamos el ya conocido proceso para encontrar una *carta de Brinkmann* $\{u, v, x^i\}$ asociada a una *descomposición de Brinkmman* $\{u, v\}$, de tal forma que la métrica de la variedad se escribe de la siguiente manera:

$$g = -2du(dv + H(u, x^k)du + W_i(u, x^k)dx^i) + g_{ij}(u, x^k)dx^i dx^j, \;\; k = 2, \ldots, n-1. \qquad (2)$$

Éstas proporcionan una foliación espacial $\overline{\mathcal{M}}$ asociada de dimensión $(n-2)$, caracterizada por valores constantes en u y v, y una foliación temporal $\mathcal{U}$ de dimensión 2 no necesariamente ortogonal a la primera, generada por ∂_u, ∂_v, así como otras distribuciones de interés. Para mayor claridad, en la Sección 4.2 se explican brevemente las relaciones entre los campos tensoriales de estas distribuciones. En la Sección 4.3 se hace un riguroso estudio de la foliación $\overline{\mathcal{M}}$. En particular, se introducen tres operadores $\overline{\nabla}$, $\overline{d}$ y $\dot{\ }$ adaptados a las foliaciones $\overline{\mathcal{M}}$ y $\mathcal{U}$, y se relacionan con la conexión ∇, la diferencial exterior d y la geometría de la foliación $\overline{\mathcal{M}}$. En la Sección 4.4, cálculos *à la Cartan* en bases locales introducidas en la Sección 2.1.6 junto con la Subsección 4.3.6 proporcionan expresiones manejables de las uno-formas de conexión y el tensor de curvatura R de un espacio de Brinkmann. Para ayudar a la simplificación de posteriores expresiones, en la Sección 4.5 se introduce un cuarto operador diferencial D_0 de naturaleza transversa, que complementa los tres anteriores. Después, en el Teorema 4.6.2 de la Sección 4.6 se facilita una versión de una Teorema clásico de Eisenhart [30] adaptado a nuestro problema, cuyo enunciado involucra tanto métricas u-dependientes en las hojas de la foliación, como las derivadas intrínseca $\overline{\nabla}$ y transversa D_0. Esto nos permite obtener condiciones suficientes para que la u-familia de métricas riemannianas $\overline{g} = g_{ij}(u, x^k)dx^i dx^j$ sea simultáneamente reducible. En el Capítulo 5 se presentan algunas propiedades de los espacios propios simétricos de orden r. Luego, se define la familia de los espacios generalizados de Cahen-Wallach como un caso particular de los r-ésimo-simétricos.

Finalmente en el Capítulo 6, en la Sección 6.1 damos posibles caracterizaciones para los espacios semi-riemannianos segundo-simétricos en comparación con los resultados ya conocidos para los localmente simétricos y en la Sección 6.2 comentamos la existencia de un campo vectorial paralelo luminoso definido globalmente. En la Sección 6.3 resolvemos las ecuaciones de segundo-simetría sobre los espacios de Brinkmann en varios pasos. Primero, en la Subsección 6.3.1 se dan expresiones manejables para las dos primeras derivadas covariantes del tensor de curvatura R junto con las ecuaciones de segundo-simetría. Se enfatiza y se controla explícitamente la dependencia de todos los objetos en cada carta de Brinkmann y su comportamiento bajo cambio de cartas. En segundo lugar, demostramos que las ecuaciones de segundo-simetría implican que la foliación $\overline{\mathcal{M}}$ debe ser localmente simétrica para toda carta de Brinkmann $\{u, v, x^i\}$ asociada a una descomposición fija $\{u, v\}$ (Proposición 6.3.7), por lo que se deben satisfacer severas restricciones en la curvatura. En tercer lugar, usando los resultados algebraicos auxiliares sobre espacios vectoriales dados en la Subsección 2.1.4, se fijan explícitamente dichas restricciones en el Teorema 6.3.8. Es más, en (b) del Teorema 6.3.8 se define un campo tensorial dos-covariante $\tilde{A}$ en $\overline{\mathcal{M}}$, asociado únicamente a la descomposición de $\{u, v\}$ pero no al resto de las coordenadas de la carta de Brinkmann. Toda la información del tensor de curvatura queda, por tanto, codificada en $\tilde{A}$ y en las hojas de $\overline{\mathcal{M}}$. Estos resultados se resumen en la Subsección 6.3.2. En cuarto lugar, en la Sección 6.3.3 se demuestra que es posible una reorganización de las ecuaciones de segundo-simetría en dos bloques independientes asociados a diferentes espacios de Brinkmann $(M^{[m]}, g^{[m]})$ con $m \in \{1, 2\}$ (Teorema 6.3.19 y Proposición 6.3.20). Esto se consigue aplicando nuestra versión del Teorema de Eisenhart y demostrando que la métrica se reduce adecuadamente a una parte Ricci-llana y otra no Ricci-llana. De hecho, la primera es llana como consecuencia de un resultado conocido [1] y el operador $\tilde{A}$ actúa únicamente en esta parte llana (Proposición 6.3.15). Finalmente, en la Subsección 6.3.4 se completa la demostración del teorema principal. El primer espacio $(M^{[1]}, g^{[1]})$ se calcula directamente y conduce al requerido espacio de Cahen-Wallach generalizado, obteniendo así la expresión de la parte (M_1, g_1) en el Teorema 1. Las ecuaciones de segundo-simetría para el segundo espacio $(M^{[2]}, g^{[2]})$ se convierten en ecuaciones equivalentes a las de simetría local de tal forma que $(M^{[2]}, g^{[2]})$ colapsa a un espacio de Brinkmann localmente simétrico, por lo que la clasificación de Cahen-Wallach nos permite determinar la parte riemanniana localmente simétrica (M_2, g_2) en el Teorema 1. De esta forma queda demostrado el Teorema 1. Un análisis juicioso junto con ciertos resultados técnicos (Corolario 6.2.2, Lema 5.4.3) implicarán la afirmación de carácter global del Teorema 1 (véase Teorema 6.3.2).

Por último, la notación y las convenciones que se van a usar en esta tesis están resumidas en la sección 1.2.

Chapter 1

Introduction

Euclidean Geometry, the geometry based in the book *Elements*, written in 300 BC by the Greek mathematician Euclid, endured as the basis of Geometry without changes until the XIXth century. The Russian mathematician Lobachevski presented a research in 1829 in which the V postulate, *the parallel postulate*, of that book is not satisfied. This way, he developed a geometry which is referred to as *hyperbolic geometry*, giving rise to *non-Euclidean Geometry*. Three years later the Hungarian mathematician Bolyai also came to the conclusion that the postulate was independent of the other axioms of the book and that different consistent geometries could be constructed on its negation. In his work not only the hyperbolic geometry was treated. It is also well-known that Gauss arrived to analogous conclusions even though he did not publish them.

It was Riemann, also a German mathematician, who made the most radical contribution in this area in the XIXth century. He presented a general work in which the geometry of the space could vary from point to point. With his work he opened up research areas combining analysis with geometry, and *Riemannian Geometry* originated under this new viewpoint.

Later, in the XXth century, another amazing milestone was developed in the area of geometry. From the observation of nature, physics tries to understand the behaviour of the universe by making use of mathematics. And a relevant example of this is the application of the Riemannian geometry to the General Relativity Theory of Gravitation. Exhibiting his insightful intuition, the German-born physicist Einstein in his work [28] published in 1905 put forward radical novel ideas about the nature of light in vacuum. He realized that all inertial observers should be physically equivalent (first postulate of *Special Relativity*), and that the speed of the light in vacuum should be equal for all of them (second postulate). At the same time, he rescued the Lorentz transformations between inertial systems, which were conceived by Lorentz to understand the Michelson-Morley experiment to measure the speed of light. Lorentz transformations contained two hypothesis: contraction of rigid motions and dilatation of time through *ether*. Although the essence of the Special Relativity is contained in these transformations, Einstein deduced them solely from the previous postulates.

The Russian mathematician and physicist Minkowski in 1907 realized that the theory of Special Relativity could be better understood in a four-dimensional space, known nowadays as *Lorentz-Minkowski space-time*, in which time and space are not separated entities but they intermingle in a four-dimensional *space-time*, where the Lorentz transformations can be neatly represented. The beginning part of his address delivered at the 80th Assembly of German Natural Scientists and physicists (September 21, 1908) is now famous:

> *"The views of space and time which I wish to lay before you have sprung from the soil of experimental physics, and therein lies their strength. They are radical. Henceforth space by itself, and time by itself, are doomed to fade away into mere shadows, and only a kind of union of the two will preserve an independent reality."*

His dead in 1909 at the age of 44 prevented him from learning how much his ideas would revolutionize both geometry and the theory of gravitation from 1915 on. Indeed, in 1915, when gravity was included in his theory (and so *the Theory of General Relativity* was developed), in his work [29] Einstein inferred that the gravitational field (a physical concept) is the natural effect of the existence of curvature in the space-time (a geometrical concept). Thus, Einstein learned about the new branch of geometry started by Riemann and about tensor calculus in order to formulate the new equations of the gravitational field properly. These equations are given by the formula:

$$\mathrm{Ric} - \frac{1}{2}Sg + \Lambda g = \frac{8\pi G}{c^4}T.$$

where g is the metric tensor, Ric its Ricci tensor, $S = \mathrm{tr}_g(\mathrm{Ric})$ its scalar curvature, Λ is Einstein's cosmological constant, G is Newton's gravitational constant, c the speed of light in vacuum, and T the stress–energy tensor, or energy-momentum tensor. This tensor equation relates the Ricci tensor (part of the curvature of the four-dimensional space-time) with T, that describes the density and flux of energy and momentum in the universe.

Thus, combining Riemann's ideas for geometry with the ideas of Lorentz and Minkowski, a new branch in pure mathematics was developed, *Lorentzian Geometry*. From that moment hereinafter both branches, Lorentzian Geometry and General Relativity, have been very productive. Undoubtely, there exists a continuous feed-back between them, giving rise to a new branch as an intersection of both of them, which shall be called *mathematical relativity*.

One of the virtues of the mathematical approach to Lorentzian Geometry is its contribution to the resolution of many problems that theoretical physicists encounter in their work on the theory of the gravitational field. In many occasions, though, the geometers may have not considered a particular problem or a specific question arising naturally in the gravitational theory, and then the interplay and collaboration between both branches, that is between mathematicians and physicists, is very helpful and often necessary. There are also instances of geometrical problems solved, or launched, by physicists, as well as ideas that arise on the mathematical approach and spread quickly to the physical applications.

This memoir tries to provide a tiny grain on this interplay, by giving the classification of the Lorentzian *second-order symmetric spaces*, that is to say, the Lorentzian manifolds that satisfy that $\nabla\nabla R = 0$, where R is the curvature tensor. As Senovilla, who introduced such spaces, points out in [79], they may "have important applications in physics, in particular in theories concerning gravitation". For example, among other facts, they can be solutions in higher-order Lagrangian physical theories including gravity and they can provide examples of exact solutions for backgrounds for 11-dimensional supergravity and relatives via M-theory [13, 33]. In his work, he also provided a fundamental approximation to these spaces. He proved that if $\nabla R \neq 0$, then a second-order symmetric space must have a parallel lightlike vector field. Thus, this family of particular spaces is contained in the set of Brinkmann spaces. In this thesis, by making a meticulous preliminary study of Brinkmann spaces, a complete classification of Lorentzian second-order symmetric spaces is performed. The results of this work are gathered in [10, 11, 12]. The four-dimensional case was first announced in the Spanish Relativity Meetings celebrated in Bilbao, 7-11 September '09 (see [10]) and in the same meetings next year in Granada (6-10 September '10) the general n-dimensional case was considered (see [11]).

From the purely mathematical viewpoint, the interest of these Lorentzian manifolds is the following. A venerable result in Differential Geometry (Nomizu and Ozeki [64], Tanno [87]) states that, for a Riemannian manifold (M, g), the vanishing of the r-th covariant derivative of its curvature tensor R,

$$\nabla^r R \, (= \nabla \overset{(r)}{\cdots} \nabla R) \equiv 0, \qquad r \geq 2, \tag{1.1}$$

implies the vanishing of the first one, i.e., that (M, g) is *locally symmetric*. But when (M, g) is a Lorentzian manifold, the equality (1.1) does not imply $\nabla R = 0$. Thus, a ladder of logical generalizations of Lorentzian locally symmetric spaces is given by (1.1). When $r = 2$, the aforementioned second-order spaces appear. We call these semi-Riemannian manifolds *rth order symmetric* (or *rth-symmetric* for short) spaces[1]. As a consequence, the standard generalization of Riemannian locally symmetric spaces are the semi-symmetric spaces, introduced by Cartan [20] and defined by the commutativity of the covariant derivatives applied to R:

$$R(X, Y)R := (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})R = 0 \qquad \text{for all vector fields } X, Y. \tag{1.2}$$

Their structure was determined by Szabó locally in [84] and globally later in [85].

The classification of Riemannian locally symmetric spaces is known since Cartan's work [19] (see also [46, 8]), and the classification of the Lorentzian simply-connected symmetric spaces was carried out by Cahen and Wallach in [17]. Extensions to other signatures and to non-simply-connected cases are also available, see Cahen and Parker

[1] rth-order symmetric spaces were introduced in [79] and termed as r-symmetric for short. However, a different notion of *3-symmetric space*, introduced by Gray [39], was already available and somehow spread in the literature (for example, see the recent article [38]). Thus, we have preferred to use the ordinal (*3rd-symmetric*, say) to avoid any possible confusion.

[16], Neukirchner [63] and specially Kath and Olbrich [49, 50]. Lorentzian semi-symmetric spaces have also been studied in the literature, see for instance their classification in four dimensions [31, 40] and references therein. Nevertheless, prior to the paper [79] by Senovilla, the 2nd-symmetric spaces had not been studied systematically. As pointed out in this reference, simple examples of *proper rth-symmetric* —rth-symmetric but not $(r-1)$th-symmetric— Lorentzian spaces can be constructed within the class of n-dimensional plane waves (see Section 5.4). They constitute a straightforward generalization of the locally symmetric *Cahen-Wallach spaces* [17] and, as we will prove, they essentially exhaust the whole class of proper 2nd-symmetric Lorentzian spaces.

It is worth pointing out that the 2nd-symmetric spaces are treatable from the viewpoint of the local group of symmetries of the manifold, because the condition $\nabla^2 R = 0$ can be expressed in terms of the infinitesimal holonomy algebra of the manifold (as this algebra is generated by the image of the curvature two-form and its first derivative see, for example, [52, Th. 9.2 Ch. III]); in fact, the main result in [79] is a property of this holonomy group. With the help of this property and the well-established results on locally symmetric spaces by Cahen and Wallach, our proof will be completely self-contained, by solving the equations of 2nd-symmetry crudely. Specifically, the main result we will prove is:

Theorem 1.1. *An n-dimensional 2nd-symmetric non-locally symmetric Lorentzian space (M, g) is locally isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$ where (M_2, g_2) is a non-flat Riemannian symmetric space and (M_1, g_1) is a proper generalized Cahen-Wallach space of order 2, defined as $M_1 = \mathbb{R}^{d+2}$ ($d \geq 0$) endowed with the metric*

$$g_1 = -2du \left(dv + du \sum_{i,j=2}^{d+1} p_{ij}(u) x^i x^j \right) + \sum_{i=2}^{d+1} (dx^i)^2,$$

where $(u, v, x^2, \ldots, x^{d+1})$ are the natural coordinates of $\mathbb{R}^{d+2}$ and each function p_{ij} is affine: $p_{ij}(u) = \alpha_{ij} u + \beta_{ij}$ for some $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$ with at least one of the α_{ij} non-vanishing, and all $i, j = 2, \ldots, d+1$.

Moreover, if (M, g) is also geodesically complete and simply connected, then (M, g) is globally isometric to one such direct product.

Very roughly, the idea of the proof is the following. As commented before, the starting point is a significant result obtained by Senovilla ([79, Theorem 4.2]): *any simply-connected Lorentzian proper 2nd-symmetric space (M, g) admits a parallel lightlike vector field K.* Lorentzian spaces with such a K were obtained by Brinkmann [15] and will be studied in Chapter 4, where local bases associated to what we call *Brinkmann charts* $\{u, v, x^i\}$ will be introduced. The fact that, for any such chart, the slices with constant u and v happen to be locally symmetric suggests a reduction of the equations for 2nd-symmetry. This reduction is carried out by exploiting the integrability conditions in full, and by applying some technical algebraic properties. Then, Eisenhart-type decompositions can be used to transform the original equations in (M, g) into the 2nd-symmetry equations of two simpler spacetimes from which the stated parts (M_1, g_1) and (M_2, g_2) emerge locally. Some geometric elaborations yield the global result. In fact, along the proof it will become apparent that (M, g) admits a globally defined parallel lightlike direction. Thereupon, the global requirements complete the proof easily.

It is worth pointing out that, in parallel, a different local approach fully based in holonomy groups (including landmarks such as the classification of all the holonomy groups in Lorentzian signature [55]) has been developed by Alekseevsky and Galaev in [2, 35]. By using it, the crucial result in [79, Theorem 4.2] is revisited, and the Theorem 1.1 was provided. So, a substantial set of tools for this exciting field of research is available now. We will point out at the end of this memoir some open problems where they would be useful.

1.1 Outline of this memoir

This thesis is organized as follows. In Chapter 2, some well-known general mathematical background is given, for completeness of this memoir, together with some necessary auxiliary algebraic results and a brief explanation of the abstract index notation. We are specially careful with the latter, as we will use a powerful combination of intrinsic expressions and tensor-component computations. In Chapter 3, we review some results on locally symmetric and symmetric spaces. In particular, the Cahen-Wallach spaces are presented in Section 3.2.4 as the Lorentzian locally symmetric ones. Chapter 4 is devoted to a local study of Brinkmann spaces. It has a technical nature, but it may have interest in its own right. In Section 4.1 we revisit the known procedure to find a *Brinkmann chart* $\{u, v, x^i\}$ associated to a *Brinkmman decomposition* $\{u, v\}$, so that the metric of the manifold is written as:

$$g = -2du(dv + H(u, x^k)du + W_i(u, x^k)dx^i) + g_{ij}(u, x^k)dx^i dx^j, \quad k = 2, \ldots, n-1. \tag{1.3}$$

These yield an associated spacelike $(n-2)$-foliation $\overline{\mathcal{M}}$ characterized by constant values of u and v, and a non-necessarily orthogonal timelike 2-foliation $\mathcal{U}$ generated by ∂_u, ∂_v, as well as other distributions of interest. For the sake of clarity, the relations among tensor fields on these distributions are briefly explained in Section 4.2. In Section 4.3, a thorough study of the foliation $\overline{\mathcal{M}}$ is done. In particular, three operators $\overline{\nabla}$, $\overline{d}$ and $\dot{}$ adapted to the foliations $\overline{\mathcal{M}}$ and $\mathcal{U}$ are introduced and related to the connection ∇, the exterior differential d and the geometry of the foliation $\overline{\mathcal{M}}$. In Section 4.4, computations *à la Cartan* on local frames introduced in Section 2.1.6 together with Subsection 4.3.6 yield manageable expressions of the connection one-forms and the curvature tensor R of a Brinkmann space. To help on simplifying subsequent expressions, a fourth differential operator D_0 with a transverse nature, which complements the three previous ones, is introduced in Section 4.5. Then, a version of a classical Eisenhart theorem [30] adapted to our problem, which involves u-dependent metrics on the leaves of a foliation endowed with the intrinsic $\overline{\nabla}$ and the transverse D_0 derivatives, is provided in Theorem 4.6.2 of Section 4.6. This allows us to obtain sufficient conditions such that the u-family of Riemannian metrics $\overline{g} = g_{ij}(u, x^k)dx^i dx^j$ are simultaneously reducible. In Chapter 5 some general properties of the rth-order symmetric spaces are presented. Then, the generalized Cahen-Wallach family of Lorentzian manifolds are defined as a particular case for rth-symmetric spaces.

Finally in Chapter 6 we give some possible characterizations for semi-Riemannian second-symmetric spaces, in comparison with the known results for locally symmetric semi-Riemannian ones and comment the existence of a global parallel lighlike vector field (Section 6.1) and solve the equations of 2nd-symmetry on Brinkmann spaces in several steps (Section 6.3). Firstly, manageable expressions for the first two covariant derivatives of the curvature tensor R together with the equations of 2nd-symmetry are given in Subsection 6.3.1. The dependence of all mentioned objects on the Brinkmann chart and their behavior under changes of charts is emphasized and explicitly controlled. Secondly, we prove that the equations of 2nd-symmetry imply that the foliation $\overline{\mathcal{M}}$ must be locally symmetric for all Brinkmann charts $\{u, v, x^i\}$ associated to a fixed decomposition $\{u, v\}$ (Proposition 6.3.7), hence severe restrictions on the curvature must hold. Thirdly, using the auxiliary algebraic results on vector spaces given in Subsection 2.1.4, those restrictions are explicitly determined in Theorem 6.3.8. Moreover, a two-covariant tensor field $\tilde{A}$ on $\overline{\mathcal{M}}$, associated to the Brinkmann decomposition $\{u, v\}$ only but not to the other coordinates of the Brinkmann chart, is defined in (b) of Theorem 6.3.8. All the information of the tensor field ∇R is codified in $\tilde{A}$ and the leaves of $\overline{\mathcal{M}}$. These results are summarized in Subsection 6.3.2. Fourthly, a reorganization of the 2nd-symmetry equations into two independent blocks associated to different Brinkmann manifolds $(M^{[m]}, g^{[m]})$ with $m \in \{1, 2\}$ is proven in Subsection 6.3.3 (Theorem 6.3.19 and Proposition 6.3.20). This is achieved by applying our version of Eisenhart theorem showing that the metric is suitably reducible into a Ricci-flat part and a non-Ricci-flat one. Actually, the former is flat as a consequence of a known result [1] and the operator $\tilde{A}$ lives only in this flat part (Proposition 6.3.15). Finally, the proof of the main result is complete in Subsection 6.3.4. The first space $(M^{[1]}, g^{[1]})$ is directly computable leading to the required generalized Cahen-Wallach expression of the part (M_1, g_1) in Theorem 1.1. The 2nd-symmetry equations for the second space $(M^{[2]}, g^{[2]})$ become equivalent to the equations for local symmetry so that $(M^{[2]}, g^{[2]})$ collapses to a locally symmetric Brinkmann space, therefore the Cahen-Wallach classification allows us to determine the Riemannian locally symmetric part (M_2, g_2) in Theorem 1.1. Thus, our main theorem 1.1 is proven. Judicious interpretations alongside some technical results (Corollary 6.2.2, Lemma 5.4.3) will imply that the global counterpart of Theorem 1.1 (see Theorem 6.3.2) can be obtained from the local one.

1.2 Notation and Conventions

Vector spaces (over $\mathbb{R}$) are denoted by $\mathcal{V}, \mathcal{W}, \ldots$, while planes —two-dimensional (sub)-spaces — are denoted by π. The symbols $\vec{v}, \vec{x}, \vec{y}, \ldots$ are used for vectors (elements in $\mathcal{V}$), whereas $\omega, \beta, \tau, \ldots$ are used for covectors (elements in the dual space $\mathcal{V}^*$). A subspace generated by $\vec{v}_1, \ldots, \vec{v}_s$ is normally represented as span$\{\vec{v}_1, \ldots, \vec{v}_s\}$. Then, k-contravariant and s-covariant tensors T (multilinear maps $T : \mathcal{V}^* \times \overset{(k)}{\ldots} \times \mathcal{V}^* \times \mathcal{V} \times \overset{(s)}{\ldots} \times \mathcal{V} \longrightarrow \mathbb{R}$) will be called (k, s)-tensors. The index r is reserved for r-th symmetric spaces (see Chapter 5). A symmetric bilinear map $h : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R}$ is a *scalar product* on $\mathcal{V}$ if it is non-degenerate (i.e., whenever $h(\vec{v}, \vec{w}) = 0$ for all $\vec{w} \in \mathcal{V}$, then $\vec{v} = 0$). In this memoir, a vector space $\mathcal{V}$ endowed with a scalar product h is denoted by $(\mathcal{V}, h)$ and it is called a *semi-Euclidean space*. Then, we say that h has index q, or that its signature is $(-, \overset{(q)}{\ldots}, -, +, \overset{(n-q)}{\ldots}, +)$, where q is the dimension of the largest subspace $\mathcal{W}$ of $(\mathcal{V}, h)$ on which h is negative definite (that is, $h(\vec{w}, \vec{w}) < 0$ for any non-zero vector $\vec{w} \in \mathcal{W}$). When $q = 0$ (resp. $q = 1$), the vector space $(\mathcal{V}, h)$ is an *Euclidean space* (resp. a *Lorentz space*) and h is a positive-definite (resp. indefinite) scalar product denoted in this thesis by $\overline{g}$ (resp. g). We will use the following types of vectors in a *proper semi-Euclidean vector space* (with $0 < q < n$) :

$$\begin{cases} \text{spacelike, if } h(\vec{v}, \vec{v}) > 0, \\ \text{timelike, if } h(\vec{v}, \vec{v}) < 0, \\ \text{lightlike, if } h(\vec{v}, \vec{v}) = 0 \text{ and } \vec{v} \neq \vec{0}, \\ \text{null, if } h(\vec{v}, \vec{v}) = 0 \text{ or } \vec{v} = \vec{0}. \end{cases}$$

Consider a subspace $\mathcal{W}$ of a semi-Euclidean space $(\mathcal{V}, h)$. The restriction of h to $\mathcal{W}$ is a bilinear form on $\mathcal{W}$ which will be also denoted by h, or $h_{|\mathcal{W}}$, and may be degenerate (i.e., there can exist a non-zero vector $\vec{w} \in \mathcal{W}$ such that $h(\vec{w}, \vec{u}) = 0$ for all $\vec{u} \in \mathcal{W}$), in which case $\mathcal{W}$ is a *degenerate subspace* of $(\mathcal{V}, h)$. If it is non-degenerate, then $(\mathcal{W}, h)$ is a *semi-Euclidean subspace* of $(\mathcal{V}, h)$. The *orthogonal subspace* $\mathcal{W}^\perp$ to $\mathcal{W}$ in $(\mathcal{V}, h)$ is defined by

$$\mathcal{W}^\perp = \{\vec{v} \in \mathcal{V} : h(\vec{v}, \vec{w}) = 0, \forall \vec{w} \in \mathcal{W}\}.$$

In general, $\mathcal{W}^\perp$ is not complementary to $\mathcal{W}$ in $\mathcal{V}$, though dim$\mathcal{W}^\perp$ + dim$\mathcal{W} = n$ and $(\mathcal{W}^\perp)^\perp = \mathcal{W}$ (see for example [65]).

$N, N', M, \ldots$ will denote (connected) manifolds, usually n-dimensional if otherwise is not specified. For simplicity, they will be implicitly assumed to be differentiable of class C^l with $l = \infty$, but one only needs $l = r+3$ for r-th symmetric spaces. Accordingly, all objects are assumed to be as differentiable as necessary depending on l. Indices written in greek small letters $\alpha, \beta, \lambda, \ldots$ will run from 0 to $n-1$, while those in latin small letters starting at i $(i, j, k, \ldots)$ will run from 2 to $n-1$, and the usual summation convention is used (except when otherwise is especified):

$$\sum_\alpha x_\alpha x^\alpha \equiv x_\alpha x^\alpha.$$

Open and coordinate neighborhoods are both denoted by U and normal neighborhoods by W. Thus, a coordinate chart is indicated without distintion as $\{(U, \varphi = (x^\alpha))\}$, $\{(U, \varphi)\}$ or simply with its coordinate functions $\{x^\alpha\}$. When working on a Brinkmann space (M, g) the coordinates x^0 and x^1 will also be written as u and v, respectively, according to (4.1). Indeed, most of our computations will be local in such spaces, namely, in some appropriate neighborhood U of any point $p \in M$, but we will not specify the neighborhood —as we have already done with the charts. Therefore, in Brinkmann spaces we will use the notation as if $U = M$ except if there were some possibility of confusion.

If $C^\infty(N)$ denotes the set of all smooth functions $f : N \longrightarrow \mathbb{R}$ on a manifold N, the partial derivatives for any $f \in C^\infty(N)$ are denoted by

$$f_{,\alpha} \equiv \frac{\partial f}{\partial x^\alpha}, \quad \dot{f} \equiv \frac{\partial f}{\partial u} \quad \text{and} \quad f_{,i} \equiv \frac{\partial f}{\partial x^i}. \tag{1.4}$$

Then, if $\Gamma(D)$ denotes the $C^\infty(N)$-module of sections of a subbundle D with projection $\Pi : D \to N$, we use the notation $X \in \Gamma(D)$ to indicate that X lies in D, i.e., $X_p \in D_p$ for all $p \in N$. If the base were not evident, we will use the notation $\Gamma(N, D)$. In the case of (anti-symmetric) s-forms the notation will be simplified so that $\Lambda^s(D)$ denotes the space of all the s-forms. Furthermore, when $s = 1$, the superscript s will be omitted. In the particular cases when $D = TN$, the $C^\infty(N)$-module of vector fields $\Gamma(TN)$ will be indicated as $\mathfrak{X}(N)$ and $\Lambda^s(TN)$ as $\Lambda^s(N)$, while if D is the (involutive) distribution $D(\mathcal{F})$ associated to a foliation $\mathcal{F}$, then $\mathfrak{X}(\mathcal{F}) := \Gamma(T\mathcal{F})$ and $\Lambda^s(\mathcal{F}) := \Lambda^s(T\mathcal{F})$. Notationally, there will be no distinction between (k, s)-tensors in a vector space $\mathcal{V}$ and (k, s)-tensor fields (sections of the (k, s)-tensor bundle) in a manifold N, neither between covectors and one-forms (sections of T^*N the dual space of TN), or between scalar products and metrics. As an exception, vector fields will be denoted by $X, Y, Z, V, \ldots \in \mathfrak{X}(N)$, and $X_p \in T_pN$ for all p in the domain of X.

A *local (moving) frame* is a vector field basis on TN defined in an open neighborhood $U \subset N$. Some local frame $\{V_\alpha\}$ on TN (or on some of its subbundles) plus its dual frame (or coframe) $\{\xi^\alpha\}$ on T^*N are used to write tensor equations in components. Of course, these bases are not necessarily holonomic, i.e., associated to specific coordinates $\{x^\alpha\}$, for which we use the standard notation $\{\partial_\alpha\}, \{dx^\alpha\}$. We will keep $\{E_\alpha\}, \{\theta^\alpha\}$ to denote a *partly null frame* and its coframe (see Definition 2.1.1). Sometimes the abstract index notation is also used (see Section 2.1.3 for further information). Either in components or in abstract index notation, the symmetrization (respectively anti-symmetrization) in some indices of a tensor field T is denoted by round (respectively square) brackets that enclose the mentioned indices to be symmetrized (respectively antisymmetrized).

We write

$$2\omega^\alpha \beta^\beta = \omega^\alpha \otimes \beta^\beta + \omega^\beta \otimes \beta^\alpha$$

for any $\omega, \beta \in \Lambda(N)$. For the *wedge product*, the convention is

$$\beta^1 \wedge \ldots \wedge \beta^m = \sum_{\sigma \in S_m} (-1)^{[\sigma]} \beta^{\sigma(1)} \otimes \ldots \otimes \beta^{\sigma(m)}, \quad \text{with } \beta^l \in \Lambda(N), \quad l = 1, \ldots, m$$

and S_m denotes the set of all permutations of $\{1, \ldots, m\}$. Therefore,

$$\beta = \frac{1}{s!} \beta_{\alpha_1 \ldots \alpha_s} \xi^{\alpha_1} \wedge \ldots \wedge \xi^{\alpha_s}$$

for any $\beta \in \Lambda(N)$, where $\beta_{\alpha_1\ldots\alpha_s} = \beta(V_{\alpha_1}, \ldots, V_{\alpha_s})$. The operator $\mathcal{S}[T]$ gives the symmetric part of any covariant section $T \in \Gamma(T_s^0\overline{\mathcal{M}})$ in its last two slots, C^α_β is the contraction between two indices (alternatively, C^i_j between two indices in $\{2, \ldots, n-1\}$) and $C_{\alpha\beta} = h_{\alpha\rho}C^\rho_\beta$ denotes the contraction of the α^{th} and β^{th} covariant indices (via the metric h). A smooth map $\Phi : N \longrightarrow N'$ between manifolds induces two linear transformations: the *push-forward*, denoted without distinction as $d\Phi_p : T_pN \longrightarrow T_{\Phi(p)}N'$ or $\Phi_{*|p}$, and the *pull-back* $\Phi^*_{|p} : T^*_{\Phi(p)}N' \longrightarrow T_pN$.

In this thesis, a semi-Riemannian manifold is denoted by (N, h), while a Lorentzian manifold is denoted by (M, g) and a Riemannian one by $(\overline{M}, \overline{g})$. For Lorentzian metrics, our convention on the signature is $(-, +, \ldots, +)$. Occasionally, a manifold will be called *pseudo-Riemannian* if its metric may be degenerate (possibly signature-changing). The Levi-Civita connection of a semi-Riemannian manifold (N, h) is represented by the symbol ∇, and when it is necessary to specify the metric to which it is related, it will be done by adding a superscript: ∇^h. A tensor field T is said to be *parallel* (or covariantly constant) if it satisfies

$$\nabla T = 0.$$

We will follow typical notation for covariant derivatives and their components as, for example,

$$\nabla_{\beta_{s+1}} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s} \equiv (\nabla_{V_{\beta_{s+1}}} T)(\xi^{\alpha_1}, \ldots, \xi^{\alpha_k}, V_{\beta_1}, \ldots, V_{\beta_s})$$

—see also [76, pp. 30-35]—. Then, the second order covariant derivative of T is given by the following formula:

$$\nabla^2 T(\cdot; E_1, E_2) = \nabla_{E_2}(\nabla_{E_1} T) - \nabla_{\nabla_{E_2}E_1} T. \tag{1.5}$$

The curvature tensor is defined as

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

and the components of R are denoted by $R^\alpha{}_{\beta\lambda\mu}$, which agrees with [44, 83] but differs from [76] where the same is written as $R_{\lambda\mu\beta}{}^\alpha$ (therefore, our $R_{\alpha\beta\lambda\mu}$ has a different sign with respect to [76]). The sectional curvature of a non-degenerate plane π generated by the vectors $X_p, Y_p \in T_pN$ is then given by the formula

$$K(\pi) = \frac{R(X_p, Y_p, X_p, Y_p)}{Q(X_p, Y_p)} \quad \text{where } Q(X_p, Y_p) = g(X_p, X_p)g(Y_p, Y_p) - (g(X_p, Y_p))^2$$

The metric isomorphism $\flat : TM \longrightarrow T^*M, X \mapsto g(X, \cdot)$ —to lower the index— and its inverse $\sharp : T^*M \longrightarrow TM$ —to raise the index—, are written in components or abstract index notation (see Section 2.1.3) so that

$$X_\alpha := (X^\flat)_\alpha = g_{\alpha\beta}X^\beta \text{ for all } X \in \mathfrak{X}(N),$$

$$\tau^\alpha := (\tau^\sharp)^\alpha = g^{\alpha\beta}\tau_\beta \text{ for all } \tau \in \Lambda(N).$$

Then, the one-form $X^\flat$ is said to be *metrically equivalent* to X, and when applying to a tensor field, we will denote both, the tensor T and its metrically equivalent tensor, with the same symbol T, as it is customary.

Finally, "*Proof.*"indicates the beginning of a proof, and its end is marked with the symbol ■. Remarks, conventions and examples will be emphasized with a left bar.

Chapter 2

General Background

2.1 Some Tensor Analysis

In order to make this thesis self-contained, in this Section we summarize some well-known results on Tensor Analysis.

2.1.1 Curvature preserving Linear Isometries

Let $(\mathcal{V}, h), (\mathcal{V}', h')$ be two semi-Euclidean vector spaces. Then, a *linear isometry* $L : \mathcal{V} \longrightarrow \mathcal{V}'$ is any linear isomorphism between such spaces that preserves scalar products. Therefore, the spaces $\mathcal{V}$, $\mathcal{V}'$ have the same dimension and index.

Let now (N, h),(N', h') be two semi-Riemannian manifolds. An *isometry* is a diffeomorphism $\phi : (N, h) \longrightarrow (N', h')$ that preserves metrics. ϕ is a *local isometry* if there exists a neighborhood U_p for each point p of N such that $\phi_{|U}$ is an isometry of U_p onto $\phi(U_p)$. Thus, it satisfies that $d\phi_p : T_pN \longrightarrow T_{\phi(p)}N'$ is a linear isometry for all p in N:

$$h'_{|\phi(p)}(d\phi_p(\vec{v}), d\phi_p(\vec{w})) = h_{|p}(\vec{v}, \vec{w}), \ \forall p \in N \text{ and } \forall \vec{v}, \vec{w} \in T_pN.$$

We say that a linear isometry $L : T_pN \longrightarrow T_qN'$ (respectively, a diffeomorphism ψ) *preserves curvature* if for any $\vec{x}, \vec{y}, \vec{z} \in T_pN$, it holds that

$$L(R(\vec{x}, \vec{y})\vec{z}) = R(L(\vec{x}), L(\vec{y}))L(\vec{z})$$

(resp., if $d\psi_p$ preserves curvature for all p in N). Then, notice that an isometry between semi-Riemannian manifolds always preserves curvature, but that it is not always true for linear isometries between tangent vector spaces. Obviously, for a linear isometry L at p there always exists a (local) difeomorphism ψ such that $d\psi_{|p} = L$, but ψ is not necessarily an isometry.

2.1.2 Bases and Frames

A basis of $(\mathcal{V}, h)$ formed by n mutually orthogonal unit vectors is called an *orthonormal basis*. In a Lorentz space $(\mathcal{V}, g)$, a *partly null basis* is a basis $\{\vec{l}, \vec{k}, \vec{e}_i\}$ formed by two (linearly independent) lightlike vectors $\vec{l}, \vec{k}$ with $g(\vec{l}, \vec{k}) = -1$, and $n-2$ spacelike vectors $\vec{e}_2, \ldots, \vec{e}_{n-1}$ which are orthogonal to $\vec{l}, \vec{k}$. Hence, $g_{ij} = g(\vec{e}_i, \vec{e}_j) \in \mathbb{R}$ such that g a is positive-definite scalar product in $\mathcal{W} = \text{span}\{\vec{e}_2, \ldots, \vec{e}_{n-1}\}$. A particular case of this type of basis is when the spacelike vectors $\vec{e}_2, \ldots, \vec{e}_{n-1}$ are mutually orthogonal unit vectors.

In a semi-Riemannian manifold (N, h), a local frame which is an orthonormal basis at each p of its domain of definition is called an *orthonormal frame*. Analogously, a *partly null frame* on a Lorentzian manifold (M, g) is a local frame defined on an open set U which is a partly null basis at each $p \in U$.

More precisely,

Definition 2.1.1. *By a* partly null frame $\{E_\alpha\}$ *for a Lorentzian manifold we mean a local basis of vector fields such that*

$$\begin{array}{lll} g(E_0, E_0) = 0; & g(E_1, E_1) = 0; & g(E_0, E_1) = -1; \\ g(E_0, E_i) = 0; & g(E_1, E_i) = 0; & g(E_i, E_j) = a_{ij}, \end{array}$$

for some functions $a_{ij} = a_{ji}$, which must obviously define a positive-definite metric. Its dual frame will be denoted by $\{\theta^\alpha\}$.

2.1.3 Abstract Index Notation

This section is a modest attempt to introduce abstract index notation in the shortest possible way. Much more complete introductions can be found in [89, 44]. The abstract index notation is mostly used by physicists and in Relativity texts, as well as in many classical differential geometry texts. It consists on writing tensors and tensor-equations using the expression they would have after introducing a (general) basis. Therefore, a vector $\vec{v}$ would be denoted as v^α, a one-form ω as ω_α and a (k, s)-tensor as $T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_s}$. This notation is naturally extended to frames and sections of tensor bundles on manifolds. Then, the tensor-equations written in this manner hold for any possible frame (and therefore, it is frame-independent). But when working on a particular type of frame (for example, adapted to the symmetries of a particular semi-Riemannian manifold), the obtained equations that one writes down may be valid only for the chosen frame. Consequently, equations written in abstract index notation hold between tensors (and therefore for any given frame), whereas equations in a special basis may hold only in such basis. For example, to "raise the index" via the metric h of a one-form ω_α is simply denoted in abstract index notation as ω^α, but if we choose to "raise the index" using an orthonormal frame in a Lorentzian manifold, then we have especifically that $\omega^0 = -\omega_0$, while the other components remain equal.

We list some further examples for a better understanding of this notation. Let $X, Y, Z \in \mathfrak{X}(N)$, $\omega \in \Lambda(N)$, $T \in \Gamma(T^0_2 N)$ and $S \in \Gamma(T^3_0 N)$. Then:

Intrinsic form	Abstract index form
The metric h and its inverse h^{-1}	$h_{\alpha\beta}$ and $h^{\alpha\beta}$
$h(X, \cdot) \in \Lambda(N)$	$X_\alpha := h_{\alpha\beta} X^\beta$ (lowering the index)
$h^{-1}(\omega, \cdot) \in \mathfrak{X}(N)$	$\omega^\alpha := h^{\alpha\beta}\omega_\beta$ (raising the index)
$h(T, T) \in C^\infty(N)$	$T_{\alpha\beta}T^{\alpha\beta} = h^{\alpha\rho}h^{\beta\sigma}T_{\alpha\beta}T_{\rho\sigma} = h_{\alpha\rho}h_{\beta\sigma}T^{\alpha\beta}T^{\rho\sigma}$
$h(S, S) \in C^\infty(N)$	$S_{\alpha\beta\lambda}S^{\alpha\beta\lambda} = h_{\alpha\rho}h_{\beta\sigma}h_{\lambda\mu}S^{\alpha\beta\lambda}S^{\rho\sigma\mu}$
Antisymmetrization of T	$T_{[\alpha\beta]} := \frac{1}{2}(T_{ab} - T_{ba})$
Symmetrization of S in its first and third slots	$S_{(\alpha\|\beta\|\lambda)} := \frac{1}{2}(S_{\alpha\beta\lambda} + S_{\lambda\beta\alpha})$
T anti-symmetric	$T_{[\alpha\beta]} = T_{\alpha\beta}$, $T_{\alpha\beta} = -T_{\beta\alpha}$ or $T_{(\alpha\beta)} = 0$
S symmetric in its first and third slots	$S_{(\alpha\|\beta\|\lambda)} = S_{\alpha\beta\lambda}$, $S_{\alpha\beta\lambda} = S_{\lambda\beta\alpha}$ or $S_{[\alpha\|\beta\|\lambda]} = 0$
Metrically equivalent $(1, 2)$-tensors to S	Possibilities: $S^\alpha{}_{\beta\lambda} = h^{\alpha\rho}S_{\rho\beta\lambda}$, $S_\alpha{}^\beta{}_\lambda = h^{\beta\rho}S_{\alpha\rho\lambda}$ or $S_{\alpha\beta}{}^\lambda = h^{\lambda\rho}S_{\alpha\beta\rho}$
$T \otimes S \in \Gamma(T^3_2 N)$	$T_{\alpha\beta}S^{\lambda\mu\nu}$
$C^2_1(T \otimes S) \in \Gamma(T^2_1 N)$	$T_{\alpha\beta}S^{\lambda\alpha\mu}$
$C_{13}(T \otimes T) \in \Gamma(T^0_2 N)$	$T_{\alpha\beta}T^\alpha{}_\lambda = h^{\alpha\rho}T_{\alpha\beta}T_{\rho\lambda}$
$\nabla T \in \Gamma(T^0_3 N)$	$\nabla_\alpha T_{\beta\lambda}$ (abuse of notation for $(\nabla T)_{\beta\lambda\alpha}$)
$\nabla_X T \in \Gamma(T^0_2 N)$	$X^\alpha \nabla_\alpha T_{\beta\lambda} = \nabla_X T_{\beta\lambda}$
$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z (= \nabla^2 Z(\cdot, Y, X) - \nabla^2 Z(\cdot, X, Y))$	$2X^\alpha Y^\beta \nabla_{[\alpha}\nabla_{\beta]}Z^\lambda = X^\alpha Y^\beta R^\lambda{}_{\mu\alpha\beta}Z^\mu$
$R(X, Y)R = 0$ (Semi-symmetry condition)	$\nabla_{[\alpha}\nabla_{\beta]}R^\lambda{}_{\mu\nu\rho} = 0$
First Bianchi identity	$R_{\alpha[\beta\lambda\mu]} = 0$ or $R_{\alpha\beta\lambda\mu} + R_{\alpha\lambda\mu\beta} + R_{\alpha\mu\beta\lambda} = 0$
Second Bianchi identity	$\nabla_{[\alpha}R_{\beta\lambda]\mu\rho} = 0$ or $\nabla_\alpha R_{\beta\lambda\mu\rho} + \nabla_\beta R_{\lambda\alpha\mu\rho} + \nabla_\lambda R_{\alpha\beta\mu\rho} = 0$

In addition, since $2X^\alpha Y^\beta \nabla_{[\alpha}\nabla_{\beta]}Z^\lambda = X^\alpha Y^\beta R^\lambda{}_{\mu\alpha\beta}Z^\mu$ holds for any X, Y, Z one can write $2\nabla_{[\alpha}\nabla_{\beta]}Z^\lambda = R^\lambda{}_{\mu\alpha\beta}Z^\mu$ for all Z, which is called *the Ricci identity*. This identity can be extended to any tensor field T as

$$2\nabla_{[\lambda}\nabla_{\mu]}T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = \sum_{i=1}^{r} R^{\alpha_i}{}_{\rho\lambda\mu}T^{\alpha_1\ldots\alpha_{i-1}\rho\alpha_{i+1}\ldots\alpha_r}_{\beta_1\ldots\beta_s} - \sum_{i=1}^{s} R^\rho{}_{\beta_i\lambda\mu}T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_{i-1}\rho\beta_{i+1}\ldots\beta_s}. \tag{2.1}$$

When working using both notations, one realizes that in some cases abstract index notation arises to be a much more powerful tool, while other times computations in intrinsic notation are easier to handle. An ilustrative example of the usefulness of abstract index notation is the generalized Kronecker delta $\delta^{\alpha_1\ldots\alpha_s}_{\beta_1\ldots\beta_s}$ which is defined in the following equivalent manners:

(i) $\delta^{\alpha_1\ldots\alpha_s}_{\beta_1\ldots\beta_s} = \begin{cases} 1, & \text{if } \{\alpha_1\ldots\alpha_s\} \text{ is an even permutation of } \{\beta_1\ldots\beta_s\} \\ -1, & \text{if } \{\alpha_1\ldots\alpha_s\} \text{ is an odd permutation of } \{\beta_1\ldots\beta_s\} \\ 0, & \text{otherwise.} \end{cases}$

(ii) $\delta^{\alpha_1\ldots\alpha_s}_{\beta_1\ldots\beta_s} = \sum_\sigma (-1)^{[\sigma]} \delta^{\alpha_1}_{\beta_{\sigma(1)}} \cdot \ldots \cdot \delta^{\alpha_s}_{\beta_{\sigma(s)}}$, where σ is a permutation of $\{1,\ldots,s\}$.

(iii) $\delta^{\alpha_1\ldots\alpha_s}_{\beta_1\ldots\beta_s} = \det \begin{pmatrix} \delta^{\alpha_1}_{\beta_1} & \ldots & \delta^{\alpha_1}_{\beta_s} \\ \vdots & \ddots & \vdots \\ \delta^{\alpha_s}_{\beta_1} & \ldots & \delta^{\alpha_s}_{\beta_s} \end{pmatrix}$, where δ^α_β is the usual Kronecker delta.

For example, one of the properties of this (s,s)-tensor field is that

$$T_{[\beta_1\ldots\beta_s]} = \frac{1}{s!} \delta^{\alpha_1\ldots\alpha_s}_{\beta_1\ldots\beta_s} T_{\alpha_1\ldots\alpha_s}.$$

So it corresponds with the identity endomorphism on the space of s-forms $\Lambda^s(N)$, up to a constant.

2.1.4 Auxiliary Algebraic Results

Tensors in this section will be denoted in abstract index form. For proofs in this section we use arguments inspired in [79, Lemma 4.1].

Proposition 2.1.2. *Let $(\mathcal{V}, \overline{g})$ be an n-dimensional Euclidean vector space and $T_{\alpha\beta\lambda}$ a $(0,3)$-tensor such that:*

(a) it is anti-symmetric in the last two indices: $T_{\alpha[\beta\lambda]} = T_{\alpha\beta\lambda}$,

(b) it satisfies a cyclic identity: $T_{\alpha\beta\lambda} + T_{\beta\lambda\alpha} + T_{\lambda\alpha\beta} = 0$.

If

$$T_{(\alpha\beta)}{}^{\rho} T_{\rho\lambda\mu} = 0 \tag{2.2}$$

holds, then $T_{\alpha\beta\lambda} = 0$.

Proof. Observe that, on using (a), (b) can be rewritten as $T_{[\alpha\beta\lambda]} = 0$. Suppose first that there does not exist a non-vanishing vector $\vec{v} \in \mathcal{V}$ such that $v^\rho T_{\rho\lambda\mu} = 0$. Define the set of vectors $\{Q^\rho(\alpha,\beta)\}_{\alpha,\beta=0}^{n-1}$ by $Q^\rho(\alpha,\beta) = T_{(\alpha\beta)}{}^{\rho}$ for each pair (α,β). Then, equation (2.2) can be rewritten as $Q^\rho(\alpha,\beta) T_{\rho\lambda\mu} = 0$, that is, $Q^\rho(\alpha,\beta) = 0$ for all α,β by our assumption. Consequently, $T_{(\alpha\beta)\lambda} = 0$, i.e., $T_{\alpha\beta\lambda} = T_{[\alpha\beta]\lambda}$. Using this fact and (a), $T_{\alpha\beta\lambda}$ is totally anti-symmetric (i.e., $T_{\alpha\beta\lambda} = T_{[\alpha\beta\lambda]}$), and so it vanishes by (b).

Suppose then that there does exist a non-vanishing vector $\vec{v} \in \mathcal{V}$ such that $v^\rho T_{\rho\lambda\mu} = 0$. Without loss of generality, we can assume that $||\vec{v}|| = 1$. Define $b_{\alpha\beta} = T_{\alpha\beta\rho}v^\rho$. Observe that $b_{\alpha\beta}v^\alpha = 0$ and by (a) and (b), it follows that $b_{\alpha\beta}v^\beta = 0$. Therefore, the orthogonal splitting of $T_{\alpha\beta\lambda}$ with respect to $\vec{v}$ must be:

$$T_{\alpha\beta\lambda} = a_{\alpha\beta\lambda} + b_{\alpha\beta}v_\lambda - b_{\alpha\lambda}v_\beta, \tag{2.3}$$

where

$$a_{\alpha\beta\lambda}v^\alpha = a_{\alpha\beta\lambda}v^\beta = a_{\alpha\beta\lambda}v^\lambda = 0, \quad b_{\alpha\beta} = T_{\alpha\beta\lambda}v^\lambda; \;\; b_{\alpha\beta}v^\alpha(= T_{\alpha\beta\lambda}v^\lambda v^\alpha) = 0, \quad b_{\alpha\beta}v^\beta = 0,$$

and by (a) and (b) one can easily deduce that

$$a_{[\alpha\beta\lambda]} = 0, \;\; a_{\alpha[\beta\lambda]} = a_{\alpha\beta\lambda}.$$

Since (b) implies $v^\alpha(T_{\alpha\beta\lambda} + T_{\beta\lambda\alpha} + T_{\lambda\alpha\beta}) = 0$, we obtain by (2.3) that $b_{[\beta\lambda]} = 0$, i.e., $b_{\alpha\beta}$ is a symmetric two-covariant tensor. But (2.2) implies $v^\alpha v^\mu T_{(\alpha\beta)}{}^\rho T_{\rho\lambda\mu} = 0$, which in turn implies $b_\beta{}^\rho b_{\rho\nu} = 0$ on using (2.3) once more. Contracting the indices β, ν and using that $b_{\alpha\beta}$ is symmetric, we have that $b_{\alpha\beta}b^{\alpha\beta} = 0$, and as the inner product is positive-definite, that $b_{\alpha\beta} = 0$. Hence, $T_{\alpha\beta\lambda} = a_{\alpha\beta\lambda}$ so that $T_{\alpha\beta\lambda}$ must be actually totally orthogonal to $\vec{v}$ (that is, $\vec{v}$ belongs to the kernel of $T_{\alpha\beta\lambda}$). As the tensor $a_{\alpha\beta\lambda}$ satisfies the symmetries (a) and (b) of $T_{\alpha\beta\lambda}$ but in the $(n-1)$-dimensional space $(\text{span}\{\vec{v}\})^\perp$, the result follows by induction. ■

An immediate consequence, to be used later, is

Corollary 2.1.3. *Let $\mathcal{V}$ be an n-dimensional Euclidean vector space and $T_{\alpha\beta\lambda\mu}$ a $(0,4)$-tensor such that:*

(a) it is anti-symmetric in the last two indices: $T_{\mu\alpha[\beta\lambda]} = T_{\mu\alpha\beta\lambda}$,

(b) it satisfies the cyclic identity: $T_{\mu\alpha\beta\lambda} + T_{\mu\beta\lambda\alpha} + T_{\mu\lambda\alpha\beta} = 0$.

If $T_{\mu(\alpha\beta)}{}^\rho T_{\nu\rho\lambda\mu} = 0$ holds, then $T_{\mu\alpha\beta\lambda} = 0$.

Proof. Let us define for any $\vec{v} \in \mathcal{V}$ the tensor $v^\mu T_{\mu\alpha\beta\lambda}$. This tensor satisfies all the conditions in Proposition 2.1.2, so $v^\mu T_{\mu\alpha\beta\lambda} = 0$ for all $\vec{v} \in \mathcal{V}$, that is, $T_{\mu\alpha\beta\lambda} = 0$. ■

Similar results in Lorentz vector spaces are presented below.

Proposition 2.1.4. *Let $(\mathcal{V}, g)$ be an n-dimensional Lorentz vector space and $T_{\alpha\beta\lambda}$ a $(0,3)$-tensor such that:*

(a) it is anti-symmetric in the last two indices: $T_{\alpha[\beta\lambda]} = T_{\alpha\beta\lambda}$,

(b) it satisfies a cyclic identity: $T_{\alpha\beta\lambda} + T_{\beta\lambda\alpha} + T_{\lambda\alpha\beta} = 0$.

If

$$T_{(\alpha\beta)}{}^{\rho}T_{\rho\lambda\mu} = 0$$

holds, then there exists a lightlike vector $\vec{l}$ *such that* $l^{\alpha}T_{\alpha\beta\lambda} = 0$ *and* $l^{\beta}T_{\alpha\beta\lambda} = 0$.

Proof. Following the same argument as in Proposition 2.1.2 (see formula (2.3)), one has that the splitting of $T_{\alpha\beta\lambda}$ with respect to a fixed non-vanishing vector $\vec{v}$ such that $v^{\rho}T_{\rho\lambda\mu} = 0$ (which must exist) gives:

$$T_{\alpha\beta\lambda} = a_{\alpha\beta\lambda} + b_{\alpha\beta}v'_{\lambda} - b_{\alpha\lambda}v'_{\beta},$$

where if $\vec{v}$ is lightlike $\vec{v}\,' = -\vec{k}$ with $\vec{k}$ a lightlike vector such that $g(\vec{v},\vec{k}) = -1$, otherwise assume that $\vec{v}$ is unit with $g(\vec{v},\vec{v}) = \pm 1$ and so $\vec{v}\,' = \pm\vec{v}$; in both cases a and b are determined by:

$$a_{\alpha\beta\lambda}v^{\alpha} = a_{\alpha\beta\lambda}v^{\beta} = a_{\alpha\beta\lambda}v^{\lambda} = 0, \quad b_{\alpha\beta} = T_{\alpha\beta\lambda}v^{\lambda};\; b_{\alpha\beta}v^{\alpha} = 0, \quad b_{\alpha\beta}v^{\beta} = 0,$$

and

$$a_{[\alpha\beta\lambda]} = 0,\; a_{\alpha[\beta\lambda]} = a_{\alpha\beta\lambda}, \tag{2.4}$$

$$b_{[\alpha\beta]} = 0,\; b_{\beta}{}^{\rho}b_{\rho\lambda} = 0, \tag{2.5}$$

Define for arbitrary vectors $\vec{u},\vec{u}\,'$ the vectors $v^{\alpha} = b_{\beta}{}^{\alpha}u^{\beta}, v'^{\alpha} = b_{\beta}{}^{\alpha}u'^{\beta}$ (that is, $\vec{v}^{\flat} = b(\vec{u},\cdot)$ and $\vec{v}\,'^{\flat} = b(\vec{u}\,',\cdot)$). Since $b_{\alpha\beta}$ is symmetric, it follows by second in (2.5) that $\vec{v},\vec{v}\,'$ are mutually orthogonal lightlike vectors, so for any $\vec{u}$ the vector $\vec{v}$ must be always proportional to the same lightlike vector $\vec{l}$. Hence, since $b_{\alpha\beta}$ is symmetric, it follows that

$$b_{\alpha\beta} = \Lambda l_{\alpha}l_{\beta}. \tag{2.6}$$

Thus, $g(\vec{l},\vec{v}) = 0$ or $\Lambda = 0$. Let $\vec{k}\,'$ be a lightlike vector such that $g(\vec{l},\vec{k}\,') = -1$. If $\vec{v}$ is not lightlike, from $v^{\beta}T_{(\alpha\beta)}{}^{\rho}T_{\rho\lambda\mu} = 0$ and $T_{(\alpha\beta)}{}^{\rho}T_{\rho\lambda\mu}k'^{\lambda}v^{\mu} = 0$ one obtains, respectively,

$$\Lambda l^{\rho}a_{\rho\lambda\mu} = 0, \qquad \Lambda a_{(\alpha\beta)}{}^{\rho}l_{\rho} = 0,$$

which together with conditions (2.4) imply that

$$\Lambda a_{\alpha\beta\lambda}l^{\beta} = 0.$$

Therefore, either $\Lambda = 0$ and thus $b_{\alpha\beta} = 0$, or $a_{\alpha\beta\lambda}$ is totally orthogonal to $\vec{l}$ and thus $l^{\rho}T_{\rho\lambda\mu} = 0$. In the former case, $T_{\alpha\beta\lambda} = a_{\alpha\beta\lambda}$ so that $T_{\alpha\beta\lambda}$ must be actually totally orthogonal to $\vec{v}$ and the tensor $a_{\alpha\beta\lambda}$ satisfies the symmetries (a) and (b) of $T_{\alpha\beta\lambda}$ but in the $(n-1)$-dimensional space $(\text{span}\{\vec{v}\})^{\perp}$.

All in all, it only remains to assume that $\vec{v}$ is lightlike. In this case $\vec{v} = \vec{l}$ and $\vec{k}\,' = \vec{k}$. Therefore

$$T_{\alpha\beta\lambda} = a_{\alpha\beta\lambda} + \Lambda l_{\alpha}l_{\beta}k_{\lambda} - \Lambda l_{\alpha}l_{\lambda}k_{\beta},$$

and $T_{(\alpha\beta)}{}^{\rho}T_{\rho\lambda\mu}l^{\lambda}k^{\mu} = 0$ yields $\Lambda = 0$. Consequently, $T_{\alpha\beta\lambda} = a_{\alpha\beta\lambda}$. ■

With slight differences in the proof, one can show the following:

Proposition 2.1.5. *Let $\mathcal{V}$ be an n-dimensional Lorentz vector space and $T_{\alpha\beta\lambda\mu\nu}$ a $(0,5)$-tensor such that:*

(a) it is anti-symmetric in the second and the third indices: $T_{\alpha[\beta\lambda]\mu\nu} = T_{\alpha\beta\lambda\mu\nu}$,

(b) it satisfies the cyclic identity: $T_{\alpha\beta\lambda\mu\nu} + T_{\beta\lambda\alpha\mu\nu} + T_{\lambda\alpha\beta\mu\nu} = 0$.

If $T_{(\alpha\beta)}{}^{\rho}{}_{\mu\nu} T_{\rho\lambda\gamma\tau\delta} = 0$ holds, then there exists a lightlike vector $\vec{l}$ such that $l^{\alpha} T_{\alpha\beta\lambda\mu\nu} = 0$ and $l^{\beta} T_{\alpha\beta\lambda\mu\nu} = 0$.

Proof. Observe that, as in the proof of the previous propositions, there exists a non-zero vector $\vec{v}$ such that $v^{\rho} T_{\rho\beta\lambda\mu\nu} = 0$, otherwise $T_{(\alpha\beta)\rho\mu\nu} = 0$ which together with (b) implies that $T_{\alpha\beta\rho\mu\nu} = 0$.

Let us define for any arbitrary two-form $F_{\alpha\beta}$ the tensor

$$\overset{F}{T}_{\alpha\beta\lambda} = T_{\alpha\beta\lambda\mu\nu} F^{\mu\nu}.$$

As this tensor satisfies all the conditions in Proposition 2.1.4, following this time the arguments of the proof of Proposition 2.1.4 one has that the splitting of $\overset{F}{T}_{\alpha\beta\lambda}$ with respect to such vector $\vec{v}$ gives:

$$\overset{F}{T}_{\alpha\beta\lambda} = \overset{F}{a}_{\alpha\beta\lambda} + \overset{F}{b}_{\alpha\beta}\, v'_{\lambda} - \overset{F}{b}_{\alpha\lambda}\, v'_{\beta}, \tag{2.7}$$

with same properties as in the proof of Proposition 2.1.4. Particularly, for each two-form F there exists a lightlike vector $\vec{l}^{(F)}$ such that

$$b_{\alpha\beta} = \Lambda(F) l^{(F)}_{\alpha} l^{(F)}_{\beta}.$$

Furthermore, since the condition $T_{(\alpha\beta)}{}^{\rho}{}_{\mu\nu} T_{\rho\lambda\gamma\tau\delta} = 0$ can be reexpressed for any two-forms F and G as

$$\overset{F}{T}_{(\alpha\beta)}{}^{\rho}\, \overset{G}{T}_{\rho\lambda\gamma} = 0,$$

and this last condition also implies that

$$\overset{F}{b}_{\beta}{}^{\rho}\, \overset{G}{b}_{\rho\lambda} = 0, \qquad \text{for arbitrary two-forms } F, G,$$

it follows that the lightlike vectors $\vec{l}^{(F)}$ and $\vec{l}^{(G)}$ are mutually orthogonal, and thus, proportional. Hence,

$$\overset{F}{b}_{\alpha\beta} = \Lambda(F) l_{\alpha} l_{\beta}$$

for a fixed lightlike vector field $\vec{l}$ independent of F.

Therefore, $g(\vec{l},\vec{v})=0$ or $\Lambda(F)=0$. Let $\vec{k}\,'$ be a lightlike vector such that $g(\vec{l},\vec{k}\,')=-1$. If $\vec{v}$ is not lightlike, from $v^\beta \overset{F}{T}_{(\alpha\beta)}{}^{\rho}\, \overset{G}{T}_{\rho\lambda\mu}=0$ and $\overset{F}{T}_{(\alpha\beta)}{}^{\rho}\, \overset{G}{T}_{\rho\lambda\mu}\, k'^{\lambda}v^{\mu}=0$ for any $F,G\in\Lambda^2(N)$ one obtains, respectively,

$$\Lambda(F)l^\rho \overset{G}{a}_{\rho\lambda\mu}=0, \qquad \Lambda(F)\overset{G}{a}_{(\alpha\beta)}{}^{\rho}l_\rho=0,$$

which together with conditions (2.4) applied to $\overset{G}{a}_{\alpha\beta\lambda}$ imply that

$$\Lambda(F)\overset{G}{a}_{\alpha\beta\lambda}\, l^\beta=0.$$

Consequently, either $\Lambda(F)=0$ for all $F\in\Lambda^2(N)$, or all the tensors $\overset{G}{a}_{\alpha\beta\lambda}$ are totally orthogonal to $\vec{l}$ for all $G\in\Lambda^2(N)$ and thus $l^\rho T_{\rho\beta\lambda\mu\nu}=0$. In the former case, $\overset{F}{T}_{\alpha\beta\lambda}=\overset{F}{a}_{\alpha\beta\lambda}$ so that $\overset{F}{T}_{\alpha\beta\lambda}$ must be actually totally orthogonal to $\vec{v}$ for all $F\in\Lambda^2(N)$ and the tensors $\overset{F}{a}_{\alpha\beta\lambda}$ restricted to the $(n-1)$-dimensional space $(\mathrm{span}\{\vec{v}\})^\perp$ satisfy the symmetries (a) and (b) of $\overset{F}{T}_{\alpha\beta\lambda}$.

All in all, it only remains to assume that $\vec{v}$ is lightlike. In this case $\vec{v}=\vec{l}$ and $\vec{k}\,'=\vec{k}$. Therefore for an arbitrary two-form F,

$$\overset{F}{T}_{\alpha\beta\lambda}=\overset{F}{a}_{\alpha\beta\lambda}+\Lambda(F)l_\alpha l_\beta k_\lambda-\Lambda(F)l_\alpha l_\lambda k_\beta,$$

where $\overset{F}{a}_{\alpha\beta\lambda}$ is totally orthogonal to $\vec{l}$. Besides, $\overset{F}{T}_{(\alpha\beta)}{}^{\rho}\,\overset{F}{T}_{\rho\lambda\mu}\, l^\lambda k^\mu=0$ yields $\Lambda(F)=0$. Consequently, $\overset{F}{T}_{\alpha\beta\lambda}=\overset{F}{a}_{\alpha\beta\lambda}$. ■

2.1.5 The Lie Derivative of a Connection

Let N be a smooth manifold and $\nabla,\widetilde{\nabla}$ two affine connections. Then, though $\nabla,\widetilde{\nabla}$ are not tensor fields, their difference $\widetilde{\nabla}-\nabla$ is a $(1,2)$-tensor field. In fact, if given a coordinate chart $\{x^\alpha\}$, the Christoffel symbols associated to $\nabla,\widetilde{\nabla}$ are, respectively[1],

$$\Gamma^\alpha_{\beta\lambda}(=(\nabla_{\partial_\lambda}\partial_\beta)dx^\alpha), \qquad \widetilde{\Gamma}^\alpha_{\beta\lambda}(=(\widetilde{\nabla}_{\partial_\lambda}\partial_\beta)dx^\alpha),$$

and the components of the tensor field $\widetilde{\nabla}-\nabla$ in the coordinate basis associated to the chart x^α are:

$$(\widetilde{\nabla}-\nabla)^\alpha_{\beta\lambda}=\widetilde{\Gamma}^\alpha_{\beta\lambda}-\Gamma^\alpha_{\beta\lambda}.$$

Let now (N,h) be a semi-Riemannian manifold with Levi-Civita connection ∇ and Z a vector field with local flow $\{\varphi_t\}$. It is possible to define a new connection $\widetilde{\nabla}=(\varphi_t)^*\nabla$ on M associated to Z in the following way:

$$(((\varphi_t)^*\nabla)_X Y)(\omega)_{|p} := \left[(\nabla_{X^*}Y^*)(\omega^*)\right]_p, \forall X,Y\in\mathfrak{X}(N), \forall\omega\in\Lambda(N) \text{ and } \forall p\in N, \quad (2.8)$$

[1]Observe that the second covariant index λ in $\Gamma^\alpha_{\beta\lambda}$ corresponds to the derivation direction while in some mathematical books as for example [65, 26], this index is usually the first covariant index β.

where

$$\begin{aligned} X^*_p &= (d\varphi_{-t}X)_{|p} = d\varphi_{-t|\varphi_t(p)}X_{\varphi_t(p)}, \\ Y^*_p &= (d\varphi_{-t}Y)_{|p} = d\varphi_{-t|\varphi_t(p)}Y_{\varphi_t(p)}, \\ \omega^*_p &= (\varphi^*_t\omega)_{|p} = \varphi^*_{t|\varphi_t(p)}\omega_{\varphi_t(p)}. \end{aligned}$$

The Christoffel symbols of this new connection in a given coordinate chart $\{x^\alpha\}$ are, then:

$$\widetilde{\Gamma}^\alpha_{\beta\lambda} = (\partial_\lambda\varphi^\sigma_{-t})(\partial_\rho\varphi^\alpha_t)(\partial_\sigma\partial_\beta\varphi^\rho_{-t}) + (\partial_\lambda\varphi^\sigma_{-t})(\partial_\beta\varphi^\rho_{-t})(\partial_\mu\varphi^\alpha_t)\Gamma^\mu_{\sigma\rho},$$

where $\partial_\alpha = \frac{\partial}{\partial x^\alpha}$ and $\varphi_t = (\varphi^\alpha_t)$. Then, it is possible to define the Lie derivative $\mathcal{L}_Z\nabla$ of ∇ along Z using the usual infinitesimal definition (see, for example, [9]), which will be a $(1,2)$-tensor field:

Definition 2.1.6. *Let (N,h) be a semi-Riemannian manifold. The Lie derivative of its Levi-Civita connection ∇ along a vector field Z is defined as:*

$$\mathcal{L}_Z\nabla = \lim_{t\to 0}\frac{(\varphi_t)^*\nabla - \nabla}{t}, \text{ with } (\varphi_t)^*\nabla \text{ as in (2.8).}$$

The components of $\mathcal{L}_Z\nabla$ in a given coordinate chart $\{x^\alpha\}$ are:

$$\begin{aligned} (\mathcal{L}_Z\nabla)^\alpha_{\beta\lambda} &= \lim_{t\to 0}\frac{((\varphi_t)^*\nabla - \nabla)^\alpha_{\beta\lambda}}{t} = \lim_{t\to 0}\frac{\widetilde{\Gamma}^\alpha_{\beta\lambda} - \Gamma^\alpha_{\beta\lambda}}{t} \\ &= \lim_{t\to 0}\frac{1}{t}(\partial_\lambda\varphi^\sigma_{-t})(\partial_\rho\varphi^\alpha_t)(\partial_\sigma\partial_\beta\varphi^\rho_{-t}(\varphi_t(x))) \\ &\quad + \lim_{t\to 0}\frac{1}{t}((\partial_\lambda\varphi^\sigma_{-t})(\partial_\beta\varphi^\rho_{-t})(\partial_\mu\varphi^\alpha_t)\Gamma^\mu_{\sigma\rho}(\varphi_t(x)) - \Gamma^\alpha_{\beta\lambda}(x)) \end{aligned}$$

Using that $(\partial_\beta\varphi^\alpha_{\pm t})_{|t=0} = \delta^\alpha_\beta$, one gets

$$\begin{aligned} (\mathcal{L}_Z\nabla)^\alpha_{\beta\lambda} &= \frac{d}{dt}\left[(\partial_\lambda\varphi^\sigma_{-t})(\partial_\rho\varphi^\alpha_t)(\partial_\sigma\partial_\beta\varphi^\rho_{-t}(\varphi_t(x)))\right]_{|t=0} \\ &\quad + \frac{d}{dt}\left[(\partial_\lambda\varphi^\sigma_{-t})(\partial_\beta\varphi^\rho_{-t})(\partial_\mu\varphi^\alpha_t)\Gamma^\mu_{\sigma\rho}(\varphi_t(x))\right]_{|t=0} \end{aligned}$$

Now, using

$$\frac{d}{dt}\left(\partial_\beta\varphi^\alpha_{\pm t}\right)_{|t=0} = \partial_\beta\left(\frac{d\varphi^\alpha_{\pm t}}{dt}_{|t=0}\right) = \partial_\beta Z^\alpha,$$

$$\frac{d}{dt}\left(\Gamma^\alpha_{\beta\lambda}(\varphi_t(x))\right)_{|t=0} = \partial_\rho\Gamma^\alpha_{\beta\lambda}(x)\frac{d\varphi^\rho_t}{dt}_{|t=0} = Z^\rho\partial_\rho\Gamma^\alpha_{\beta\lambda}(x),$$

one easily obtains that

$$(\mathcal{L}_Z\nabla)^\alpha_{\beta\lambda} = \partial_\lambda\partial_\beta Z^\alpha + Z^\sigma\partial_\sigma\Gamma^\alpha_{\lambda\beta} + \Gamma^\alpha_{\sigma\beta}\partial_\lambda Z^\sigma + \Gamma^\alpha_{\lambda\sigma}\partial_\beta Z^\sigma - \Gamma^\sigma_{\lambda\beta}\partial_\sigma Z^\alpha.$$

Using the expressions $\partial_\alpha Z^\beta = \nabla_\alpha Z^\beta - \Gamma^\beta_{\sigma\alpha} Z^\sigma$ and $R^\alpha{}_{\beta\lambda\mu} = \partial_\lambda \Gamma^\alpha_{\mu\beta} - \partial_\mu \Gamma^\alpha_{\lambda\beta} + \Gamma^\alpha_{\lambda\rho}\Gamma^\rho_{\mu\beta} - \Gamma^\alpha_{\mu\rho}\Gamma^\rho_{\lambda\beta}$, one gets

$$(\mathcal{L}_Z \nabla)^\alpha_{\beta\lambda} = \nabla_\lambda \nabla_\beta Z^\alpha + R^\alpha{}_{\beta\sigma\lambda} Z^\sigma,$$

i.e.,

$$\mathcal{L}_Z \nabla = \nabla^2 Z + R(Z, \cdot). \tag{2.9}$$

It follows that

Proposition 2.1.7. *Let (N, h) be a semi-Riemannian manifolds. Then:*

$$\mathcal{L}_Z \nabla(\cdot, Y, X) = \mathcal{L}_Z(\nabla_X Y) - \nabla_{\mathcal{L}_Z X} Y - \nabla_X \mathcal{L}_Z Y, \quad \forall X, Y, Z \in \mathfrak{X}(N). \tag{2.10}$$

Proof. In a given chart $\{x^\alpha\}$:

$$\begin{aligned}
(\mathcal{L}_Z \nabla)^\alpha_{\beta\lambda} Y^\beta X^\lambda \partial_\alpha &= (\nabla^2 Z)(\cdot; Y, X) + R(Z, X)Y \\
&= \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z + \nabla_Z(\nabla_X Y) - \nabla_X(\nabla_Z Y) - \nabla_{[Z,X]} Y \\
&= [Z, \nabla_X Y] - \nabla_{[Z,X]} Y - \nabla_X [Z, Y].
\end{aligned}$$

and the result follows from $\mathcal{L}_Z X = [Z, X]$ for all vector fields X, Z. ∎

Corollary 2.1.8. *Let (N, h) be a semi-Riemannian manifolds. Then:*

$$\mathcal{L}_Z \nabla(\cdot, Y, \cdot) = \mathcal{L}_Z(\nabla Y) - \nabla \mathcal{L}_Z Y, \quad \forall Y \in \mathfrak{X}(N),$$

$$-\mathcal{L}_Z \nabla(\omega, \cdot, \cdot) = \mathcal{L}_Z(\nabla \omega) - \nabla \mathcal{L}_Z \omega, \quad \forall \omega \in \Lambda(N).$$

Proof. Let X, Y be two vector fields. Then, the first assertion follows from $\mathcal{L}_Z(\nabla Y)(\cdot, X) = \mathcal{L}_Z(\nabla_X Y) - \nabla_{\mathcal{L}_Z X} Y$ on the one hand, $(\nabla \mathcal{L}_Z Y)(\cdot, X) = \nabla_X \mathcal{L}_Z Y$ on the other hand, and Proposition 2.1.7.

Now, let ω be a one-form. Then, by Proposition 2.1.7,

$$\begin{aligned}
\mathcal{L}_Z \nabla(\omega, Y, X) &= (\mathcal{L}_Z(\nabla_X Y) - \nabla_{\mathcal{L}_Z X} Y - \nabla_X \mathcal{L}_Z Y)(\omega) \\
&= Z(\nabla_X Y(\omega)) - (\nabla_X Y)(\mathcal{L}_Z \omega) - \mathcal{L}_Z X(Y(\omega)) + Y(\nabla_{\mathcal{L}_Z X} \omega) \\
&\quad -X(\mathcal{L}_Z Y(\omega)) + (\mathcal{L}_Z Y)(\nabla_X \omega) \\
&= [X(Z(\omega(Y))) - X(\mathcal{L}_Z Y(\omega)) - (\nabla_X Y)(\mathcal{L}_Z \omega)] \\
&\quad - [Z(X(\omega(Y))) - Z(\nabla_X Y(\omega)) - (\mathcal{L}_Z Y)(\nabla_X \omega) - Y(\nabla_{\mathcal{L}_Z} \omega)] \\
&= [X((\mathcal{L}_Z \omega)(Y)) - (\mathcal{L}_Z \omega)(\nabla_X Y)] \\
&\quad - [Z((\nabla_X \omega)(Y)) - (\nabla_X \omega)(\mathcal{L}_Z Y) - (\nabla_{\mathcal{L}_Z X} \omega)(Y)] \\
&= [\nabla_X (\mathcal{L}_Z \omega)(Y)] - (\mathcal{L}_Z \nabla \omega)(Y; X) = (\nabla \mathcal{L}_Z \omega - \mathcal{L}_Z(\nabla \omega))(Y; X).
\end{aligned}$$

∎

2.1.6 Connection and Curvature *à* la Cartan

Here, we will explain a well-known technique for the computation of the connection and the curvature tensor of a semi-Riemannian manifold based on the E. Cartan approach to geometry in terms of calculus of differential forms (see, for example, [21]). We also give three examples. For this section, the conventions in [62, 22] are followed, and [66] has been used as a reference for our computations.

General formalism

Let (N, h) be a semi-Riemannian manifold and $\{V_\alpha\}$ a local frame with coframe $\{\xi^\alpha\}$, so that

$$h = h_{\alpha\beta}\xi^\alpha\xi^\beta \quad \text{and} \quad \xi^\alpha(V_\beta) = \delta^\alpha_\beta, \quad \text{with} \quad h_{\alpha\beta} = h(V_\alpha, V_\beta).$$

To determine the connection by a differential approach, one defines the connection one-forms:

Definition 2.1.9. *The set of* n^2 *one-forms* $\{\omega^\alpha_\beta\}^{n-1}_{\alpha,\beta=0}$ *defined as*

$$\nabla_X V_\alpha = \omega^\beta_a(X)V_\beta, \quad \forall\alpha, \beta \in \{0, \ldots, n-1\}, \forall X \in \mathfrak{X}(N)$$

is the set of connection one-forms *of the frame* $\{V_\alpha\}$.

The connection one-forms also satisfy the dual formula

$$\omega^\alpha_\beta(X) = \xi^\alpha(\nabla_X V_\beta).$$

The functions $\gamma^\alpha_{\beta\lambda} := \omega^\alpha_\beta(V_\lambda)$ are named *the coefficients of the connection* in the coframe $\{\xi^\alpha\}$. Therefore, when the frame corresponds to a coordinate basis $\{\partial_\alpha\}$, the functions $\gamma^\alpha_{\beta\lambda}$ will be denoted $\Gamma^\alpha_{\beta\lambda}$, since they coincide with its Christoffel symbols.

Then, a simple computation shows that the components of the covariant derivative of any tensor field $T \in T^k_s N$ in the basis $\{V_\alpha\}$ can be expressed by the formula:

$$(\nabla T)^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s} = dT^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s} + \sum_{l=1}^{k}\omega^{\alpha_l}_\rho T^{\alpha_1\ldots\alpha_{l-1}\rho\alpha_{l+1}\ldots\alpha_k}_{\beta_1\ldots\beta_s} - \sum_{l=1}^{k}\omega^\rho_{\beta_l} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_{l-1}\rho\beta_{l+1}\ldots\beta_s}, \tag{2.11}$$

or, equivalently,

$$\begin{aligned}(\nabla T)^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s\beta_{s+1}} &\equiv \nabla_{\beta_{s+1}} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s} \\ &= V_{\beta_{s+1}}(T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s}) + \sum_{l=1}^{k}\gamma^{\alpha_l}_{\rho\beta_{s+1}} T^{\alpha_1\ldots\rho\ldots\alpha_k}_{\beta_1\ldots\beta_s} - \sum_{l=1}^{s}\gamma^\rho_{\beta_l\beta_{s+1}} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\rho\ldots\beta_s}.\end{aligned} \tag{2.12}$$

Therefore, given the connection one-forms of a frame, the connection ∇ of the manifold is defined locally. Indeed, introducing $T = h$ in (2.11) the condition $\nabla h = 0$ simply reads as

$$dh_{\alpha\beta} = \omega^\rho_\alpha h_{\rho\beta} + \omega^\rho_\beta h_{\alpha\rho}. \tag{2.13}$$

Denoting $\omega_{\alpha\beta} := h_{\alpha\rho}\omega^{\rho}_{\beta}$, the equation (2.13) yields the "anti-symmetry" relation

$$dh_{\alpha\beta} = \omega_{\beta\alpha} + \omega_{\beta\alpha}. \tag{2.14}$$

The following formulae allow to calculate the connection one-forms from the coframe $\{\xi^{\alpha}\}$:

Proposition 2.1.10 (The First Structural Equation). $0 = d\xi^{\alpha} + \omega^{\alpha}_{\beta} \wedge \xi^{\beta}$ *for all* α *in* $\{0, \ldots, n-1\}$.

Proof. On the one hand, from the expression

$$d\tau(X, Y) = X\tau(Y) - Y\tau(X) - \tau([X, Y])$$

for all $\tau \in \Lambda(N)$ and $X, Y \in \mathfrak{X}(N)$, and using $\xi^{\alpha}(V_{\beta}) = \delta^{\alpha}_{\beta}$, it follows that

$$d\xi^{\alpha}(V_{\beta}, V_{\lambda}) = -\xi^{\alpha}([V_{\beta}, V_{\lambda}]).$$

On the other hand, using again $\xi^{\alpha}(V_{\beta}) = \delta^{\alpha}_{\beta}$, and $\omega^{\alpha}_{\beta}(V_{\lambda}) = \xi^{\alpha}(\nabla_{V_{\lambda}} V_{\beta})$, it follows that

$$(\omega^{\alpha}_{\rho} \wedge \xi^{\rho})(V_{\beta}, V_{\lambda}) = \omega^{\alpha}_{\lambda}(V_{\beta}) - \omega^{\alpha}_{\beta}(V_{\lambda}) = \xi^{\alpha}(\nabla_{V_{\beta}} V_{\lambda} - \nabla_{V_{\lambda}} V_{\beta}) = \xi^{\alpha}([V_{\beta}, V_{\lambda}]).$$

■

Moreover,

Corollary 2.1.11. *The connection one-forms* $\{\omega^{\alpha}_{\beta}\}^{n-1}_{\alpha,\beta=0}$ *associated to a frame* $\{V_{\alpha}\}$ *are unique, and are defined by the formula:*

$$\begin{aligned} \omega^{\alpha}_{\beta}(V_{\lambda}) &= \frac{1}{2}\Big(\xi^{\alpha}([V_{\lambda}, V_{\beta}]) + h^{\alpha\rho}\left(h_{\beta\sigma}\xi^{\sigma}([V_{\rho}, V_{\lambda}]) - h_{\lambda\sigma}\xi^{\sigma}([V_{\beta}, V_{\rho}])\right) \\ &\quad + h^{\alpha\rho}\left(dh_{\rho\lambda}(V_{\beta}) + dh_{\rho\beta}(V_{\lambda}) - dh_{\beta\lambda}(V_{\rho})\right)\Big) \end{aligned}$$

Proof. From the first structural equations,

$$d\xi^{\alpha}(V_{\beta}, V_{\lambda}) = \omega^{\alpha}_{\beta}(V_{\lambda}) - \omega^{\alpha}_{\lambda}(V_{\beta}). \tag{2.15}$$

So contracting with $g_{\mu\alpha}$ the equation (2.15) and cyclically permuting the indices μ, β, λ, one gets (recall that $\omega_{\alpha\beta} = h_{\alpha\rho}\omega^{\rho}_{\beta}$)

$$\begin{aligned} h_{\mu\alpha}d\xi^{\alpha}(V_{\beta}, V_{\lambda}) &= \omega_{\mu\beta}(V_{\lambda}) - \omega_{\mu\lambda}(V_{\beta}), \\ h_{\beta\alpha}d\xi^{\alpha}(V_{\lambda}, V_{\mu}) &= \omega_{\beta\lambda}(V_{\mu}) - \omega_{\beta\mu}(V_{\lambda}), \\ h_{\lambda\alpha}d\xi^{\alpha}(V_{\mu}, V_{\beta}) &= \omega_{\lambda\mu}(V_{\beta}) - \omega_{\lambda\beta}(V_{\mu}). \end{aligned}$$

Substract the last one to the sum of the first two and use (2.14) to get

$$2\omega_{\mu\beta}(V_{\lambda}) = h_{\mu\alpha}d\xi^{\alpha}(V_{\beta}, V_{\lambda}) + h_{\beta\alpha}d\xi^{\alpha}(V_{\lambda}, V_{\mu}) - h_{\lambda\alpha}d\xi^{\alpha}(V_{\mu}, V_{\beta}) + dh_{\mu\lambda}(V_{\beta}) - dh_{\beta\lambda}(V_{\mu}) + dh_{\mu\beta}(V_{\lambda}).$$

Finally, use $h_{\alpha\rho}h^{\rho\beta} = \delta^{\beta}_{\alpha}$ and $d\xi^{\alpha}(V_{\beta}, V_{\lambda}) = -\xi^{\alpha}([V_{\beta}, V_{\lambda}]))$ to obtain the desired expression for $\omega^{\alpha}_{\beta}(V_{\lambda}) = h^{\alpha\mu}\omega_{\mu\beta}(V_{\lambda})$. ■

Therefore, the connection ∇ is uniquely determined (locally) by the connection one-forms. The above formula is a well-known generalization to arbitrary frames of the Koszul formula for the Christoffel symbols.

Now, to determine the curvature by a differential approach, one defines the curvature two-forms:

Definition 2.1.12. *The set of n^2 two-forms $\{\Omega^\alpha_\beta\}_{\alpha,\beta=0}^{n-1}$ defined by*

$$\Omega^\alpha_\beta(X,Y) = \xi^\alpha(R(X,Y)V_\beta), \quad \forall\alpha,\beta \in \{0,\dots,n-1\} \text{ and } \forall X,Y \in \mathfrak{X}(N),$$

where R is the curvature tensor field of M, form the set of curvature two-forms *of the frame $\{V_\alpha\}$.*

The curvature two-forms also satisfy the dual formula

$$\Omega^\alpha_\beta(X,Y)V_\alpha = R(X,Y)V_\beta,$$

and the components of the curvature tensor R (and its metrically equivalent $(0,4)$-tensor field) with respect to the frame $\{V_\alpha\}$ are

$$R^\alpha{}_{\beta\lambda\mu} = \Omega^\alpha_\beta(V_\lambda,V_\mu) \quad \text{and} \quad R_{\alpha\beta\lambda\mu} = h_{\alpha\rho}\Omega^\rho_\beta(V_\lambda,V_\mu),$$

so by the anti-symmetry of R in the first two slots as a $(0,4)$-tensor field, one gets

$$\Omega^\alpha_\beta = -h^{\alpha\sigma}h_{\beta\rho}\Omega^\rho_\sigma.$$

The following formulae give a straighforward method to calculate the curvature two-forms from the connection one-forms.

Proposition 2.1.13 (The Second Structural Equation). *$\Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\rho \wedge \omega^\rho_\beta$ for all α,β in $\{0,\dots,n-1\}$.*

Proof. Apply $\nabla_{V_\lambda}V_\alpha = \omega^\beta_a(V_\lambda)V_\beta$ systematically and use

$$d\tau(X,Y) = X\tau(Y) - Y\tau(X) - \tau([X,Y])$$

for all $\tau \in \Lambda(N)$ and $X,Y \in \mathfrak{X}(N)$, to get

$$\Omega^\alpha_\beta(V_\lambda,V_\mu)V_\alpha = \left(d\omega^\alpha_\beta + \omega^\alpha_\rho \wedge \omega^\rho_\beta\right)(V_\lambda,V_\mu)V_\alpha$$

as follows:

$$\begin{aligned}
\Omega^\alpha_\beta(V_\lambda,V_\mu)V_\alpha &= R(V_\lambda,V_\mu)V_\beta = \nabla_{V_\lambda}(\nabla_{V_\mu}V_\beta) - \nabla_{V_\mu}(\nabla_{V_\lambda}V_\beta) - \nabla_{[V_\lambda,V_\mu]}V_\beta \\
&= \nabla_{V_\lambda}(\omega^\alpha_\beta(V_\mu)V_\alpha) - \nabla_{V_\mu}(\omega^\alpha_\beta(V_\lambda)V_\alpha) - \omega^\alpha_\beta([V_\lambda,V_\mu])V_\alpha \\
&= \left(V_\lambda(\omega^\alpha_\beta(V_\mu)) - V_\mu(\omega^\alpha_\beta(V_\lambda)) - \omega^\alpha_\beta([V_\lambda,V_\mu])\right)V_\alpha \\
&\quad + \left((\omega^\alpha_\rho \otimes \omega^\rho_\beta - \omega^\rho_\beta \otimes \omega^\alpha_\rho)(V_\lambda,V_\mu)\right)V_\alpha \\
&= \left(d\omega^\alpha_\beta + \omega^\alpha_\rho \wedge \omega^\rho_\beta\right)(V_\lambda,V_\mu)V_\alpha.
\end{aligned}$$

■

All this is summarized in Table 2.1.

CARTAN APPROACH :

$\{V_\alpha\}$ a local frame on a semi-Riemannian manifold (N,h) with coframe $\{\xi^\alpha\}$ such that $h = h_{\alpha\beta}\xi^\alpha\xi^\beta$ and $\xi^\alpha(V_\beta) = \delta^\alpha_\beta$

- Connection one-forms:
$\omega^\alpha_\beta(X) = \xi^\alpha(\nabla_X V_\beta), \quad \forall \alpha, \beta$

- Curvature two-forms:
$\Omega^\alpha_\beta(X,Y) = \xi^\alpha(R(X,Y)V_\beta), \quad \forall \alpha, \beta$

- First structural equation:
$0 = d\xi^\alpha + \omega^\alpha_\beta \wedge \xi^\beta, \quad \forall \alpha$

- Second structural equation:
$\Omega^\alpha_\beta = d\omega^\alpha_\beta + \omega^\alpha_\rho \wedge \omega^\rho_\beta, \quad \forall \alpha, \beta$

- Components of the connection ∇:
$\gamma^\alpha_{\beta\lambda} = \omega^\alpha_\beta(V_\lambda), \quad \forall \alpha, \beta, \lambda$

- Components of the curvature R:
$R^\alpha{}_{\beta\lambda\mu} = \Omega^\alpha_\beta(V_\lambda, V_\mu), \quad \forall \alpha, \beta, \lambda, \mu$

- Additional data:
$dh_{\alpha\beta} = \omega^\rho_\alpha h_{\rho\beta} + \omega^\rho_\beta h_{\alpha\rho}, \quad \forall \alpha, \beta$

$\Omega^\alpha_\beta = -h^{\alpha\sigma} h_{\beta\rho} \Omega^\rho_\sigma, \quad \forall \alpha, \beta$

$$\begin{aligned}
\omega^\alpha_\beta(V_\lambda) = \gamma^\alpha_{\beta\lambda} &= \frac{1}{2}\Big(\xi^\alpha([V_\lambda, V_\beta]) + h^{\alpha\rho}\left(h_{\beta\sigma}\xi^\sigma([V_\rho, V_\lambda]) - h_{\lambda\sigma}\xi^\sigma([V_\beta, V_\rho])\right) \\
&\quad + h^{\alpha\rho}\left(dh_{\rho\lambda}(V_\beta) + dh_{\rho\beta}(V_\lambda) - dh_{\beta\lambda}(V_\rho)\right)\Big) \\
\nabla_{\beta_{s+1}} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s} &= V_{\beta_{s+1}}(T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\beta_s}) + \sum_{l=1}^{k} \gamma^{\alpha_l}_{\rho\beta_{s+1}} T^{\alpha_1\ldots\rho\ldots\alpha_k}_{\beta_1\ldots\beta_s} - \sum_{l=1}^{s} \gamma^{\rho}_{\beta_l\beta_{s+1}} T^{\alpha_1\ldots\alpha_k}_{\beta_1\ldots\rho\ldots\beta_s}
\end{aligned}$$

Table 2.1: Cartan approach to ∇ and R in terms of calculus of differential forms.

Some relevant special frames

In order to make the computations for a particular given frame, one first tries to obtain the maximum information from the equation $dh_{\alpha\beta} = \omega^\rho_\alpha h_{\rho\beta} + \omega^\rho_\beta h_{\alpha\rho}$, which provides relations between different connection one-forms, and other possible properties of the space (as for example, the existence of a parallel vector field). Then, one uses this information together with the First Structural Equation, in order to obtain all the connection one-forms. Finally, it only remains to calculate the curvature two-forms with the Second Structural Equation.

(a) Orthonormal frames on semi-Riemannian manifolds. Let $\{V_\alpha\}$ be an orthonormal frame on a semi-Riemannian manifold (N,h) with signature $(\varepsilon_0, \ldots, \varepsilon_{n-1})$ and

$\varepsilon_\alpha^2 = 1$ for all $\alpha = 0, \ldots, n-1$, so that

$$h = \varepsilon_\alpha (\xi^\alpha)^2, \quad \text{or, equivalently} \quad h_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta} = \varepsilon_\beta \delta_{\alpha\beta} \quad \text{(no summation in } \alpha \text{ nor in } \beta).$$

Then, the equations in the first row of *additional data* in Table 2.1, (see [66, page 49], which also provides an example) become:

$$\omega_\beta^\alpha = -\varepsilon_\alpha \varepsilon_\beta \omega_\alpha^\beta; \quad \omega_\alpha^\alpha = 0; \quad \Omega_\beta^\alpha = -\varepsilon_\alpha \varepsilon_\beta \Omega_\alpha^\beta \text{ and } \Omega_\alpha^\alpha = 0,$$

for all α, β, with $\alpha \neq \beta$ and no-summation in β, and the equations in the second row become

$$\omega_\beta^\alpha(V_\lambda) = \frac{1}{2}(\xi^\alpha([V_\lambda, V_\beta]) + \varepsilon_\alpha \varepsilon_\beta \xi^\beta([V_\alpha, V_\lambda]) - \varepsilon_\alpha \varepsilon_\lambda \xi^\lambda([V_\beta, V_\alpha]),$$

with no sumation either in β or in λ.

(b) Partly null frames on Lorentzian manifolds. Let $\{E_\alpha\}$ be a partly null frame on a Lorentzian manifold (M, g) with coframe $\{\theta^\alpha\}$ such that

$$g = -2\theta^0\theta^1 + a_{ij}\theta^i\theta^j \left(\equiv -\theta^0 \otimes \theta^1 - \theta^1 \otimes \theta^0 + a_{ij}\theta^i \otimes \theta^j\right)$$

so that $g_{01} = g_{10} = -1$, the functions $\{a_{ij}\}_{i,j=2}^{n-1}$ define a positive-definite metric, and the remaining components of g vanish. Then, the equations in the first row of *additional data* in Table 2.1 become:

$$\omega_0^1 = \omega_1^0 = 0; \qquad \omega_i^0 = a_{ij}\omega_1^j; \qquad w_i^1 = \omega_0^j a_{ij}; \qquad da_{ij} = \omega_i^k a_{kj} + \omega_j^k a_{ki}$$

$$\Omega_i^0 = a_{ij}\Omega_1^j; \qquad \Omega_i^1 = a_{ij}\Omega_0^j; \qquad \Omega_j^i = -a^{ik}a_{js}\Omega_k^s,$$

for all i, j, and the equations in the second row become

$$\begin{aligned}
\omega_i^0(E_\lambda) &= \frac{1}{2}\left(\theta^0([E_\lambda, E_i]) - a_{ij}\theta^j([E_1, E_\lambda]) + g_{\lambda\sigma}\theta^\sigma([E_i, E_1]) + da_{i\lambda}(E_1)\right), \\
\omega_i^1(E_\lambda) &= \frac{1}{2}\left(\theta^1([E_\lambda, E_i]) - a_{ij}\theta^j([E_0, E_\lambda]) + g_{\lambda\sigma}\theta^\sigma([E_i, E_0]) + da_{i\lambda}(E_0)\right), \\
\omega_j^i(E_\lambda) &= \frac{1}{2}\left(\theta^i([E_\lambda, E_j]) + a^{is}(a_{jl}\theta^l([E_s, E_\lambda]) + g_{\lambda\sigma}\theta^\sigma([E_j, E_s]))\right) \\
&\quad + \frac{1}{2}\left(a^{is}(dg_{s\lambda}(E_j) + da_{sj}(E_\lambda) - dg_{i\lambda}(E_s))\right)
\end{aligned}$$

A non-trivial example of this case appears in Section 4.4 for Brinkmann spaces.

(c) Particular case of a Partly Null frame. Let $\{E_\alpha\}$ be a partly null frame on a Lorentzian manifold (M,g) with coframe $\{\theta^\alpha\}$ such that

$$g = -2\theta^0\theta^1 + \sum_{i=2}^{n-1}(\theta^i)^2 \left(\equiv -\theta^0 \otimes \theta^1 - \theta^1 \otimes \theta^0 + \sum_{i=2}^{n-1} \theta^i \otimes \theta^i \right)$$

so that

$$g_{01} = g_{10} = -1; \quad g_{ij} = \delta_{ij}; \quad \text{otherwise zero.}$$

This is a particular case for partly null frames, with $a_{ij} = \delta_{ij}$. the equations in the first row of *additional data* in Table 2.1 become:

$$\omega_0^1 = \omega_1^0 = 0; \qquad \omega_i^0 = \omega_1^i; \qquad w_i^1 = \omega_0^i; \qquad \omega_i^j = -\omega_j^i$$

$$\Omega_i^0 = \Omega_1^i; \qquad \Omega_i^1 = \Omega_0^i; \qquad \Omega_j^i = -\Omega_i^j,$$

for all i,j, and the equations in the second row become

$$\begin{aligned}
\omega_i^0(V_\lambda) &= \frac{1}{2}\left(\theta^0([V_\lambda,V_i]) - \theta^i([V_1,V_\lambda]) + g_{\lambda\sigma}\theta^\sigma([V_i,V_1])\right),\\
\omega_i^1(V_\lambda) &= \frac{1}{2}\left(\theta^1([V_\lambda,V_i]) - \theta^i([V_0,V_\lambda]) + g_{\lambda\sigma}\theta^\sigma([V_i,V_0])\right),\\
\omega_j^i(V_\lambda) &= \frac{1}{2}\left(\theta^i([V_\lambda,V_j]) + (\theta^j([V_i,V_\lambda]) + g_{\lambda\sigma}\theta^\sigma([V_j,V_i]))\right)
\end{aligned}$$

Example 2.1.14. Let (M,g) be the 4-dimensional Brinkmann space (see Chapter 4)

$$ds^2 = -2du(dv + Hdu + W_2dx^2 + W_3dx^3) + f^2((dx^2)^2 + (dx^3)^2)$$

where H, W_2, W_3, f are arbitrary functions on u, x^2, x^3. Take the null frame

$$\{V_0,V_1,V_2,V_3\} = \{\partial_u - H\partial_v, \partial_v, -f^{-1}W_2\partial_v + f^{-1}\partial_2, -f^{-1}W_3\partial_v + f^{-1}\partial_3\}$$

with coframe

$$\{\theta^0,\theta^1,\theta^2,\theta^3\} = \{du, dv + Hdu + W_2dx^2 + W_3dx^3, fdx^2, fdx^3\}.$$

Since θ^0 is a parallel one-form, it also holds that $\omega_\alpha^0 = \omega_1^\alpha = 0$, so the only non-vanishing connection one-forms are

$$\omega_2^1 = \omega_0^2, \quad \omega_3^1 = \omega_0^3 \text{ and } \omega_3^2 = -\omega_2^3.$$

Using the notation (1.4) for partial derivatives, put $h_i := H_{,i} - \dot{W}_i$ for any $i = 2,3$, and $t_{23} := \frac{1}{2}(W_{2,3} - W_{3,2})$. The First Structural Equation yields

$$\begin{aligned}
h_2dx^2 \wedge du + h_3dx^3 \wedge du - 2t_{23}dx^2 \wedge dx^3 &= -f\omega_2^1 \wedge dx^2 - f\omega_3^1 \wedge dx^3, &(2.16)\\
\dot{f}du \wedge dx^2 - f_{,3}dx^2 \wedge dx^3 &= -\omega_2^1 \wedge du - f\omega_3^2 \wedge dx^3, &(2.17)\\
\dot{f}du \wedge dx^3 - f_{,2}dx^3 \wedge dx^2 &= -\omega_3^1 \wedge du + f\omega_3^2 \wedge dx^2. &(2.18)
\end{aligned}$$

From (2.17) and (2.18) it is easy to deduce

$$\omega^1_2 = \dot{f}dx^2 + (?)dx^3 + (?)du; \qquad \omega^1_3 = (?)dx^2 + \dot{f}dx^3 + (?)du,$$

$$\omega^2_3 = f^{-1}f_{,3}dx^2 - f^{-1}f_{,2}dx^3 + (?)du,$$

where $(?)$ means "to be determined". Now, using (2.16) one gets

$$\omega^1_2 = \dot{f}dx^2 - f^{-1}t_{23}dx^3 + f^{-1}h_2du,$$

$$\omega^1_3 = f^{-1}t_{23}dx^2 + \dot{f}dx^3 + f^{-1}h_3du.$$

Substituting these values on (2.17) or (2.18), one obtains the remaining unknown in ω^2_3, so that:

$$\omega^2_3 = f^{-1}f_{,3}dx^2 - f^{-1}f_{,2}dx^3 - f^{-2}t_{23}du.$$

Therefore,

$$\begin{array}{rcl} \omega^1_2 & = & \dot{f}f^{-1}\theta^2 - f^{-2}t_{23}\theta^3 + f^{-1}h_2\theta^0, \\ \omega^1_3 & = & f^{-2}t_{23}\theta^2 + \dot{f}f^{-1}\theta^3 + f^{-1}h_3\theta^0, \\ \omega^2_3 & = & f^{-2}\left(f_{,3}\theta^2 - f_{,2}\theta^3 - t_{23}\theta^0\right). \end{array}$$

Now, use the Second Structural Equation to compute the curvature two-forms and obtain the components of the curvature. Observe that for $i = 2, 3$ one gets

$$\Omega^0_i = 0,$$

and

$$\Omega^1_2 = d\omega^1_2 - \omega^1_2 \wedge \omega^2_3,$$

$$\Omega^1_3 = d\omega^1_3 + \omega^1_2 \wedge \omega^2_3$$

$$\Omega^2_3 = d\omega^2_3$$

are the only non-vanishing curvature two-forms. For example,

$$\begin{array}{rcl} \Omega^2_3 & = & \{f^{-1}\dot{f}_{,3} - f^{-2}\dot{f}f_{,3} - (f^{-2}t_{23})_{,2}\}du \wedge dx^2 \\ & & -\{f^{-1}\dot{f}_{,2} - f^{-2}\dot{f}f_{,2} + (f^{-2}t_{23})_{,3}\}du \wedge dx^3 \\ & & +\{f^{-2}((f_{,3})^2 + (f_{,2})^2) - f^{-1}(f_{,2,2} + f_{,3,3})\}dx^2 \wedge dx^3 \\ & & \\ & = & f^{-1}\{f^{-1}\dot{f}_{,3} - f^{-2}\dot{f}f_{,3} - (f^{-2}t_{23})_{,2}\}\theta^0 \wedge \theta^2 \\ & & -f^{-1}\{f^{-1}\dot{f}_{,2} - f^{-2}\dot{f}f_{,2} + (f^{-2}t_{23})_{,3}\}\theta^0 \wedge \theta^3 \\ & & +\{f^{-4}((f_{,3})^2 + (f_{,2})^2) - f^{-3}(f_{,2,2} + f_{,3,3})\}\theta^2 \wedge \theta^3. \end{array}$$

Therefore, as $R^\alpha{}_{\beta\lambda\mu}$ correspond to half of the components of Ω^α_β for the term $\theta^\lambda \wedge \theta^\mu$, i.e., $\Omega^\alpha_\beta = \frac{1}{2} R^\alpha{}_{\beta\lambda\mu}\theta^\lambda \wedge \theta^\mu$, it follows that

$$\begin{aligned} R^2{}_{302}(= -R^1{}_{223}) &= f^{-1}\{f^{-1}\dot{f}_{,3} - f^{-2}\dot{f}f_{,3} - (f^{-2}t_{23})_{,2}\}, \\ R^2{}_{303}(= R^1{}_{332}) &= -f^{-1}\{f^{-1}\dot{f}_{,2} - f^{-2}\dot{f}f_{,2} + (f^{-2}t_{23})_{,3}\}, \\ R^2{}_{323} &= f^{-3}\{f^{-1}((f_{,3})^2 + (f_{,2})^2) - (f_{,2,2} + f_{,3,3})\}. \end{aligned}$$

For example if, say, $f(u, x^2, x^3) = \frac{1}{1+\frac{k(u)}{4}((x^2)^2+(x^3)^2)}$ (i.e., the slices $\{u, v\} = \{u_0, v_0\}$ are of constant curvature) then $R^2{}_{323} = k(u)$.

2.2 Distributions and Foliations

Roughly speaking, a foliation corresponds to a partition of N into inmersed submanifolds of the same dimension. One can talk about foliations using foliated atlases, involutive distributions and differential forms. The general notion of a foliation was defined by C. Ehresmann and G.Reebs in [27] and has been studied over the last forty years. Non-constancy in the dimension of the leaves correspond to foliations with singularities, which have been studied, for example, in [60, 61, 73, 91]. Foliations appear in a natural way in the theory of submersions, non-singular system of differential equations, fiber bundles, symplectic geometry, etc.

In this Section, a brief introduction on foliations is given. To that end, first the local and global definitions of a distribution are given, together with its possible metric structure. Then, foliations are presented in order to connect them with the theory of distributions. This part is mostly based on the books [75, 88, 6, 60, 9].

2.2.1 Distributions and Frobenius Theorem

Definition: local and global viewpoints

Let N be an n-dimensional manifold, and TN its tangent vector bundle with base N and (canonical) projection $\Pi : TN \longrightarrow N$ which maps each X_p to $p \in N$. The fiber $\pi^{-1}(p)$ of each $p \in N$ is the tangent space T_pN. Then, each local chart $\{(U, \varphi = (x^\alpha))\}$ of N defines a local chart $\{(U^*, \phi = (x^0, \ldots, x^{n-1}, v^0, \ldots, v^{n-1}))\}$ in TN, where $U^* = \pi^{-1}(U)$ and $\phi : U^* \longrightarrow \mathbb{R}^{2n}$ is the natural bundle isomorphism from U^* onto $\varphi(U) \times \mathbb{R}^n$.

Then, an *m-distribution* is globally defined either as a vector subbundle of TN of rank m or as a global section of the Grassman bundle $G_m(N)$. As a vector subbundle D of TN of rank m with base N and projection $\Pi' : D \longrightarrow N$, it satisfies that, for each $p \in N$ there exists a local chart $\{(U, \varphi = (x^\alpha))\}$ at p such that the associated local chart (U^*, ϕ) in TN satisfies $\phi(U^* \cap D) = \varphi(U) \times \mathbb{R}^m \times \{\vec{0}\}$, and the fiber $\pi'^{-1}(p) = D_p$ is an m-dimensional subspace of T_pN. Indeed, if for each $p \in N$ the set $G_m(p)$

consists of all the m-dimensional vector subspaces of T_pN, it follows that any section of $G_m(N) = \bigcup_{p\in N} G_m(p)$ —the *Grassmann bundle* over N— defines an m-distribution, and, conversely, any m-distribution defines a section of $G_m(N)$.

Locally a distribution on N can be represented using either local frames or differential systems. Choose on a coordinate neighborhood $U \subset N$ m linearly independent vector fields $\{V_0, \ldots, V_{m-1}\}$. Then, the mapping $p \to D_p = \text{span}\{V_{0|p}, \ldots, V_{m-1|p}\}$ for $p \in U$ defines an m-distribution on U. As any m-distribution on N is locally represented at a point p by m linearly independent vector fields, they satisfy on the intersection of two coordinate neighborhoods $U \cap \widetilde{U}$ the relation

$$\widetilde{V}_i = \sum_{j=0}^{m-1} a_i^j V_j, \text{ for } i = 0, \ldots, m-1,$$

where a_i^j are smooth functions such that $(a_i^j(p))$ is a non-singular $m \times m$ matrix for any $p \in U \cap \widetilde{U}$(see for example [6]).

Assume now that on a coordinate neighborhood $U \subset N$ there exist $m' = n - m$ linearly independent one-forms $\{\omega^m, \ldots, \omega^{n-1}\}$. Then, the mapping $p \to D_p$ for $p \in U$ such that D_p corresponds to the m-dimensional subspace of T_pN consisting on all the solutions X_p of the system

$$\omega^m(X) = 0, \ldots, \omega^{n-1}(X) = 0 \tag{2.19}$$

defines an m-distribution on U. Analogously, any m-distribution on N is given locally at a point p by a differential system (2.19) whose representative one-forms on the intersection of two coordinate neighborhoods $U, \widetilde{U}$ are related by

$$\widetilde{\omega}^i = \sum_{j=m}^{n-1} \Lambda_j^i \omega^j, \text{ for } i = m, \ldots, n-1,$$

where Λ_i^j are smooth functions on the intersection of two coordinate neighborhoods $U \cap \widetilde{U}$ such that $(\Lambda_i^j(p))$ is a non-singular $m' \times m'$ matrix for any $p \in U \cap \widetilde{U}$.

The Integrability Problem for a Distribution

The integrability problem for a distribution is related with the possible existence of solutions for a partial differential equation, in the following sense. A smooth function f in N is a *first integral* of D, if for every $X \in \Gamma(D)$, $X(f) = 0$ holds, or, equivalently, $df(X) = 0$. If finding $n - m$ functionally independent first integrals were feasable, then, one would have a complete local description of D.

Let D be an m-distribution on N. A k-dimensional submanifold N' of N with $0 < k \leq m$ is an *integrable submanifold of* D if $T_pN' \subset D_p$ for all $p \in N'$. More precisely, if $\varphi : N' \longrightarrow N$ is the immersion of the manifold, which is asumed to be injective, then $\varphi^*(T_pN') \subset D_{\varphi(p)}$. Thus, the maximum dimension of N' is m.

Then, D is a *(completely) integrable distribution* if there is an m-dimensional integral submanifold of D passing through each $p \in N$. In other words, for any $p \in N$ there exists a local chart $\{(U, \varphi = (x^\alpha))\}$ on N such that all the submanifolds of U given by the equations

$$x^m = const., \dots, x^{n-1} = const. \tag{2.20}$$

are integral manifolds of D ([9, 6]). Hence, $\{x^m, \dots, x^{n-1}\}$ are functionally independent first integrals and they give a complete local analysis of D. A connected submanifold given by (2.20) is called a *plaque* of D. Observe that any connected integral manifold of D lying in U is a submanifold of one of the plaques of D. Indeed, there is a unique *maximal connected integral submanifold of* D passing throgh each $p \in N$, that is, a unique m-dimensional connected integral submanifold which is not contained in any larger connected integral submanifold (see [9, page 154]).

Based on the above definitions, we have the following:

Theorem 2.2.1. *Let D be an m-distribution on N. Then, the following assertions are equivalent:*

- *D is an integrable distribution.*
- *For any $p \in N$ there exists a chart $\{(U, \varphi = (x^\alpha))\}$ such that D is given on U by the differential system*
$$dx^m = 0, \dots, , dx^{n-1} = 0.$$
- *For any $p \in N$ there exists a chart $\{(U, \varphi = (x^\alpha))\}$ such that $D = span\{\frac{\partial}{\partial x^0}, \dots, \frac{\partial}{\partial x^{m-1}}\}$ on U.*

It is said that D is an *involutive distribution* if $[X, Y] \in \Gamma(D)$ for any $X, Y \in \Gamma(D)$. Then, the integrability equations for an m-dimensional distribution D were found by Frobenius:

Theorem 2.2.2 (Frobenius Theorem). *D is an integrable distribution if and only if D is an involutive distribution.*

The proof of this theorem can be found in many books. See [90] for a classical version, or [6, 9, 60]. The key point of the proof is based mostly on the following result:

Lemma 2.2.3. *Let $p \in N$ and U' be an open subset with $p \in U'$. Then, if D is a local involutive m-distribution defined on U' by m linearly independent vector fields $\{E_0, \dots, E_{m-1}\}$, there exists a local chart $\{(U, \varphi = (x^\alpha))\}$ at p such that $E_i = \frac{\partial}{\partial x^i}$, for $i = 0, \dots, m-1$.*

See [6, Lemma 1.6 in page 64] for a stronger result and its proof.

Semi-Riemannian Distributions

Let N be an n-dimensional manifold endowed with an m-distribution D, and denote $L^2_S(D_p, \mathbb{R})$ the real tensor space of all symmetric bilinear forms $h_p : D_p \times D_p \longrightarrow \mathbb{R}$. Consider the tensor bundle over N

$$L^2_S(D, \mathbb{R}) = \bigcup_{p \in N} L^2_S(D_p, \mathbb{R}).$$

Accordingly to the notation in this thesis, a *semi-Riemannian metric of index q on D* is a section $h : N \longrightarrow L^2_S(D, \mathbb{R})$ of $L^2_S(D, \mathbb{R})$ such that h_p is non-degenerate of index q on D_p for each $p \in N$, and it is said that *D is a semi-Riemannian distribution*. When $q = 0$, that is, h_p is positive-definite for all $p \in N$, h is a *Riemannian metric on D* denoted by $\overline{g}$, and D is a *Riemannian distribution*. Finally, if $q = 1$, then D is a *Lorentzian distribution* and the metric is denoted by g. In particular, if $D = TN$, then h becomes a semi-Riemannian metric on N.

Consider now an m-distribution D on an n-dimensional semi-Riemannian manifold (N, h). Then, h induces a global section of $L^2_S(D, \mathbb{R})$, which we denote by the same symbol h except if otherwise is specified. Then, two cases arise: either (D, h) is a semi-Riemannian distribution on N or (D_p, h_p) is a degenerate subspace of (T_pN, h_p) for some $p \in N$. This last case, of course, cannot occur when (N, h) is a Riemannian manifold, in which case all the possible distributions are Riemannian distributions. For a semi-Riemannian distribution, consider the vector subbundle

$$D^\perp = \bigcup_{p \in N} D^\perp_p,$$

where $D^\perp_p$ is the complementary orthogonal subspace to D_p in (T_pN, h_p). Since $D^\perp_p$ is also non-degenerate, h induces a semi-Riemannian metric h' on $D^\perp$, and therefore $(D^\perp, h')$ is a semi-Riemannian distribution too.

2.2.2 Foliations

In order to motivate the definition of a foliation on a manifold N, we first of all present the natural foliation on $\mathbb{R}^n$. Take $\mathbb{R}^n$ as an n-dimensional manifold with the global chart $(\mathbb{R}^n, I_{\mathbb{R}^n})$ and consider two positive integers m, m' such that $n = m + m'$. Then, identify $\mathbb{R}^n$ with the Cartesian product $\mathbb{R}^m \times \mathbb{R}^{m'}$. If $c = (c^m, \ldots, c^{n-1})$ is a point in $\mathbb{R}^{m'}$, denote by $\mathbb{R}^m_c$ the affine m-dimensional subspace of $\mathbb{R}^n$ passing through the point $(0, \ldots, 0, c^m, \ldots, c^{n-1}) \in \mathbb{R}^n$, and parallel to $\mathbb{R}^m$, i.e.,

$$\mathbb{R}^m_c = \{(x^0, \ldots, x^{n-1}) \in \mathbb{R}^n : \quad x^m = c^m, \ldots, x^{n-1} = c^{n-1}\}.$$

Then, an *(m, c)-plaque* P^m_c in $\mathbb{R}^n$ is the intersection of $\mathbb{R}^m_c$ with an open ball in $\mathbb{R}^n$. Consider now the disjoint partition

$$\mathbb{R}^n = \bigcup_{c \in \mathbb{R}^{m'}} \mathbb{R}^m_c, \tag{2.21}$$

which suggests the following definition: the family $\{\mathbb{R}^m_c\}_{c\in\mathbb{R}^{m'}}$ is *the model foliation on* $\mathbb{R}^n$ of dimension m (or codimension m'), and each $\mathbb{R}^m_c$ is a *leaf* of the foliation. Indeed, each such leaf is an m-dimensional submanifold of $\mathbb{R}^n$.

The definition of a foliation involving local automorphisms of $\mathbb{R}^n$ was given in [72] (see also [88],[6]). Let $(x_1, x_2) = (x^0, \dots, x^{m-1}, x^m, \dots, x^{n-1})$ denote the standard coordinates in $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{m'}$ with $x_1 \in \mathbb{R}^m$ and $x_2 \in \mathbb{R}^{m'}$, and $\phi : U \longrightarrow U'$ be a local diffeomorphism of open connected subsets of $\mathbb{R}^n$. Then, ϕ is a *local automorphism of* $\mathbb{R}^n$ if it takes the form

$$\phi(x_1, x_2) = (\phi_1(x_1, x_2), \phi_2(x_2)).$$

This condition is equivalent to

$$\frac{\partial \phi^i}{\partial x^j} = 0, \qquad \text{for all } i = m, \dots, n-1 \text{ and } j = 0, \dots, m-1.$$

Observe that the map ϕ_2 induces a local diffeomorphism of $\mathbb{R}^{m'}$. If we denote as $\Gamma_{n,m'}$ the family of local automorphisms of $\mathbb{R}^n$, then $\Gamma = \Gamma_{n,m'}$ is a pseudogroup of transformations of $\mathbb{R}^n$, that is to say, the following properties hold:

(i) if $\phi \in \Gamma$, then $\phi^{-1} \in \Gamma$,

(ii) if $\phi : U \longrightarrow U'$ and $\phi' : U' \longrightarrow U''$ belong to Γ, then $\phi' \circ \phi \in \Gamma$,

(iii) if $\phi \in \Gamma$, then its restriction to any open connected subset also belongs to Γ,

(iv) if $\phi : U \longrightarrow U'$ is a local diffeomorphism of $\mathbb{R}^n$ which coincides on a neighborhood of each point of U with an element on Γ, then $\phi \in \Gamma$.

The partition (2.21) can be generalized to manifolds as follows: let N be an n-dimensional manifold and $\mathcal{F} = \{L_\lambda\}_{\lambda\in\Lambda}$ a partition of N, i.e., each L_λ is a connected subset of N such that

$$N = \bigcup_{\lambda\in\Lambda} L_\lambda \quad \text{and} \quad L_\lambda \cap L_{\lambda'} = \varnothing, \qquad \text{for } \lambda \neq \lambda'.$$

Consider an integer $0 < m < n$ and a local chart (U, ϕ). Then, (U, ϕ) is an *m-foliated chart* if whenever $L_\lambda \cap U \neq 0$ for some $\lambda \in \Lambda$, then each connected component of $L_\lambda \cap U$ is mapped by ϕ onto an (m, c)-plaque of $\mathbb{R}^n$. Such components $\phi^{-1}(P^m_c)$ are called *the plaques of $\mathcal{F}$ in U* and will be denoted by N^λ_c. An *m-foliated atlas associated to $\mathcal{F}$* on N is a collection of m-foliated charts whose domains cover N. When such a foliated atlas exists, we say that the partition $\mathcal{F}$ of N is a *foliation of dimension* m or *codimension* m' of N. Then, each subset $\{L_\lambda\}$, for $\lambda \in \Lambda$ is called a *leaf* of the foliation. Any leaf of the foliation $\mathcal{F}$ is an m-dimensional immersed submanifold of N, and each N^λ_c is the domain on some local chart of L_λ.

From now on, when dealing with a fixed m-foliation, we will omit the "m" from the names. The definition given above yields the global view of a foliation. The maximal foliated atlas represents all possible local views of it.

Observe that the transition maps between diferent foliated charts on a foliated atlas of dimension m belong to $\Gamma_{n,m'}$. To prove this, consider two foliated charts $\{(U, \varphi = (x^\alpha))\}$ and $\{(\widetilde{U}, \widetilde{\varphi} = (y^\alpha))\}$ such that $U \cap \widetilde{U} \neq \varnothing$, and let $N_c^\lambda, N_{\tilde{c}}^\lambda$ be two plaques in U and $\widetilde{U}$ respectively, such that $N_c^\lambda \cap N_{\tilde{c}}^\lambda \neq \varnothing$. Then, since each of them is mapped to an (m, c)-plaque of R^n, necessarily

$$\frac{\partial}{\partial x^a} = \frac{\partial y^b}{\partial x^a}\frac{\partial}{\partial y^b}, \qquad \text{for } a, b = 0, \ldots, m-1. \tag{2.22}$$

As $U \cap \widetilde{U}$ is covered by intersections of plaques of $\mathcal{F}$, it follows that (2.22) is satisfied in all $U \cap \widetilde{U}$. But in general

$$\frac{\partial}{\partial x^\alpha} = \frac{\partial y^\beta}{\partial x^\alpha}\frac{\partial}{\partial y^\beta},$$

so one deduces that

$$\frac{\partial y^j}{\partial x^i} = 0, \text{ for } j = m, \ldots, n-1 \text{ and } i = 0, \ldots, m-1.$$

The last m' coordinates $x_2 = (x^m, \ldots, x^{n-1})$ of a foliated chart are known as the *transverse coordinates*. Indeed, the connected components of the sets $x_2 = \text{const.}$ in a foliated chart are the plaques of $\mathcal{F}$, the map $x_2 \to (x_1, x_2)$ is a smooth embedding and the plaques are (connected) m-dimensional submanifolds of N. Then, the leaves of the foliation turn out to be (injective) immersed submanifolds composed by a union of plaques.

Example 2.2.4 (Submersions). A smooth map $\pi : M \longrightarrow N$ of smooth manifolds is a *submersion* if for all $p \in M$ the differential $d\pi_p$ is onto. Then, the inverse images of all the points of N constitute a family of closed submanifolds of N of the same dimension $dim(M) - dim(N)$, which are the leaves of the foliation $\mathcal{F}$ of N defined by π.

Indeed, if $\{(U, \varphi = (x^\alpha))\}$ is a foliated chart and $p : \mathbb{R}^n \longrightarrow \mathbb{R}^{m'}$ the natural projection, then the map $f = p \circ \varphi$ is a submersion which defines the plaques of $\mathcal{F}$ on U. There exists another approach to foliations using a family of submersions together with a family of local diffeomorphisms on $\mathbb{R}^{m'}$, called the transition functions that satisfy a cocycle condition, developed by Haefliger in [43] (see also [88], [45]).

2.2.3 Relation between Foliations and Distributions

Let $\mathcal{F}$ be a foliation of dimension m on a manifold N. Then, it is possible to define an m-distribution $D(\mathcal{F})$ in the following way: for each $p \in N$ take the leaf L_λ passing through p and define $D_p = T_pL_\lambda$. If $\{U, \varphi\}$ is a foliated atlas on $\mathcal{F}$, then locally D_p is spanned by $\{\frac{p}{\partial x^0}, \ldots, \frac{p}{\partial x^{m-1}}\}$ and, therefore, $D(\mathcal{F})$ is an integrable distribution.

Conversely, let D be an integrable distribution. Then, for any $p \in N$ there exists a chart $\{(U, \varphi)\}$ on N such that the submanifold of U given by $x^m = const., \ldots, x^{n-1} = const.$ are integral submanifolds of D. Then, the leaves of the foliation are defined as the maximal integral submanifolds of D. The leaf through p is constructed as the sets of points q accesible from p by piecewise smooth paths for which the tangent vectors belong to D.

From now on, we will denote $D(\mathcal{F})$ as $T\mathcal{F}$. Nevertheless, the sections of the subbundle $\pi : T\mathcal{F} \to N$ will be denoted as $\mathfrak{X}(\mathcal{F})$, and analogously for other tensor fields. We will say that $\mathcal{F}$ is a *Riemannian, Lorentzian or semi-Riemannian foliation* if $D(\mathcal{F}) = T\mathcal{F}$ has such character as a distribution. Then, given a semi-Riemannian metric h, the distribution $T\mathcal{F}^{\perp}$ orthogonal to $T\mathcal{F}$ may be non-involutive but its bundle of tensors will still be denoted by $T^r_s(T\mathcal{F}^{\perp})$.

2.3 Holonomy and Reducibility in semi-Riemannian Manifolds

In this section we briefly summarize some well-known notions about when a semi-Riemannian manifold is (locally) isometric to a direct product. The conditions for this to happen are given in terms of the holonomy group of the manifold. After that, we prove a result due to Eisenhart that relates the existence of parallel two-covariant tensor fields with holonomy in Riemannian manifolds, and state similar results for arbitrary signature.

Let N be a semi-Riemannian manifold and $p \in N$. The parallel transport along any loop (a closed curve starting and ending at p) is an automorphism on T_pN, and the set ψ_p constituted by all such automorphisms is a group called *the (homogeneous) holonomy group of N*. In fact, it is possible to prove that the holonomy group does not depend on the point p, so from now on it will be denoted simply by ψ.

2.3.1 Reducibility in semi-Riemannian Manifolds

The holonomy group ψ of a manifold N is said to be *reducible* (by extension, N is called *reducible* too) if there exists a proper subspace $\mathcal{W}_p$ of T_pN with $p \in N$ invariant by the holonomy group. Otherwise, it is *irreducible*. If the invariant subspace $\mathcal{W}_p$ is non-degenerate, then the holonomy group and the manifold are said to be *non-degenerately reducible*.

In the case of positive-definite metrics, a classical result due to de Rham states that if the manifold is reducible, then N is locally isometric to the direct product of the maximal integral manifolds for the integrable distributions obtained by parallel translation over N of the invariant subspace $\mathcal{W}_p$ and its orthogonal. More precisely,

Theorem 2.3.1 (The de Rham decomposition Theorem,[74, 52]). *A connected, simply connected and complete Riemannian manifold $(\overline{M}, \overline{g})$ is isometric to the direct product $\overline{M}_0 \times$*

$\overline{M}_1 \times \ldots \times \overline{M}_s$, where $\overline{M}_0$ is a Euclidean space (possibly of dimension 0) and $\overline{M}_1, \ldots, \overline{M}_s$ are simply connected, complete, irreducible Riemannian manifolds. Such decomposition is unique up to reordering of factors.

Therefore, a reducible Riemannian manifold can be expressed locally as a product of irreducible manifolds plus a Euclidean one. Due to this theorem, when proving some property on Riemannian manifolds one can always assume irreducibility as long as the property also holds for Euclidean spaces. In fact, in this local chart, all geometrical objects (the curvature tensor, the Christoffel symbols, the Ricci tensor, the conection itself, etc.) associated to the connection are decompose accordingly into a sum of terms, each of them belonging to a factor. The proof of these theorems can be found in the original paper due to de Rham [74], and in the book [52].

The de Rham decomposition theorem does not hold in general for other signatures (as reducible and non-degenerately reducible are not equivalent), and it is necessary to assume non-degenerate reducibility:

Theorem 2.3.2 (Wu's non-degenerate reducibility, [92]). *Let (N, h) be a connected, complete and simply connected semi-Riemannian manifold. If N is non-degenerately reducible, N is isometric to a direct product $N_1 \times N_2$.*

Each N_i in the theorem above is the maximal integral submanifold of the distribution T_i obtained by parallel translation of the holonomy-invariant subspace $\mathcal{W}_p$ and its orthogonal $(\mathcal{W}_p)^{\perp}$ over N. The proof of this theorem can be found in [92], and uses the holonomy theorem of Ambrose-Singer [3] (apart from other facts) instead of the "piecing together local isometries into a global one", that is used in the Riemannian case. Of course, if the manifold is non-degenerately reducible, it is locally the product of irreducible manifolds and a flat one (either Euclidean or Lorentzian), as in the Riemannian case.

Example 2.3.3. *If there exists a parallel vector field X on a semi-Riemannian manifold (N, h), then span$\{X\}$ is invariant by the holonomy group and, thus, the manifold is reducible. Actually if X is not lightlike, the manifold is non-degenerately reducible (also called decomposable, see [68, 83]) and the metric can be decomposed into*

$$h = cX^\flat \otimes X^\flat + (h - cX^\flat \otimes X^\flat)$$

where $c = \frac{1}{h(X,X)}$ and $X^\flat \otimes X^\flat$ is the projector to span$\{X\}$.

2.3.2 Parallel Symmetric two-covariant Tensor Fields

Here we summarize some results which relate the existence of a parallel $(0, 2)$-tensor field to the holonomy group of a manifold. But fist we need some basic notions on parallel traslation along geodesics:

Definition 2.3.4. *Let (N, h) be a semi-Riemannian manifold, $p \in N$, $\vec{v} \in T_pN$ and W a normal neighborhood of p. We define* the geodesical extension $V : N \longrightarrow TN$ of $\vec{v}$ in W *as follows: at each $q \in W$ the vector $V|_q$ is obtained by ∇-parallelly transporting $\vec{v}$ along the unique geodesic $\gamma_q : [0,1] \to W$ from p to q, i.e., the geodesic $\gamma(\tau; p, \vec{x})$ (notation as in page 45) for a vector $\vec{x}$ such that $\gamma(0) = p$ and $\gamma(1) = q$.*

This construction satisfies the following:

Lemma 2.3.5. *Any geodesical extension V on W of a vector $\vec{v} \in T_pN$ is a smooth vector field. Moreover, if $\vec{v}, \vec{v}' \in T_pN$ are orthogonal, then so are their geodesical extensions V and V'.*

Proof. Let us prove that V depends smoothly on q. Let $\{\vec{e}_\alpha\}$ be an orthonormal frame of T_pN, $\vec{x} \in T_pN$ such that $\gamma_q : [0,1] \to W$ is the unique geodesic from p to q satisfying $\gamma_{pq}(\tau) = \gamma(\tau; p, \vec{x})$ for some $\vec{x} = x^\alpha \vec{e}_\alpha$. In normal coordinates $\{x^\alpha\}$ associated to $\{\vec{e}_\alpha\}$ on W, $\gamma_q(\tau) = (\tau x^0, \ldots, \tau x^{n-1}) = \tau\vec{x}$. Consequently $\gamma_q(\tau)$ (and thus $\gamma_q'(0) = \vec{x}$) depends smoothly on $q(\equiv (x^0, \ldots, x^{n-1}))$ and on τ. By construction, since $\gamma(\tau; p, \vec{x}) = \gamma(1; p, \tau\vec{x})$ implies $\gamma_q(\tau) = \gamma_{\tau q}(1)$,

$$V_{\gamma_q(\tau)} = V_{\tau q} = X_{\gamma_q(\tau)}$$

where X is the parallel transport of $\vec{v}$ along γ_q. Therefore, V is parallel along any geodesic that starts at p. One can rewrite this condition in normal coordinates as:

$$\frac{d}{d\tau}(V^\alpha_{\gamma_q(\tau)}) + \Gamma^\alpha_{\beta\sigma}(\gamma_q(\tau))V^\beta_{\gamma_q(\tau)} q^\sigma = 0, \ \forall \tau \in [0,1] \tag{2.23}$$

$\Gamma^\alpha_{\beta\sigma}$ is smooth on q, because it is a combination of derivatives of smooth functions on q (the components of the metric g). Consequently, the matrix $A^\alpha_\beta = \Gamma^\alpha_{\beta\sigma}(\gamma_q(\tau))q^\sigma$ depends smoothly on q. So we can always choose a sufficiently small neighbourhood U in where the solution V of (2.23) is smooth on q.

For the last assertion, let $q \in W$ and $\gamma_q : [0,1] \to W$ be the unique geodesic from p to q. If X, Y are the parallel transports of $\vec{v}, \vec{v}'$ along γ_q, respectively, then, $h(X,Y) = h(\vec{v}, \vec{v}') = 0$ along γ_q. So,

$$h_q(V_q, V'_q) = h_q(X(1), Y(1)) = 0, \ \ \forall q \in U. \qquad \blacksquare$$

Then, for positive-definite metrics, we have the Classical Eisenhart Theorem:

Theorem 2.3.6 (The Classical Eisenhart Theorem 1923, [30]). *If a Riemannian manifold $(\overline{M}, \overline{g})$ admits a symmetric two-covariant tensor field $L \in \Gamma(T^0_2\overline{M})$ not proportional to the metric $\overline{g}$ such that $\nabla^{\overline{g}} L = 0$, then*

- *$\overline{g}$ is reducible: $\overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)} \oplus \ldots \oplus \overline{g}^{(s)}$ (with each $\overline{g}^{(m)}$ not necessarily irreducible).*
- *$L = \sum_{m=1}^{s} \lambda_m \, \overline{g}^{(m)}$ for some constants λ_m.*

Proof. Let p be any point of $\overline{M}$. We construct an orthonormal basis of eigenvector fields $V_i \in \mathfrak{X}(\overline{M})$ as we describe now: consider the eigenvalue problem of L_p with respect to $\overline{g}_p$ on the vector space $T_p\overline{M}$, i.e.,

$$L_p(.\,,\vec{v}) - \lambda g_p(.\,,\vec{v}) = 0. \tag{2.24}$$

Since $L_{|p}$ is symmetric and $\overline{g}_{|p}$ is positive-definite, the eigenvalues are real and the eigenspaces associated to them are mutually orthogonal. Take an orthonormal basis $\{\vec{v}_i\}_{i=1}^n$ of eigenvectors of L_p in $T_p\overline{M}$. Extend this basis to a normal neighborhood W of p by taking the geodesical extensions V_i of each $\{\vec{v}_i\}$. Clearly, if $\vec{v}_i$ is a λ-eigenvector, then $V_i|_q$ is an eigenvector of L_q with the same eigenvalue λ, because the one-forms on γ_q defined as $\tau \mapsto L_{\gamma_q(\tau)}(.\,,V_{\gamma_q(\tau)})$ and $\tau \mapsto \lambda\overline{g}_{\gamma_q(\tau)}(.\,,V_{\gamma_q(\tau)})$ are parallel and coincide at p due to (2.24). Besides, $\{V_i\}$ is an orthonormal basis on W.

Therefore, the eigenvalues and the dimension of each eigenspace remain constant all over $\overline{M}$. If we denote any λ-eigenvector field as $V^{(\lambda)}$, and $\lambda_1,\dots,\lambda_s$ are the eigenvalues of L with corresponding multiplicities $m_1,\dots,m_s$, we can reorder the vector fields so that

$$\{V_1^{(\lambda_1)},\dots V_{m_1}^{(\lambda_1)},\dots V_1^{(\lambda_s)},\dots,V_{m_s}^{(\lambda_s)}\}$$

is an orthonormal basis of $T\overline{M}$.

Let λ be one of the eigenvalues and let us prove that the distribution S_λ generated by its eigenvectors is involutive. Taking the ∇_V covariant derivative of $L(\cdot\,,V_i^{(\lambda)}) = \lambda g(\cdot,V_i^{(\lambda)})$ for any $V \in \mathfrak{X}(\overline{M})$ and using that L is ∇-parallel, we have that $\nabla_V V_i^{(\lambda)}$ lies in S_λ, and so does $[V_i^{(\lambda)},V_j^{(\lambda)}]$. Analogously, the distribution $S_\lambda^\perp$ which assigns to each point p' the orthogonal of $(S_\lambda)_{p'}$ in $T_{p'}\overline{M}$ is involutive. So there are m_λ functionally independent functions on $\overline{M}$ which are solutions of the equation $X(f) = 0, \forall X \in S_\lambda^\perp$. Such functions $x_\lambda^i, i = 1,\dots m_\lambda$ will complete a cooordinate chart for $\overline{M}$ when the construction is repeated for all the eigenvalues $\lambda = \lambda_l, l = 1,\dots,s$, in the following sense: in the constructed orthonormal basis $\{V_i\}_{i=1}^n = \{V_1^{(\lambda_1)},\dots V_{m_1}^{(\lambda_1)},\dots V_1^{(\lambda_s)},\dots,V_{m_s}^{(\lambda_s)}\}$ of $T\overline{M}$, grad$(x^{i_\lambda}) = \sum_{j=1}^n \overline{g}(\text{grad}(x^{i_\lambda}),V_j)V_j$, $g(\text{grad}(x^{i_\lambda}),V_j) = dx^{i_\lambda}(V_j) = V_j(x^{i_\lambda})$ and if $V_j \notin S_\lambda$ by definition $V_j(x^{i_\lambda}) = 0$. So grad(x^{i_λ}) belongs to the eigenspace S_λ. Therefore, since dx^{i_λ} are functionally independent, the set $\{\text{grad}(x^{i_\lambda})\}$ is a frame of S_λ. Also, $\overline{g}(\text{grad}(x^{i_\lambda}),\frac{\partial}{\partial x^{j_\lambda}}) = dx^{i_\lambda}(\frac{\partial}{\partial x^{j_\lambda}}) = \delta_{j_\lambda}^{i_\lambda}$, so grad$(x^{i_\lambda})$ is pointwise proportional to $\frac{\partial}{\partial x^{i_\lambda}}$. Consequently, $\{\frac{\partial}{\partial x^{i_\lambda}}\}$ constitute a frame for S_λ.

So we can take the family of n-functions

$$\{x^i\}_{i=1}^n = \bigcup_{\lambda;i_\lambda=1}^{m_\lambda}\{x^{i_\lambda}\}$$

as coordinate functions of the manifold. Then, in these coordinates, if $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^{j'}}$ belong to diferent eigenspaces, since these are mutually orthogonal,

$$g_{ij'} = g(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^{j'}}) = 0.$$

Let Γ^i_{jk} be the Christoffel symbols for the coordinates $\{x^i\}$, x^i be a coordinate function associated to the eigenvalue λ and $x^{j'}, x^{k'}$ associated to a different eigenvalue λ'. Then $\nabla_{\frac{\partial}{\partial x^{j'}}} \frac{\partial}{\partial x^i} \in S_\lambda$ implies $\Gamma^{k'}_{ij'} = 0$. Therefore,

$$\frac{\partial g_{j'k'}}{\partial x^i} = 0.$$

So the metric is reducible.

From $L(\cdot\,, V_i^\lambda) = \lambda g(\cdot, V_i^\lambda)$, it follows that $L(V_i^\lambda, V_j^{\lambda'}) = \lambda' g(V_i^\lambda, V_j^{\lambda'}) = \lambda' \delta_{ij} \delta^{\lambda\lambda'}$. Therefore, $L = \sum_{m=1}^s \lambda_m \, g^{(m)}$ for the frame $\{V_i\}$ and in consequence, for any frame. ■

Corollary 2.3.7. *If an irreducible Riemannian manifold $(\overline{M}, \overline{g})$ admits a symmetric two-covariant tensor field $L \in \Gamma(T_2^0 \overline{M})$ such that $\nabla^{\overline{g}} L = 0$, then L is (constantly) proportional to the metric.*

For Lorentzian manifolds, we have the following result:

Proposition 2.3.8. *If a Lorentzian manifold (M, g) admits a symmetric two-covariant tensor field $L \in \Gamma(T_2^0 M)$ not proportional to the metric g such that $\nabla^g L = 0$, then (M, g) is reducible. Furthermore (M, g) is not non-degenerately reducible if only if there exists a parallel lightlike vector field (up to a constant of proportionality) which is the unique parallel vector field.*

Remember that the classification of a self-adjoint endomorphism in a Lorentzian manifold is done in terms of Segre types (see Appendix B), which are (up to degeneracies):

- $[1, 1 \ldots 1]$ if there exists a timelike eigenvector,
- $[211 \ldots 1]$ if there exists a double lightlike eigenvector,
- $[31 \ldots 1]$ if there exists a triple lightlike eigenvector,
- $[z\overline{z}1 \ldots 1]$ if there are no timelike nor lightlike eigenvectors, and there exist a pair of complex conjugate eigenvectors.

Then, the proof of Proposition 2.3.8 is done in terms of the possible Segre types of L, and shows that if L has Segre type other than $[(1, 1 \ldots 1)]$ (which is also excluded because L is not proportional to g), $[(211 \ldots 1)]$ or $[(31 \ldots 1)]$, the manifold is non-degenerately reducible (Lemma 2.3.9). The cases $[(211 \ldots 1)]$ and $[(31 \ldots 1)]$ are proven to have a unique parallel lightlike vector field, and the latter, having also a parallel spacelike vector field, allows a non-degenerate decomposition of the metric (Lemma 2.3.10).

Lemma 2.3.9. *Let (M, g) be a simply-connected Lorentzian manifold which admits a symmetric two-covariant tensor field $L \in \Gamma(T_2^0 M)$ not proportional to the metric g such that $\nabla^g L = 0$. If L has an eigenvalue with an associated spacelike eigenspace, then the manifold is non-degenerately reducible.*

Proof. Let λ be the eigenvalue of L with an associated spacelike eigenspace S_λ of dimension m_λ. Then, at each $p \in M$ there exists an orthonormal basis of spacelike eigenvectors $\{\vec{v}_1, \ldots, \vec{v}_{m_\lambda}\}$ for S_λ, which can be extended to a frame $\{V_1, \ldots, V_{m_\lambda}\}$ in a starshaped neighborhood by geodesical extension (see Definition 2.3.4). One can prove that the projector P on S_λ defined as $P = \sum_{A=1}^{m_\lambda} V_A^\flat \otimes V_A^\flat$ is a parallel symmetric $(0,2)$-tensor field (concretely, S_λ is the image of its metrically associated self-adjoint endomorphism $\hat{P} = \sum_{A=1}^{m_\lambda} V_A^\flat \otimes V_A$, and $S_\lambda^\perp$ its kernel), as follows. Let X, Y be any two vector fields in M. Observe that since $L(\cdot, V_A) = \lambda g(\cdot, V_A), \forall A$, the eigenvalue λ is constant by Lemma B.12 and L is parallel, $\nabla_X V_A$ belongs to S_λ, so

$$\nabla_X V_A^\flat = \sum_{B=1}^{m_\lambda} f_{AB}(X) V_B^\flat, \qquad \text{for some family of one-forms } \{f_{AB}\}_{A,B=1}^{m_\lambda}$$

Moreover, $g(V_A, V_B) = \delta_{AB}$ implies $g(\nabla_X V_A, V_B) + g(V_A, \nabla_X V_B) = 0$, so

$$(f_{AB} + f_{BA})(X) = 0.$$

Thus,

$$\nabla_X P = \sum_{A,B} (f_{AB} + f_{BA})(X) V_B^\flat \otimes V_A^\flat = 0$$

and g is non-degenerately reducible to

$$g = P + (g - P).$$

This ensures the local non-degenerate reducibility and, by standard considerations on simple connectedness, the global one. ∎

Lemma 2.3.10. *Let (M, g) be a simply connected Lorentzian manifold which admits a symmetric two-covariant tensor field $L \in \Gamma(T_2^0 M)$ not proportional to the metric g such that $\nabla^g L = 0$. If L has a unique eigenvalue, then:*

(1) there exists a unique parallel lightlike vector field, and

(2) the manifold is non-degenerately reducible only if there exists another parallel vector field.

Proof. If L has a unique eigenvalue λ with eigenspace S_λ, then λ is real and the possible Segre types of L are $[(1,1\ldots1)]$, $[(211\ldots1)]$ and $[(31\ldots1)]$. The case $[(1,1\ldots1)]$ is excluded because L is not proportional to the metric. Then, at each $p \in M$ there exists a basis of eigenvectors $\{K_p, V_{i|p}\}$ that span S_λ and such that

$$K_p \text{ is lightlike}, \qquad g(K_p, V_{i|p}) = 0, \qquad g(V_{i|p}, V_{j|p}) = \delta_{ij}$$

and $i, j = 2, \ldots, n-1$ for $[(211\ldots1)]$ while $i, j = 3, \ldots, n-1$ for $[(31\ldots1)]$. By parallel transporting it along geodesics it is possible to extend to a frame $\{K, V_i\}$ in a starshaped neighborhood (see Definition 2.3.4).

Let us prove that K is the required unique parallel lightlike vector field. Uniqueness comes from the fact that if it were not unique, there would exist a parallel timelike eigenvector field and the Segre type would be $[(1,1\dots 1)]$. To prove the existence, add a new lightlike vector field K' and a new spacelike vector field V_2 if necessary to the frame of S_λ to obtain a *null frame* in TM (recall Theorem B.10), so that:

$$g = -K^\flat \otimes K'^{\,\flat} - K'^{\,\flat} \otimes K^\flat + \sum_{i=2}^{n-1} V_i^\flat \otimes V_i^\flat$$

and

$$L = \begin{cases} \lambda(g \pm K^\flat \otimes K^\flat), & \text{for Segre type } [(211\dots 1)], \\ \lambda g + K^\flat \otimes V_2^\flat - V_2^\flat \otimes K^\flat, & \text{for Segre type } [(31\dots 1)], \end{cases}$$

Let X be any vector field. $\nabla_X L = 0$ implies that

$$\nabla_X(K^\flat \otimes K^\flat) = 0, \qquad \text{for Segre type } [(211\dots 1)], \tag{2.25}$$

$$\nabla_X(K^\flat \otimes V_2^\flat - V_2^\flat \otimes K^\flat) = 0, \qquad \text{for Segre type } [(31\dots 1)]. \tag{2.26}$$

Thus:

- for Segre type $[(211\dots 1)]$, from (2.25) automatically K is parallel (observe that since $K \neq 0$ it follows that $\nabla_X K^\flat = \tau(X)K^\flat$ and substituting this in (2.25) one obtains that $\tau(X) = 0$ for all X).
- for Segre type $[(31\dots 1)]$, applying V_2 to (2.26):

  $$0 = g(V_2, V_2)\nabla_X K^\flat + g(\nabla_X V_2, V_2)K^\flat - g(K, V_2)\nabla_X V_2^\flat - g(\nabla_X K, V_2)V_2^\flat.$$

  Since V_2 is a unit spacelike vector field (so $g(V_2, \nabla_X V_2) = 0$) orthogonal to K, it follows that

  $$\nabla_X K^\flat = g(\nabla_X K, V_2)V_2^\flat, \tag{2.27}$$

  i.e., $\nabla_X K$ is proportional to V_2. Applying $\nabla_X K$ to (2.26), one obtains that

  $$0 = g(\nabla_X K, \nabla_X K)V_2^\flat + g(\nabla_X K, K)\nabla_X V_2^\flat + g(\nabla_X K, \nabla_X V_2)K^\flat + g(\nabla_X K, V_2)\nabla_X K^\flat.$$

  Sustituting (2.27) in the above expression, one gets

  $$(g(\nabla_X K, V_2))^2 V_2^\flat = 0,$$

  so $g(\nabla_X K, V_2) = 0$ and K is parallel.

Nevertheless, for example in Segre type $[(31\dots 1)]$, the basis vector field X_2 is also parallel (apply (K', K'), (K', K) and (K', V_i) for all $i = 3, \dots, n-1$ to (2.26) and recall that $g(V_2, \nabla_X V_2)$ to prove it), and the manifold is non-degenerately reducible by Example 2.3.3. So in order for the manifold not to be non-degenerately reducible the existence of another parallel vector field in Segre type $[(211\dots 1)]$ and in general must be excluded. ■

And, as said, the proof of Proposition 2.3.8 is deduced from the above lemmas.

Corollary 2.3.11. *If an irreducible Lorentzian manifold (M, g) admits a symmetric two-covariant tensor field $L \in \Gamma(T_2^0 M)$ such that $\nabla^g L = 0$, then L is (constantly) proportional to the metric.*

There also exists an analogous result for semi-Riemannian manifolds, which extend both theorems above:

Proposition 2.3.12 (Tanno 1967,[86]). *Let (N, h) be an irreducible semi-Riemannian manifold of signature (q, q') and assume that any of these two cases is satisfied:*

- *dim $N = 2$ or odd,*
- *dim $N \geq 4$ is even and $q \neq q'$.*

If it admits a symmetric two-covariant tensor field L such that $\nabla L = 0$, then $L = \lambda h$ for some constant λ.

Chapter 3

Locally Symmetric and Symmetric Spaces

In this Chapter a succint review on locally symmetric and symmetric spaces is given, mostly based on the books [65, 46]. The complete classification of these spaces for Riemannian signature was seted by Cartan in two different ways: the first one based on the holonomy group of the manifold [19], and the second based on the geodesic symmetry at a point (see [20] and [46]), to be developed here for semi-Riemannian manifolds (Proposition 3.1.5(iii) and Definition 3.2.1 below). Then, symmetric spaces are proved to be homogeneous spaces, thus they have a transitive group of isometries, and can be represented as a coset space G/H, where G is a connected Lie Group with an involutive automorphism σ whose fixed points are H. Finally, the particular cases for different signatures are summarized in the last sections.

3.1 Locally Symmetric Spaces

Definition 3.1.1. *A semi-Riemannian manifold* (N, h) *is said to be* locally symmetric *if the curvature tensor field* R *satisfies that its first covariant derivative vanishes, i.e., if* $\nabla R = 0$*.*

Next, we review some other well-known characterizations for locally symmetric semi-Riemannian manifolds. We have summarized them in two propositions in the following way: the first one involves parallelism of the curvature (Proposition 3.1.2), and the second one local properties on the space (Proposition 3.1.5).

Proposition 3.1.2. *For a semi-Riemannian manifold* (N, h) *the following properties are equivalent:*

(i) (N, h) *is locally symmetric.*

(ii) If X, Y *and* Z *are parallel vector fields along a curve* α*, then the vector field* $R(X, Y)Z$ *is also parallel along* α*.*

(iii) *The sectional curvature is invariant under parallel translation, i.e.: for any non-degenerate plane $\Pi_p \subset T_pN$, its parallel transport $\tau \mapsto \Pi_\alpha(\tau)$ along any curve $\alpha(\tau)$ that starts at p has constant (non-degenerate) sectional curvature.*

Proof. To prove that (i) implies (ii), observe that if X, Y and Z are parallel vector fields along a curve α, and α' is its tangent vector field, then the following equality is satisfied:

$$(\nabla_{\alpha'} R)(X,Y)Z = \nabla_{\alpha'}(R(X,Y)Z). \tag{3.1}$$

For (ii)$\Longrightarrow$(iii), let $\alpha(\tau)$ be a curve starting at $p \in N$ with tangent vector field α' and $\Pi_p \subset T_pN$ a non-degenerate plane generated by the vectors $\vec{x}, \vec{y} \in T_pN$. Let $X(\tau), Y(\tau)$ be the parallel transports of $\vec{x}, \vec{y}$ along α with $X(0) = \vec{x}, Y(0) = \vec{y}$, so that $\Pi_\alpha = \text{span}\{X, Y\}$. Then, the sectional curvature $K(\Pi_\alpha)$ is given by the formula

$$K(\Pi_\alpha) = \frac{R(X,Y,X,Y)}{Q(X,Y)} \quad \text{where } Q(X,Y) = g(X,X)g(Y,Y) - (g(X,Y))^2$$

The function $Q(X,Y)$ is always constant and different from zero along α by the parallelness of X, Y and g. If $R(X,Y)Z$ is parallel along α, so it is $R(X,Y,X,Y)$, and the required implication is proven.

For (iii)$\Longrightarrow$(i), let $\vec{z}, \vec{t} \in T_pN$ and denote by $Z(\tau), T(\tau)$ their parallel transports along α, with $Z(0) = \vec{z}, T(0) = \vec{t}$. Then, the curvature tensor can always be written in terms of sectional curvatures in this way: first, observe that $R(X,Y,X,Y) = K(< X,Y >) \cdot Q(X,Y)$. Then expand $R(X+Z,Y,X+Z,Y), R(X,Y+T,Z,Y+T)$ and $R(Y,X+T,Z,X+T)$ to obtain the expressions

$$\begin{aligned}
R(X+Z,Y,X+Z,Y) &= 2R(X,Y,Z,Y) + R(X,Y,X,Y) + R(Z,Y,Z,Y) \\
R(X,Y+T,Z,Y+T) &= R(X,Y,Z,Y) + R(X,T,Z,T) + R(X,Y,Z,T) + R(X,T,Z,Y) \\
R(Y,X+T,Z,X+T) &= R(Y,X,Z,X) + R(Y,T,Z,T) + R(Y,X,Z,T) + R(Y,T,Z,X)
\end{aligned}$$

From the first equation one obtains an expression for $R(X,Y,Z,Y)$ in terms of sectional curvatures and functions of the type $Q(X,Y)$ (which are invariant along α). Substracting the last two expressions and applying the first Bianchi identity one gets

$$\begin{aligned}
3R(X,Y,Z,T) &= R(X,Y+T,Z,Y+T) + R(X,Y,Z,Y) + R(X,T,Z,T) \\
&\quad -R(Y,X+T,Z,X+T) - R(Y,X,Z,X) - R(Y,T,Z,T)
\end{aligned}$$

Therefore, (iii) also implies that $R(X,Y,Z,T)$ is a constant function along α. Then, let $\vec{v} \in T_pN$ be any vector and take the curve α so that $\alpha'(0) = \vec{v}$. From (3.1), it follows that

$$(\nabla_{\vec{v}} R)(\vec{x},\vec{y},\vec{z},\vec{t}) = \frac{d}{d\tau}_{|\tau=0}(R(X,Y,Z,T)) \text{ for any } \vec{v},\vec{x},\vec{y},\vec{z},\vec{t} \in T_pN$$

so by the hypothesis on (iii) the condition in (i) is satisfied. ■

Before stating the second proposition, we need some basic definitions. Let p be in N, U be a neighborhood of p in N and $\vec{v} \in T_pN$. We denote as $\gamma(\tau; p, \vec{v})$ the unique inextendible geodesic on U starting at p with initial velocity $\gamma'(0) = \vec{v}$ evaluated at τ. Observe that $\gamma(\tau; p, \lambda\vec{v}) = \gamma(\lambda\tau; p, \vec{v})$, since the later curve $\alpha(\tau) = \gamma(\lambda\tau; p, \vec{v})$ is a geodesic with the same initial data as the former. Specifically, $\gamma(1; p, \vec{v}) = \gamma(\lambda; p, \frac{1}{\lambda}\vec{v})$.

Let now $\widetilde{W} \subset T_pN$ be a small starshaped neighborhood ($\vec{v} \in \widetilde{W} \Longrightarrow tv \in \widetilde{W},\ \forall t \in [0,1]$) of the origin $\vec{0}$. We define the *exponential mapping* $exp_p : \widetilde{W} \longrightarrow M$ as $exp_p(\vec{v}) = \gamma(1; p, \vec{v})$ for all $\vec{v} \in \widetilde{W}$ (the neighborhood $\widetilde{W}$ must be chosen small enough for γ to be defined on $\tau = 1$). In fact, $\widetilde{W}$ can be chosen so that exp_p restricted to $\widetilde{W}$ is a diffeomorphism. This is because the differential map of exp_p at the origin is identifiable to the identity mapping on T_pN, so applying the inverse function theorem exp_p is a local diffeomorphism around $\vec{0}$. Moreover, choosing an orthonormal basis in T_pN and identifying T_pN with $\mathbb{R}^n$, the mapping $(exp_p)^{-1}$ together with $exp(\widetilde{W})$ becomes a coordinate chart for N. The neighborhood $W = exp(\widetilde{W})$ of p is called a *normal neighborhood*, and the coordinates of the manifold given by this map are called *normal coordinates centered at* p.

It is possible to obtain a description of exponential maps in terms of Jacobi fields ([65, Proposition 6 in page 217]). A *Jacobi field* $V(\tau) \equiv V(\gamma(\tau))$ *on a geodesic* $\gamma(\tau)$ is a vector field along γ such that it satisfies the Jacobi equation

$$\nabla_{\gamma'}(\nabla_{\gamma'}V) + (R(\gamma', V)\gamma') = 0, \quad \text{for all } \tau \in \mathrm{Dom}(\gamma).$$

Now, let $p \in N$ and $\vec{v} \in T_pN$. Then, for any $\vec{w} \in T_{\vec{v}}(T_pN) \simeq T_pN$ (as a vector space isomorphism) it follows that $d(exp_p)_{|\vec{v}}(\vec{w}) = V(1)$, where V is the (unique) Jacobi field on the geodesic $\gamma(\tau; p, \vec{v})$ with initial values $V(0) = 0$ and $(\nabla_{\gamma'}V)(0) = \vec{w}$ (see [65, Lemma 5 in page 217] for existence and uniqueness of such Jacobi field). This result will be useful later.

Definition 3.1.3. *Let (N, h) be a semi-Riemannian manifold, $p \in N$ and W a normal neighborhood of p. The* local geodesic symmetry s_p *with respect to p is defined as the map $s_p : W \longrightarrow W$ such that for each $q \in W$, $s_p(q) = \gamma(-1)$, where γ is the geodesic on W through p and q such that $\gamma(0) = p$ and $\gamma(1) = q$.*

Then, the local geodesic symmetry with respect to p can also be defined as $s_p = exp_p \circ L \circ (exp_p)^{-1}$, where L is the linear isometry of T_pN defined as $\vec{v} \longrightarrow -\vec{v}$ (see [65, page 221-223] for further details). In particular, s_p sends $exp_p(v)$ to $exp_p(-v)$. So by the reasoning above:

Lemma 3.1.4. *The local geodesic symmetry s_p of a semi-Riemannian manifold (N, h) satisfies:*

(1) It is a diffeomorphism.

(2) $(ds_p)_{|p} = L = -Id_{|T_pN}$.

(3) In normal coordinates centered at p, $s_p(q) = -q$.

(4) It reverses geodesics through p: if γ is a geodesic such that $\gamma(0) = p$, then $s_p(\gamma(s)) = \gamma(-s)$ (observe that this last property uniquely determines s_p).

We are now able to state other well-known characterizations for locally symmetric semi-Riemannian manifolds, involving local properties of the manifold.

Proposition 3.1.5. *For a semi-Riemannian manifold (N, h) the following conditions are equivalent:*

(i) (N, h) is locally symmetric.

(ii) If $L : T_pN \longrightarrow T_qN$ is a linear isometry that preserves curvature there exist small normal neighborhoods W of p and W' of q and an isometry $\phi : W \to W'$ such that $d\phi_{|p} = L$.

(iii) The local geodesic symmetry s_p is an isometry at any $p \in N$.

Proof. To prove that (i) implies (ii), define the diffeomorphism $\phi = exp_q \circ L \circ (exp_p)^{-1}$. Clearly, $d\phi_{|p} = L$. So one only has to prove that ϕ is an isometry, i.e., that for any $p' \in W$ and any $\vec{v} \in T_{p'}N$,

$$h_{p'}(\vec{v}, \vec{v}) = h_{\phi(p')}(d\phi_{|p'}(\vec{v}), d\phi_{|p'}(\vec{v})).$$

To that end, Jacobi vector fields are used (see [65, Theorem 14 in page 222]). Let $\vec{v}_0 = (exp_p)^{-1}(p') \in (exp_p)^{-1}(W)$ and $\vec{w}_0 \in T_{\vec{v}_0}(T_pN)$ be the unique vector such that

$$d(exp_p)_{\vec{v}_0}(\vec{w}_0) = \vec{v}.$$

Then, by the description of exponential maps in terms of Jacobi fields presented in page 45 one has, on one hand, that $\vec{v} = V(1)$ with V the unique Jacobi field along the geodesic $\gamma(\tau; p, \vec{v}_0)$ such that $V(0) = 0$ and $(\nabla_{\gamma'} V)(0) = \vec{w}_0$. On the other hand, as L is linear,

$$dL_{|\vec{v}_0}(\vec{w}_0) = L(\vec{w}_0) \in T_{L(\vec{v}_0)}(T_qN) \simeq T_qN,$$

and by definition

$$d\phi_{|p'}(\vec{v}) = d(exp_q)_{|L(\vec{v}_0)}(L(\vec{w}_0)).$$

Thus, as above, there exists a unique Jacobi field $\overline{V}$ along the geodesic $\overline{\gamma}(\tau; q, L(\vec{v}_0))$ such that $\overline{V}(0)$, $(\nabla_{\overline{\gamma}'} \overline{V})(0) = L(\vec{w}_0)$ and $d\phi_{|p'}(\vec{v}) = \overline{V}(1)$. Our task is to prove that

$$h(V(1), V(1)) = h(\overline{V}(1), \overline{V}(1)),$$

that is, that the corresponding Jacobi fields grow at the same rate. To that end, take an orthonormal basis $\{\vec{e}_0, \ldots, \vec{e}_{n-1}\}$ in T_pN and map it to another orthonormal basis $\{\vec{\overline{e}}_0, \ldots, \vec{\overline{e}}_{n-1}\}$ in T_qN via the linear isometry L, so that $L(\vec{e}_\alpha) = \vec{\overline{e}}_\alpha$. Propagate parallelly each basis along γ and $\overline{\gamma}$ respectively, to obtain two orthonormal frames $E(\tau) \equiv E(\gamma(\tau))$ and $\overline{E}(\tau) \equiv \overline{E}(\overline{\gamma}(\tau))$ along the geodesics. Since L a linear isometry, the components of $\vec{v}_0$ and $\vec{w}_0$ with respect to the basis $\{\vec{e}_\alpha\}$ are the same as the components of $L(\vec{v}_0)$ and $L(\vec{w}_0)$ relative to $\{\vec{\overline{e}}_\alpha\}$. So, as $\gamma'(\tau) = a^\alpha E_\alpha$ with each a^α of constant value (from the geodesic character of γ and the parallelism of the frame $\{E_\alpha\}$ along γ), it follows that $\overline{\gamma}'(\tau) = a^\alpha \overline{E}_\alpha$.

Thus, if $V = V^\alpha E_\alpha$ and $\overline{V} = \overline{V}^{\,\alpha} \overline{E}_\alpha$, the functions $V^\alpha(\tau)$ satisfy the differential equations (see [65, proof of Lemma 5 in page 217])

$$\frac{d^2 V^\alpha}{d\tau^2} = R^\alpha{}_{\beta\lambda\mu}(\gamma(\tau)) a^\beta V^\lambda a^\mu,$$

while the $\overline{V}^{\,\alpha}$-s satisfy

$$\frac{d^2 \overline{V}^{\,\alpha}}{d\tau^2} = R^\alpha{}_{\beta\lambda\mu}(\overline{\gamma}(\tau)) a^\beta \overline{V}^\lambda a^\mu,$$

both with the same initial conditions, as has been argued before. Therefore, it only remains to show that

$$R^\alpha{}_{\beta\lambda\mu}(\gamma(\tau)) = R^\alpha{}_{\beta\lambda\mu}(\overline{\gamma}(\tau)) \tag{3.2}$$

in the common domain of definition of the geodesics, so that $V^\alpha(\tau)$ and $\overline{V}^{\,\alpha}(\tau)$ satisfy the same differential equations with same initial conditions, and by the uniqueness of such solutions, $V^\alpha(\tau) = \overline{V}^{\,\alpha}(\tau)$. As $\{E_\alpha\}$ and $\{\overline{E}_\alpha\}$ are orthonormal frames, the result would follow.

Observe that, on one hand, as L preserves the metric, $R^\alpha{}_{\beta\lambda\mu}(\gamma(0)) = R^\alpha{}_{\beta\lambda\mu}(\overline{\gamma}(0))$. On the other hand, as N is locally symmetric, by Proposition 3.1.2, the functions $R^\alpha{}_{\beta\lambda\mu}(\gamma(\tau))$ and $R^\alpha{}_{\beta\lambda\mu}(\overline{\gamma}(\tau))$ are constant along the respective geodesics, so (3.2) holds.

$(ii) \Longrightarrow (iii)$: since $(ds_p)_p = -Id_{|T_pN}$ preserves curvature, we know that there exists an isometry ϕ (a priori not unique) such that $d\phi_{|p} = -Id_{|T_pN}$. In fact, since the curve $\alpha(\tau) = \phi \circ \gamma(\tau; p, v)$ is a geodesic with initial data $(p, \vec{v})$, it follows that $\phi \circ \gamma(\tau; p, v) = \gamma(\tau; p, -v) = \gamma(-\tau; p, v)$. So ϕ reverses geodesics though p, and this property uniquely determines the local geodesic symmetry at p. Therefore, $\phi = s_p$.

$(iii) \Longrightarrow (i)$:Let now p be any point at N and $\vec{x}, \vec{y}, \vec{z}, \vec{v} \in T_pN$ arbitrary. Since s_p is an isometry, it preserves curvature and covariant derivatives. So:

$$-(\nabla_{\vec{v}} R)(\vec{x}, \vec{y})\vec{z} = (\nabla_{-\vec{v}} R)(-\vec{x}, -\vec{y})(-\vec{z}) = (\nabla_{\vec{v}} R)(\vec{x}, \vec{y})\vec{z}$$

i.e., $\nabla R = 0$, as required. ■

3.2 Symmetric Spaces

It is possible to enlarge the notion of local symmetry by requiring that the local geodesic symmetries are extendible to isometries of all the manifold. Such spaces have been called *symmetric spaces*. In fact, they will turn out to be complete, in contrast to locally symmetric manifolds, which need not be complete, since any open submanifold of it is again locally symmetric. Moreover, when simply connectedness is assumed, they are the universal covering of complete connected locally symmetric spaces.

Definition 3.2.1. *A semi-Riemannian* symmetric space *is a connected semi-Riemannian manifold (N, h) such that for each p in N there is a (unique) isometry $S_p : N \longrightarrow N$ with differential map $-Id_{|T_pN}$ at p.*

Observe that S_p also reverses the geodesics through p. In fact, S_p restricted to a normal neighborhood is indeed the local geodesic symmetry s_p, so one can think of S_p as an extension to all of N of the local geodesic symmetry s_p. Hence S_p becomes *the global geodesic symmetry of N at p*. There also exists another characterization of symmetric spaces:

Lemma 3.2.2. *Let (N, h) be a semi-Riemannian manifold. Then, these conditions are equivalent:*

(i) *(N, h) is a symmetric manifold.*

(ii) *Each $p \in N$ is an isolated fixed point of an* involutive isometry *σ_p of N, i.e., for each point p in N there exists an isometry $\sigma_p : N \longrightarrow N$ such that:*

- *(fixed point) $\sigma_p(p) = p$,*
- *(isolated point) in some neighborhood of p there is no other point q' satisfying $\sigma_p(q') = q'$, and*
- *(involutive diffeomorphism) $L = (d\sigma_p)_p$ verifies that $L^2 = I_{|T_pN}$ but $L \neq I_{|T_pN}$.*

Proof. If we take $\sigma_p = s_p(= S_p)$, (ii) follows from (i). For the converse, we denote $L = (d\sigma_p)_{|p}$. Then, as $L^2 = I_{T_pN}$, it is diagonalizable and for each eigenvector $\vec{x} \in T_pN$, either $L(\vec{x}) = \vec{x}$ or $L(\vec{x}) = -\vec{x}$. Suppose there exists a vector $\vec{x}_0 \neq 0$ such that $L(\vec{x}_0) = \vec{x}_0$. Then, the curve $\alpha(\tau) = \sigma_p \circ \gamma(\tau; p, \vec{x}_0)$ is again the geodesic $\gamma(\tau; p, \vec{x}_0)$, so γ is pointwise invariant, which is a contradiction with (ii). So $L(\vec{x}) = -\vec{x}$ for all $\vec{x} \in T_pN$. Therefore, σ_p is in fact S_p and (i) follows. ■

3.2.1 Relations between Locally Symmetric and Symmetric Spaces

Proposition 3.2.3. *Any semi-Riemannian symmetric space is (geodesically) complete and locally symmetric.*

Proof. The last assertion is obvious, since $S_p = s_p$ by the reasoning above. For the first one, a geodesic curve $\alpha : [0, b) \longrightarrow N$ can be extended to all $[0, \infty)$ by making use of the fact that S_p is an isometry defined in all N which reverses geodesics through the point p in the following way: let $c \in [0, b)$ such that $b/2 < c < b$. Take the geodesic symmetry $S_{\alpha(c)}$ centered at $\alpha(c)$. Then, by the isometric character of $S_{\alpha(c)}$ and its own definition, the curve $\gamma := S_{\alpha(c)} \circ \alpha : [0, b) \longrightarrow N$ is a geodesic from some point $\gamma(0)$ to $\lim_{\tau \nearrow b} \gamma(\tau) = \alpha(2c - b)$ passing through $\gamma(2c - b) = \lim_{\tau \nearrow b} \alpha(\tau)$. So a reparametrization of γ provides the required extension of α (see Figure 3.1). ■

Then:

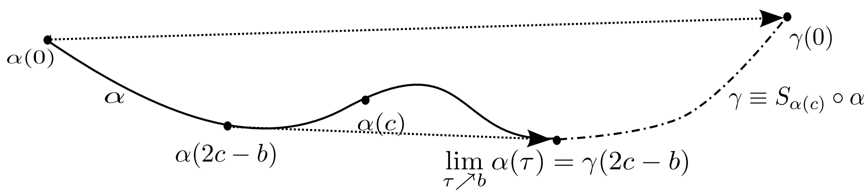

Figure 3.1: Extension of a geodesic curve α in a symmetric space.

Theorem 3.2.4 ([65], page 225). *Let N, N' be complete, connected, locally symmetric semi-Riemannian manifolds with N simply connected. If $L : T_pN \longrightarrow T_qN'$ is a linear isometry that preserves curvature, then there is a unique semi-Riemannian covering map $\phi : N \longrightarrow N'$ such that $d\phi_p = L$.*

From this theorem, one can deduce the following (see [46, page 222] for an alternative proof assuming analyticity):

Proposition 3.2.5. *Any complete, simply connected, locally symmetric semi-Riemannian manifold is symmetric.*

As a consecuence of this proposition, it follows that:

Corollary 3.2.6 ([46], page 218). *Let (N, h) be a complete semi-Riemannian locally symmetric space. If (N', h') is the universal covering space of N, with covering map $\pi : N' \longrightarrow N$ and $h = \pi^* h$, then (N', h') is a semi-Riemannian symmetric space.*

3.2.2 Symmetric spaces as coset manifolds

We are going to prove that a symmetric space is a homogeneous space. Therefore, it can be constructed in terms of Lie groups and Lie algebras (see Appendix A). This and the existence of an involutive automorphism of the Lie group makes it clear how to construct symmetric spaces from Lie group data.

Consider the isometry group $I(N) = \{\phi : N \longrightarrow N : \phi \text{ is an isometry}\}$ of a semi-Riemannian manifold (N, h) —it is a group with the composition of functions—. There is a unique way to make $I(N)$ into a Lie group such that the natural mapping $\varphi : I(N) \times N \longrightarrow N$ with $\varphi(\phi, p) = \phi(p)$ is smooth and, therefore, $I(N)$ becomes a Lie transformation group on N. This result follows[1] from general results of Palais Chapter IV in [67]. Moreover, one can prove that $I(N)$ is a transitive Lie transformation group on a symmetric space N. To that end, we will prove that any symmetric space is a homogeneous space.

Definition 3.2.7. *A semi-Riemannian manifold (N, h) is* homogeneous *if given any two points $p, q \in N$ there exists an isometry ϕ of N such that $\phi(p) = q$ (i.e., if $I(N)$ is a transitive Lie transformation group of N).*

[1]It is also possible to find the proof of this result for symmetric manifolds in [46]

To be more precise, a homogeneous space is a space that exhibits enough isometries to carry some fixed point to every point of the manifold. In particular:

Proposition 3.2.8. *A semi-Riemannian homogeneous manifold (N, h) is diffeomorphic to G/H, where $G = I(N)$ and H is the isotropy group of G at any point $p \in N$.*

Proof. It follows from Theorem A.5 ∎

In the case of semi-Riemannian symmetric manifolds, the set of isometries that make them homogeneous are the global symmetries S_p.

Proposition 3.2.9. *A semi-Riemannian symmetric manifold is homogeneous.*

Proof. A symmetric manifold, by definition, is connected, so any points p, q in N can be joined by a broken geodesic. Let now γ be a geodesic defined on $[0, 1]$. The global symmetry $S_{\gamma(1/2)}$ is an isometry that carries the point $\gamma(0)$ to the point $\gamma(1)$ (because it reverses geodesics). So by an appropriate composition of isometries it is possible to carry p to q by an isometry. ∎

Thus, symmetric spaces satisfy all the well-known properties of homogeneous spaces. For example, each tangent vector $\vec{v} \in T_pN$ for all $p \in N$ extends to a Killing vector field on N, and, by Proposition 3.2.8 symmetric spaces can be expressed as coset manifolds G/H. Now, let us focus on the metric viewpoint:

Lemma 3.2.10 ([65], page 315)**.** *Let (N, h) be a semi-Riemannian symmetric manifold and $G = I(N)$. Then, the mapping $\sigma : g \longrightarrow S_{p_0} \circ g \circ S_{p_0}$ is an involutive automorphism of G such that $(F_\sigma)_0 \subset H \subset F_\sigma$, where F_σ is the set of all fixed points of σ and $(F_\sigma)_0$ its identity component.*

At last, this result helps for the following theorem:

Theorem 3.2.11 ([65], page 315)**.** *Let G be a connected Lie group, H a closed subgroup, and $\phi : G \longrightarrow G/H$ the natural mapping, with $p = \phi(e) \in G/H$. If there exists an involutive automorphism σ of G such that $(F_\sigma)_0 \subset H \subset F_\sigma$, where F_σ is the set of all fixed points of σ, then any G-invariant metric on G/H makes G/H into a semi-Riemannian symmetric space such that $S_p \circ \pi = \pi \circ \sigma$, where S_p is the global symmetry at p.*

Then, the symmetry conditions can be expressed in terms of Lie algebras as follows:

Proposition 3.2.12 ([46], page 208)**.** *Let (N, h) be a semi-Riemannian symmetric space, $G = I(N)$, H the isotropy group of G at some point p_0 of N, and $\mathfrak{g}$, $\mathfrak{h}$ the Lie algebras of G and H respectively. Then, if σ is the involutive automorphism given in Lemma 3.2.10, we have*

$$\mathfrak{h} = \{X \in \mathfrak{g} : (d\sigma)_e X = X\}.$$

Furthermore, defining $\mathfrak{p} = \{X \in \mathfrak{g} : (d\sigma)_e X = -X\}$, then $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ (direct sum), and:

(i) $Ad_h(\mathfrak{p}) \subset \mathfrak{p}, \forall h \in H$, *i.e.,* $\mathfrak{p}$ *is* $Ad(H)$*-invariant.*

(ii) $[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}$; $[\mathfrak{h},\mathfrak{p}] \subset \mathfrak{p}$, *i.e.,* $\mathfrak{h}$ *is a subalgebra of* $\mathfrak{a}$ *and an ideal of* $\mathfrak{p}$, *and* $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{h}$.

Definition 3.2.13. *We say that* (G, H, σ, B) *is a* symmetric data, *provided that:*

(i) G *is a connected Lie group.*

(ii) H *is a closed subgroup of* G.

(iii) σ *is an involutive automorphism of* G *such that* $(F_\sigma)_0 \subset H \subset F_\sigma$, *where* F_σ *is the set of all fixed points of* σ.

(iv) B *is an* $Ad(H)$*-invariant scalar product on* $\mathfrak{p} = \{X \in \mathfrak{g} : (d\sigma)_e X = -X\}$.

Given a symmetric data (G, H, σ, B), from Proposition A.6, Theorem 3.2.11 and Lemma 3.2.12 a unique semi-Riemannian symmetric manifold is constructed.

3.2.3 Riemannian Symmetric Spaces

Every simply-connected Riemannian symmetric space can be expressed as a product of three diferent types of symmetric spaces, whose topological and geometrical properties are quite different. For these three particular cases, the group G becomes semisimple (i.e., the associated Lie algebra $\mathfrak{g}$ is semisimple), and the problem is reduced to the study of certain involutive automorphisms of semisimple Lie algebras. The three different types are the Euclidean space, and two more, the compact and non-compact type, defined as follows:

Definition 3.2.14. *Let* $\overline{M} = G/H$ *be a Riemannian symmetric manifold. Then,* $\overline{M}$ *is of* compact type *if the Killing form* B *of* G *is negative definite on* $\mathfrak{g}$ *and of* non-compact type *if* B *is negative definite on* $\mathfrak{h}$ *and positive-definite on* $\mathfrak{p}$.

A locally symmetric Riemannian manifold also satisfies the following:

Proposition 3.2.15. *Let* $(\overline{M}, \overline{g})$ *be a locally symmetric Riemannian manifold. Then,*

(1) $(\overline{M}, \overline{g})$ *is locally isometric to the direct product of a finite number of irreducible locally symmetric spaces and a Euclidean space of dimension* $d \geq 0$.

(2) If $(\overline{M}, \overline{g})$ *is irreducible then it is an Einstein manifold, i.e., Ric* $= cg$, *with* c *constant.*

(3) If $(\overline{M}, \overline{g})$ *is Ricci-flat (that is, Einstein with* $c = 0$*), then it is flat.*

Proof. (1) This a consequence of the classical de-Rham decomposition of $\overline{M}$, as any irreducible part must be locally symmetric (see Section 2.3).

(2) As the Ricci tensor in a locally symmetric space is parallel, the result follows from the classical Eisenhart Theorem 2.3.6.

(3) By hypothesis, $(\overline{M}, \overline{g})$ is locally isometric to a Ricci-flat symmetric space and, by a result in [1], this space must be flat[2]. ■

[2] In fact, Alekseevskii and Kimelfeld [1] proved that any Ricci-flat homogeneous Riemannian space is

3.2.4 Lorentzian Symmetric Spaces and other signatures

The classification of the Lorentzian simply-connected symmetric spaces was carried out by Cahen and Wallach in [17] where they obtained the following:

Theorem 3.2.16. *Any simply-connected Lorentzian symmetric space (M, g) is isometric to the product of a simply-connected Riemannian symmetric space and one of the following Lorentzian manifolds:*

(a) $(\mathbb{R}, -dt^2)$

(b) the universal cover of d-dimensional de Sitter or anti-de Sitter spaces, $d \geq 2$*,*

(c) A Cahen-Wallach space $CW^d(A) = (\mathbb{R}^d, g_A)$*,* $d \geq 2$*, where* $A = (A_{ij})$ *is a* $(d-2) \times (d-2)$ *symmetric constant matrix, and the metric is written*

$$g_A = -2du\left(dv + A_{ij}x^i x^j du\right) + \delta_{ij}dx^i dx^j, \tag{3.3}$$

where δ_{ij} *is the Kronecker delta (observe that* $CW^2 = \mathbb{L}^2$*, as* A *necessarily vanishes).*

Therefore, if a Lorentzian symmetric space admits a parallel lightlike vector field, then it is locally isometric to the product of a d*-dimensional Cahen-Wallach space and an* $(n-d)$*-dimensional Riemannian symmetric space with* $d \geq 2$*.*

Extensions to other signatures and to non-simply-connected cases are also available, see Cahen and Parker [16], Neukirchner [63] and specially Kath and Olbrich [49, 50].

flat. It is worth pointing out that, as a difference with the locally symmetric case, the locally homogeneus spaces maybe non-regular, that is, non-locally isometric to some homogeneous space [53]. However, Spiro [82] showed that all locally homogeneous spaces with non-positive Ricci curvature are regular, hence the result in [1] can be extended to the locally homogeneous case. Nevertheless, it cannot be extended to the Lorentzian case: it is easy to find a counterexample in the Cahen-Wallach spaces below.

Chapter 4

Brinkmann Spaces

Brinkmann spaces have received increasing attention in recent years, see e.g. [4, 5, 32]. In this chapter, we derive some of their properties relevant to our problem. A Lorentzian manifold (M, g) is a *Brinkmann space* if it admits a parallel lightlike vector field K.

In what follows, the lightlike parallel vector field K will always be assumed to be specified. Later in Chapter 6, the vector field K is proven to be unique for proper 2nd-symmetric Brinkmann spaces (Corollary 6.2.2).

In Section 4.1 we will introduce the *Brinkmann charts* and *Brinkmann decompositions*. Then, a decomposition of the tensor bundles of a Brinkmann space (M, g) in terms of foliations $\overline{\mathcal{M}}$,$\mathcal{U}$ associated to a fixed Brinkmann chart is given in Section 4.2. With this information at hand, an exhaustive study of the foliation $\overline{\mathcal{M}}$ is made in Section 4.3; an exterior derivative $\overline{d}$, a covariant derivative $\overline{\nabla}$ and the dot operator, plus the curvature tensor $\overline{\mathcal{R}}$ associated to $\overline{\nabla}$ are defined and some consequences on the structure of the foliation are proven. In Section 4.4, the connection ∇ and the curvature tensor R of the manifold are calculated in a partly null frame (recall Definition 2.1.1) and in Section 4.5 a new transverse operator for $\overline{\mathcal{M}}$, the operator D_0, is introduced. This new operator will be useful to simplify the expressions for ∇R and $\nabla\nabla R$ in a given partly null frame. Finally, in Section 4.6 new notions of reducibility on the foliation $\overline{\mathcal{M}}$ plus an extended Eisenhart Theorem are given.

4.1 Brinkmann charts and Brinkmann decompositions

In [15] (see also [93]) a very handful chart is presented for Brinkmann spaces: any point $p \in M$ of a Brinkmann space (M, g) admits a coordinate chart $\{x^\alpha\} = \{u, v, x^i\}$, which from now on will be called *Brinkmann chart* [1], in such a way that the metric takes the

[1]Without loss of generality, we will assume that the range of the coordinates includes $|u|, |v|, |x^i| < \epsilon$ for some $\epsilon > 0$.

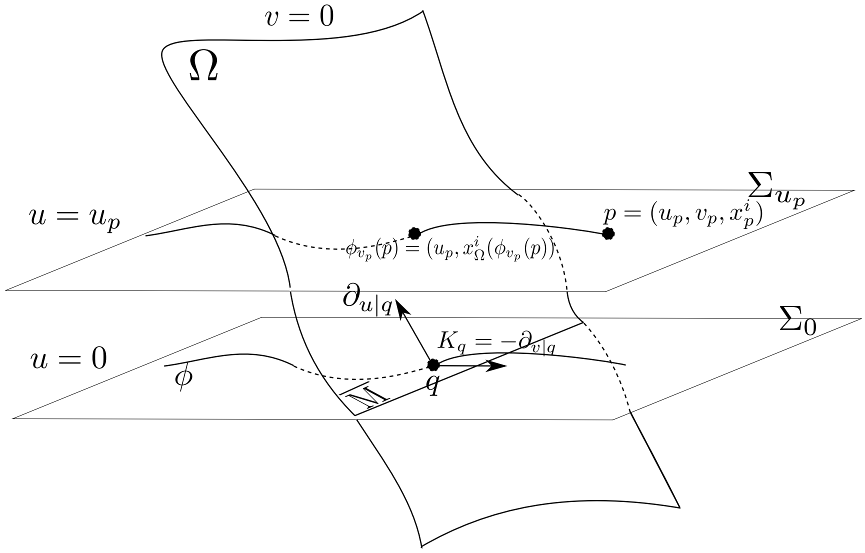

Figure 4.1: Construction of a Brinkmann chart for a fixed lightlike parallel vector field $K = -\partial_v$.

form

$$g = -2du(dv + H(u, x^k)du + W_i(u, x^k)dx^i) + g_{ij}(u, x^k)dx^i dx^j, \quad k = 2, \ldots, n-1. \tag{4.1}$$

where the functions H and W_i are arbitrary, $g_{ij} = g_{ji}$ define a positive-definite metric, otherwise they are also arbitrary, and the parallel vector field K is $-\partial_v$.

These charts can be obtained by means of the following process (see Figure 4.1). As K is parallel, by Frobenius theorem there exists a function $u : U \longrightarrow \mathbb{R}$ defined on a neighborhood U of M such that $K^\flat = du$, with the value 0 in its image. Then, each level set $\Sigma_{u_0} = u^{-1}(u_0)$ is a lightlike integral manifold of the distribution $K^\perp$ orthogonal to K. Set $\Sigma = \Sigma_0$ and choose a hypersurface Ω which is transverse to both Σ and K. The function u will serve as a coordinate for M as well as for Ω. In Ω choose a coordinate neighborhood $\{u, x^i_\Omega\}$ completing u. The coordinate v on M is defined by using the flow ϕ of K to move each point $p \in M$ with coordinates (u_p, v_p, x^i_p) to the point $\phi_{v_p}(p) \in \Sigma_{u_p} \cap \Omega$, so that $\Omega = \{v = 0\}$. Then, put $x^i_p = x^i_\Omega(\phi_{v_p}(p))$. Observe that p has been moved by using the flow of $K = -\partial_v$ so that $1 \equiv du(\partial_u) = g(\nabla u, \partial_u) = g(K, \partial_u) = -g(\partial_v, \partial_u)$.

Conversely, the expression (4.1) selects K as $-\partial_v$.

Then, a *Brinkmann decomposition* is determined locally by the pair (Σ, Ω), i.e., by the pair of functions $\{u, v\}$ constructed as above. Obviously, the same Brinkmann decomposition $\{u, v\}$ may serve as the first two coordinates for different Brinkmann charts.

4.1.1 Change of Coordinates between Brinkmann Charts

Let (M, g) be a Brinkmann chart with fixed parallel lightlike vector field K. If $\{u', v', x'^i\}$ denotes a second Brinkmann chart which overlaps $\{u, v, x^i\}$, the corresponding Σ' will also be a level set of u, and Ω' can be regarded as a graph of a function $-F(u, x^i)$ on Ω. So, the change of coordinates can be written as

$$u' = u - u_0, \quad v' = v + F(u, x^j); \quad x'^i = x'^i(u, x^j) \tag{4.2}$$

Consequently, the relations between H, W_i, g_{ij} in the original coordinates $\{u, v, x^i\}$ on a neighborhood U and H', W'_i, g'_{ij} in the new ones $\{u', v', x'^i\}$ on another neighborhood U' with $U \cap U' \neq \emptyset$ are (using notation in (1.4)):

$$H = H' + \dot{F} + W'_i \dot{x}'^i - \frac{1}{2} g'_{ij} \dot{x}'^i \dot{x}'^j \tag{4.3}$$

$$W_i = W'_j \frac{\partial x'^j}{\partial x^i} + F_{,i} - \frac{1}{2} g'_{jk} \left(\frac{\partial x'^j}{\partial x^i} \dot{x}'^k + \frac{\partial x'^k}{\partial x^i} \dot{x}'^j \right) \tag{4.4}$$

$$g_{ij} = g'_{kl} \frac{\partial x'^k}{\partial x^i} \frac{\partial x'^l}{\partial x^j}$$

Observe that the functions H and W_i do not behave as the components of a tensor field, in the following sense: define a one-form W on U whose components in the coordinate basis $\{\partial_u, \partial_v, \partial_{x^i}\}$ are W_i so that $W = W_i dx^i$. Then, when doing the change of coordinates (4.2) between Brinkmann charts, the components of the restriction of W to $U \cap U'$ in the new chart do not coincide, in general, with the values W'_i in the metric g'. Nevertheless, g_{ij} does change as a (positive-definite bilinear) $(0, 2)$-tensor field for any change of coordinates between Brinkmann charts.

4.1.2 Special Choices of Brinkmann Decompositions

The freedom in the choice of Ω makes it possible to obtain a Brinkmann decomposition $\{u, v\}$ with a Brinkmann chart such that $H \equiv 0 \equiv W_i$; in particular ∂_u is lightlike and geodesic in the associated chart, see e.g. [81]. For the sake of completeness, let us construct such a coordinate chart (see Figure 4.2). Choose some Σ as above and take any $(n-2)$-submanifold $\overline{M} \hookrightarrow \Sigma$ which is transverse to K. All such $\overline{M}$ are locally isometric, as K is a (parallel) lightlike direction in Σ. Now, consider for each $x \in \overline{M}$ the *unique lightlike direction* $\vec{l}_x$ orthogonal to $\overline{M}$ and linearly independent of K_x. In a small neighborhood, construct Ω by taking the geodesics with initial velocity $\vec{l}_x$ for all $x \in \overline{M}$. Complete the chart by choosing some local coordinates $\{x^i_{\overline{M}}\}$ in $\overline{M}$ and defining the coordinates x^i_Ω at each $y \in \Omega$ by $x^i_\Omega(y) := x^i_{\overline{M}}(x_y)$, where x_y is the unique point in $\overline{M}$ which lies on the same lightlike geodesic as y. Notice that Ω is a lightlike hypersurface, and the corresponding coordinate vector field ∂_u spans its radical, i.e., $H = 0 = W_i$, as required. The coordinate v is obtained as above, by using the flow ϕ of K to move each point $p \in M$ to a point $y \in \Sigma_{u_y} \cap \Omega$, when $u_y = u(y)$.

We emphasize that the value of H and $W_i dx^i$ depend even on the choice of the coordinates $\{x^i\}$. Moreover, given a Brinkmann decomposition $\{u, v\}$, there exists a Brinkmann chart $\{u, v, x^i\}$ with $H = 0 = W_i$ if and only if the hypersurface Ω obtained

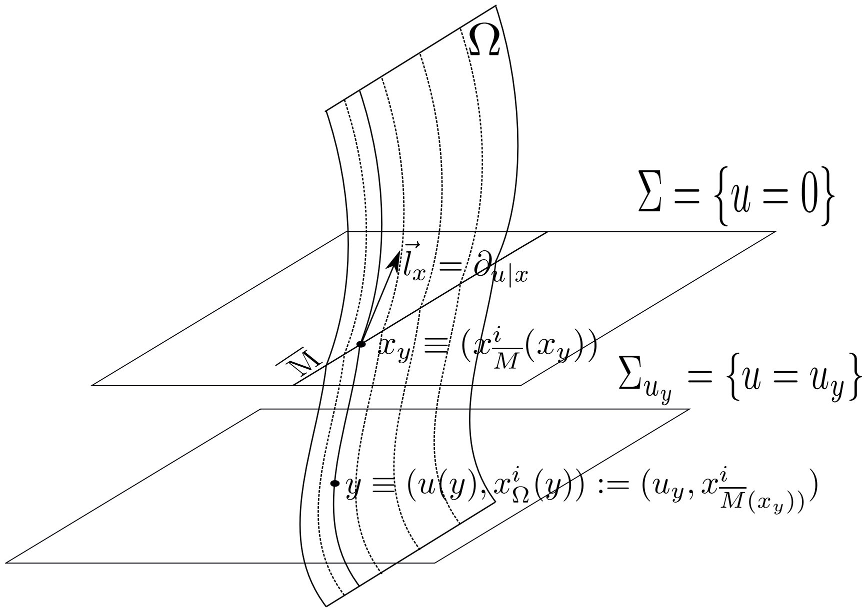

Figure 4.2: Construction of a Brinkmann chart with $H = 0 = W_i$.

as $v = 0$ is lightlike (when Ω is lightlike, the integral curves of its radical must be lightlike pregeodesics of M, because of the local maximizing properties of these curves). In spite of its simplicity, such a Brinkmann decomposition will not be especially relevant for our study. It might simplify some intermediate computations, but they are not well adapted to the generalized Cahen-Wallach spaces of order 2 (see Section 5.4) which will turn out to be, as already announced in Theorem 1.1, the essential part of proper 2nd-symmetric Lorentzian manifolds.

4.2 The Tensor Bundles decomposed by Foliations

Any Brinkmann chart allows for the decomposition of the tensor bundles in different ways. Some of them will be detailed here. Such decompositions will be constructed in terms of some transverse foliations associated to Brinkmann charts. In fact, the decomposition of the tangent bundle TM of the manifold by means of the distributions $T\overline{\mathcal{M}}, T\mathcal{U}$ associated to the foliations $\overline{\mathcal{M}}, \mathcal{U}$ (to be introduced below) will allow us to simplify the subsequent work in components substantially (as we shall see later in Section 6.3).

4.2.1 The Natural Transerve Foliations

To each Brinkmann chart we can associate canonically a partly null frame (as in Definition 2.1.1) given by

$$\{E_\alpha\} = \{\partial_u - H\partial_v,\ \partial_v,\ -W_i\partial_v + \partial_i\},\ \{\theta^\alpha\} = \{du, dv + Hdu + W_j dx^j, dx^i\},\quad i = 2, \dots, n-1 \tag{4.5}$$

which has $g(E_i, E_j) = g_{ij}$.

Each integral curve of ∂_v is labelled by $\{u = u_0, x^i = x_0^i\}$, and each hypersurface $\Omega_{v_0} \equiv \{v = v_0\}$ is a general pseudo-Riemannian hypersurface (possibly signature-changing, as studied systematically in [59]) and isometric to Ω. Recall that the vector space spanned by $\{E_0, E_1\}$ is the same as the one by $\{\partial_u, \partial_v\}$ and $E_2, \dots, E_{n-1}$ are orthogonal to $\{\partial_u, \partial_v\}$. We will repeatedly use the two natural transverse foliations associated to each Brinkmann decomposition, namely:

(1) The $(n-2)$-dimensional foliation $\overline{\mathcal{M}}$ with leaves defined by moving the Riemannian submanifold $\overline{M} = \Sigma \cap \Omega$ with the flows of ∂_u and ∂_v. Each leaf is defined by $\{u = u_0, v = v_0\}$ and represented by $\overline{M}_{(u_0,v_0)}$. The induced metric will be denoted by $\overline{g}$ ($\overline{g} : T\overline{\mathcal{M}} \times T\overline{\mathcal{M}} \to \mathbb{R}$), so that $\overline{g}_{ij} = g_{ij}$ in any Brinkmann chart. When necessary, $\overline{g}$ will be regarded as a metric on a single leaf. This foliation depends only on the Brinkmann decomposition (Σ, Ω), or equivalently on the chosen functions $\{u, v\}$.

The distribution $T\overline{\mathcal{M}}$ associated to the foliation is spanned by the vector fields $\{\partial_i\}$, while the distribution $(T\overline{\mathcal{M}})^\perp$ is spanned by $\{\partial_v, \partial_u + g^{ij} W_j \partial_i\}$, where (g^{ij}) is the inverse matrix of (g_{ij}).

(2) The 2-dimensional foliation $\mathcal{U}$ whose leaves are the surfaces obtained by moving each single point with the flows of ∂_u and ∂_v. Each leaf is given by $\{x^i = c_0^i\}$ for some constants c_0^i and the induced metric is $-2du(dv + Hdu)$. This foliation depends on the expression of ∂_u and, thereby, on the coordinates chosen on Ω for the Brinkmann chart (remember that ∂_v is fixed).

The distribution $T\mathcal{U}$ associated to the foliation is spanned by the vector fields $\{\partial_u, \partial_v\}$ (or equivalently $\{E_0, E_1\}$), while the distribution $(T\mathcal{U})^\perp$ is spanned by $\{E_i\}$. The brackets $[E_i, E_j] = (\partial_j W_i - \partial_i W_j)\partial_v \in \Gamma(T\mathcal{U})$ measure its lack of involutivity.

4.2.2 Decomposition of the Tensor Bundle

Fix a Brinkmann chart and consider the associated foliations $\overline{\mathcal{M}}$, $\mathcal{U}$ and distributions $T\overline{\mathcal{M}}$, $T\mathcal{U}$ and $(T\mathcal{U})^\perp$. The decompositions $TM = T\mathcal{U} \oplus (T\mathcal{U})^\perp$ and $TM = T\mathcal{U} \oplus T\overline{\mathcal{M}}$ (the latter, non-orthogonal sum) associated to any Brinkmann chart yield, on the one hand, the projection

$$\begin{array}{rcl} \mathcal{P}_{\overline{\mathcal{M}}} : \mathfrak{X}(M) = \Gamma\left(T\mathcal{U} \oplus (T\mathcal{U})^\perp\right) & \longrightarrow & \mathfrak{X}(\overline{\mathcal{M}}) \approx \Gamma(\{0\} \oplus T\overline{\mathcal{M}}), \\ X & \mapsto & \overline{X} \end{array}$$

so that $\mathcal{P}_{\overline{\mathcal{M}}}(\partial_u) = \mathcal{P}_{\overline{\mathcal{M}}}(\partial_v) = 0$ and $\mathcal{P}_{\overline{\mathcal{M}}}(E_i) = \partial_i$, and, on the other hand the natural inclusion $\mathcal{I}_{\overline{\mathcal{M}}} : \mathfrak{X}(\overline{\mathcal{M}}) \longrightarrow \mathfrak{X}(M)$ with the corresponding dual map, which we denote by

$$\begin{array}{rcl} \mathcal{I}^*_{\overline{\mathcal{M}}} : \Lambda(M) & \longrightarrow & \Lambda(\overline{\mathcal{M}}) \\ \beta & \mapsto & \overline{\beta}. \end{array}$$

Clearly, $\{\overline{E_i}\} = \{\partial_i\}$ is a basis in $T\overline{\mathcal{M}}$. Since $\overline{\theta^i}(\partial_j) = \theta^i(I_{\overline{\mathcal{M}}}(\partial_j)) = dx^i(\partial_j)$, its cobasis, denoted by $\{\overline{\theta}^{\,i}\}$, is $\{\overline{dx^i}\}$. It is straighforward to see that, in the given bases $\{\partial_i\}$ and $\{\overline{\theta}^{\,i}\}$, if $X = X^\alpha E_\alpha \in \mathfrak{X}(M)$ and $\beta = \beta_\alpha \theta^\alpha \in \Lambda(M)$, then $\overline{X}^i = X^i$ and $\overline{\beta}_i = \beta_i$.

Definition 4.2.1. *We define two linear homomorphisms* $^-$ *and* $^\circ$ *between spaces of sections as:*

(1) $^-\!: \Gamma(T^k_s M) \longrightarrow \Gamma(T^k_s \overline{\mathcal{M}})$*, which maps each* $T = T^{\alpha_1 \ldots \alpha_k}_{\beta_1 \ldots \beta_s} E_{\alpha_1} \otimes \ldots \otimes E_{\alpha_k} \otimes \theta^{\beta_1} \otimes \ldots \otimes \theta^{\beta_s}$ *into* $\overline{T} = T^{i_1 \ldots i_k}_{j_1 \ldots j_s} \overline{E}_{i_1} \otimes \ldots \otimes \overline{E}_{i_k} \otimes \overline{\theta}^{j_1} \otimes \ldots \otimes \overline{\theta}^{j_s}$*,*

(2) $^\circ : \Gamma(T^k_s \overline{\mathcal{M}}) \longrightarrow \Gamma(T^k_s M)$*, which maps each* $T = T^{i_1 \ldots i_k}_{j_1 \ldots j_s} \overline{E}_{i_1} \otimes \ldots \otimes \overline{E}_{i_k} \otimes \overline{\theta}^{j_1} \otimes \ldots \otimes \overline{\theta}^{j_s}$ *into* $\mathring{T} = T^{i_1 \ldots i_k}_{j_1 \ldots j_s} E_{i_1} \otimes \ldots \otimes E_{i_k} \otimes \theta^{j_1} \otimes \ldots \otimes \theta^{j_s}$*.*

By definition $\mathring{\overline{E_i}} = E_i$ and $\mathring{\overline{\theta^i}} = \theta^i$. However, in general, $\overline{\mathring{T}} = T$ but $\mathring{\overline{T}} \neq T$, because in the first case $T \in \Gamma(T^k_s \overline{\mathcal{M}})$ necessarily, but not in the second. In fact, both maps $^-$ and $^\circ$ satisfy:

(1) They are $C^\infty(M)$ linear homomorphisms between spaces of sections, they commute with contractions, tensor products and, when applicable, wedge products in a natural way, i.e., if C^i_j is the contraction between two indices in $\{2, \ldots, n-1\}$:

$$\overbrace{A+B}^{\circ} = \mathring{A} + \mathring{B}; \quad \overbrace{A \otimes C}^{\circ} = \mathring{A} \otimes \mathring{C}; \quad \overbrace{C^i_j(A)}^{\circ} = C^i_j(\mathring{A}),$$

for all $A, B \in \Gamma(T^k_s \overline{\mathcal{M}}), C \in \Gamma(T^{k'}_{s'} \overline{\mathcal{M}})$,

$$\overline{T+Q} = \overline{T} + \overline{Q}; \quad \overline{T \otimes S} = \overline{T} \otimes \overline{S}; \quad \overline{C^i_j(T)} = C^i_j(\overline{T}),$$

for all $T, Q \in \Gamma(T^k_s M), S \in \Gamma(T^{k'}_{s'} M)$ and,

$$\begin{array}{ll} \overbrace{\tau \wedge \xi}^{\circ} = \mathring{\tau} \wedge \mathring{\xi}, & \text{for all } \tau \in \Lambda^s(\overline{\mathcal{M}}), \xi \in \Lambda^{s'}(\overline{\mathcal{M}}) \\ \overline{\alpha \wedge \beta} = \overline{\alpha} \wedge \overline{\beta}, & \text{for all } \alpha \in \Lambda^s(M), \beta \in \Lambda^{s'}(M) \end{array}$$

(2) They leave invariant the $C^\infty(M)$ functions considered as (0,0)-tensor fields:

$$\mathring{f} = \overline{f} = f \text{ for all } f \in C^\infty(M).$$

(3) They have trivial expressions in the introduced bases $\{E_\alpha\}, \{\overline{E}_i\}$. Essentially, all components of the tensor fields and their images remain equal except for the fact that, in the case of $T \mapsto \overline{T}$, the components aligned with any of E_0, E_1, θ^0 or θ^1 must be dropped while, in the case of $T \mapsto \mathring{T}$, these components must be restored with value equal to zero.

Therefore, this simplifies our subsequent work in components substantially, as we will not need to distinguish notationally among different tensor fields derived from a single one: it will be enough to realize which space of sections is being considered. As important illustrative examples, observe that the functions W_i in the Brikmann expression (4.1) can be regarded as the components in the basis $\{\overline{E_i}\} = \{\partial_i\}$ of a one-form section in $\overline{\mathcal{M}}$, namely: $W = W_i\overline{\theta}^i \in \Lambda(\overline{\mathcal{M}})$ (so that $\mathring{W} = W_i dx^i \in \Lambda(M)$), as well as the components in the basis $\{E_\alpha\}$ of the one-form section $\mathring{W}$ in M. Analogously, g_{ij} can be regarded as the components of the projection $P_{\overline{\mathcal{M}}}(g) = \overline{g} \in \Gamma(T^0_2\overline{\mathcal{M}})$, which is the inherited metric on $\overline{\mathcal{M}}$. Therefore, the metric g can be rewritten as

$$g = -2du(dv + Hdu + \mathring{W}) + \mathring{\overline{g}}.$$

One should also keep in mind that, when a single leaf $\overline{M}_{(u,v)}$ is considered, g_{ij} will also denote the components of the induced metric $\overline{g}$ on the leaf, with no explicit mention to (u, v) or the underlying Brinkmann decomposition.

4.3 The foliation $\overline{\mathcal{M}}$

Using the previously obtained vector bundle decomposition associated to each Brinkmann chart, three differential operators $\overline{d}$, $\overline{\nabla}$ and $\dot{}$ ("dot") are introduced next, together with the notion of v-invariance. The operator $\overline{\nabla}$ will depend on the Brinkmann decomposition $\{u, v\}$ only, while the other two will depend on the whole Brinkmann chart $\{u, v, x^i\}$. Then, the curvature tensor $\mathcal{R}$ of the foliation $\overline{\mathcal{M}}$ associated to the operator $\overline{\nabla}$ is also defined.

4.3.1 v-invariant Tensor Fields

The variation on $\overline{\mathcal{M}}$ of $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ under displacements along the direction $E_1 = \partial_v (= -K)$ will be rather irrelevant for us. In fact, E_1 is parallel and, therefore, all the interesting objects to be used in Chapter 6 will also be invariant by its flow. More precisely:

Definition 4.3.1. *A tensor field $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ on $\overline{\mathcal{M}}$ is v-invariant if $\mathring{T}$ is Lie-parallel along the flow of E_1, that is, $\mathcal{L}_{E_1}\mathring{T} = 0$.*

Obviously as $E_1 = \partial_v$, a tensor field $T = T^{i_1 \ldots i_k}_{j_1 \ldots j_s}\partial_{i_1} \otimes \ldots \otimes \partial_{i_k} \otimes \overline{d}x^{j_1} \otimes \ldots \otimes \overline{d}x^{j_s} \in \Gamma(T^k_s\overline{\mathcal{M}})$ is v-invariant if and only if $\partial_v(T^{i_1 \ldots i_k}_{j_1 \ldots j_s}) = 0$. For example, $\overline{g}$, H and W in (4.1) are v-invariant. Indeed, if T is v-invariant then it is determined by its value on any of the hypersurfaces $\Omega_{v_0} = \{v = v_0\}$, and any section $T_{\Omega_{v_0}} \in \Gamma(\Omega_{v_0}, T^k_s\overline{\mathcal{M}})$, defined only on Ω_{v_0}, can be extended to a unique v-invariant section $T_{\overline{\mathcal{M}}} \in \Gamma(T^k_s\overline{\mathcal{M}})$.

There is an alternative definition of v-invariance.

Proposition 4.3.2. *$T \in \Gamma(T^k_s\overline{\mathcal{M}})$ is v-invariant if and only if $\mathring{T}$ is parallel along the direction E_1.*

Proof. As E_1 is parallel, $\nabla_{E_1} Q = \mathcal{L}_{E_1} Q$ for any $Q \in \Gamma(T^k_s M)$. ■

4.3.2 The $\dot{}$ (dot) Derivative

We introduce the following simple derivative:

Definition 4.3.3. The dot derivative $\dot{T} \in \Gamma(T^k_s\overline{\mathcal{M}})$ *of a tensor field* $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ *is defined as* $\dot{T} = (\overline{\mathcal{L}_{\partial_u}\mathring{T}})$.

The components of $\dot{T}$ in the coordinate basis $\{\partial_i\}$ are $\dot{T}^{i_1\ldots i_k}_{j_1\ldots j_s} = \partial_u(T^{i_1\ldots i_k}_{j_1\ldots j_s})$. Indeed, the first usage of the dot for functions was the definition (1.4). The derivative of a v-invariant section with respect to the dot operator is also v-invariant. Hence, this operator can be naturally defined for sections in $\Gamma(\Omega_{v_0}, T^k_s\overline{\mathcal{M}})$.

4.3.3 The $\overline{\mathcal{M}}$ Exterior Derivative $\overline{d}$

Definition 4.3.4. *The* $\overline{\mathcal{M}}$ exterior derivative $\overline{d} : \Lambda^s(\overline{\mathcal{M}}) \to \Lambda^{s+1}(\overline{\mathcal{M}})$ *associated to a Brinkmann chart* $\{u, v, x^i\}$ *is defined by:*

$$\overline{d}\beta = \overline{d\mathring{\beta}}, \qquad \forall \beta \in \Lambda^s(\overline{\mathcal{M}}),$$

where d is the usual exterior derivative on the manifold M.

Lemma 4.3.5. *$\forall \tau \in \Lambda^s(M)$, $\overline{d}\,\overline{\tau} = \overline{d\tau}$. In particular, $\overline{d}f = \overline{df}$ for $f \in C^\infty(M)$.*

Proof. By definition, $\overline{d}\,\overline{\tau} = \overline{d\mathring{\overline{\tau}}}$. If $\tau = \frac{1}{s!}\tau_{\alpha_1\ldots\alpha_s} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_s} \in \Lambda^s(M)$, then $\mathring{\overline{\tau}} = \frac{1}{s!}\tau_{i_1\ldots i_s} dx^{i_1} \wedge \ldots \wedge dx^{i_s} \in \Lambda^s(M)$, and $d\mathring{\overline{\tau}} = \frac{1}{s!}\partial_\alpha(\tau_{i_1\ldots i_s}) dx^\alpha \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s}$, so

$$\overline{d\mathring{\overline{\tau}}} = \frac{1}{s!}\partial_m(\tau_{i_1\ldots i_s}) dx^m \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_s},$$

which is the expression for $\overline{d\tau}$. ■

Thus $\overline{\theta^i} = \overline{dx^i} = \overline{d}x^i$, so the result above allows us to use the expressions ∂_i and $\overline{d}x^i$ instead of $\overline{E_i}$ and $\overline{\theta}^i$, respectively. Moreover, using the notation introduced in (1.4),

$$df = \dot{f} du + \partial_v f dv + \overbrace{\overline{d}f}^{\circ} . \tag{4.6}$$

Indeed, $\overline{d}$ satisfies the usual properties of the exterior derivative d on a manifold:

Proposition 4.3.6. *Let $\beta \in \Lambda^s(\overline{\mathcal{M}})$. The operator $\overline{d}$ satisfies the following properties:*

(1) It is linear and $\overline{d}(\beta \wedge \tau) = \overline{d}\beta \wedge \tau + (-1)^s \beta \wedge \overline{d}\tau$ for all $\tau \in \Lambda^q(\overline{\mathcal{M}})$.

(2) $\overline{d}(\overline{d}\beta) = 0$.

(3) If $\beta = \frac{1}{s!}\beta_{i_1 \dots i_s}\overline{d}x^{i_1} \wedge \dots \overline{d}x^{i_s}$, then $\overline{d}\beta = \frac{1}{s!}\partial_m(\beta_{i_1\dots i_s})\overline{d}x^m \wedge \overline{d}x^{i_1} \wedge \dots \overline{d}x^{i_s}$

(4) (Poincaré Lemma). If $\overline{d}\beta = 0$, then for each $p \in M$ there is a neighborhood U and a section of $(s-1)$-forms τ such that $\overline{d}\tau = \beta$ on U. Moreover, if the $\overline{d}$-closed s-form β is v-invariant, then so can be chosen the $(s-1)$-form τ.

Proof. (1) Straightforward.
(2) $\overline{d}(\overline{d}\beta) := \overline{d}(\overline{d\mathring{\beta}}) = \overline{d(d\mathring{\beta})} = 0$.
(3) Apply (1), (2) and $\overline{d}f = \overline{df}$ in $\overline{d}(\frac{1}{s!}\beta_{i_1\dots i_s}\overline{d}x^{i_1} \wedge \dots \overline{d}x^{i_s})$.
(4) If $\beta = \frac{1}{s!}\beta_{i_1\dots i_s}\overline{d}x^{i_1} \wedge \dots \overline{d}x^{i_s}$, define, for an arbitrary $(s-2)$-form Σ, the $(s-1)$-form $\tau = \frac{1}{(s-1)!}\tau_{i_2\dots i_s}\overline{d}x^{i_2} \wedge \dots \overline{d}x^{i_s} + \overline{d}\Sigma \in \Lambda^{s-1}(\overline{\mathcal{M}})$ with

$$\tau_{i_2\dots i_s} \equiv \tau_{i_2\dots i_s}(u,v,x^2,\dots,x^{n-1}) = x^{i_1}\left(\int_0^1 \sigma^{s-1}\beta_{i_1\dots i_s}(u,v,\sigma x^2,\dots,\sigma x^{n-1})d\sigma\right) + f \quad (4.7)$$

for arbitrary $f \equiv f(u,v)$. Then, $\overline{d}\tau = \frac{1}{(s-1)!}(\partial_m \tau_{i_2\dots i_s})\overline{d}^m \wedge \overline{d}x^{i_2} \wedge \dots \wedge \overline{d}x^{i_s}$ and

$$\begin{aligned}\partial_m \tau_{i_2\dots i_s} &= \frac{\partial}{\partial x^m}\left(x^{i_1}\left(\int_0^1 \sigma^{s-1}\beta_{i_1\dots i_s}(u,v,\sigma x^2,\dots,\sigma x^{n-1})d\sigma\right) + f(u,v)\right)\\ &= \int_0^1 \sigma^{s-1}\beta_{mi_2\dots i_s}(u,v,\sigma x^k)d\sigma + x^{i_1}\int_0^1 \sigma^{s-1}\frac{\partial}{\partial x^m}\left(\beta_{i_1\dots i_s}(u,v,\sigma x^k)\right)d\sigma.\end{aligned}$$

Setting $y^k = \sigma x^k$ for all $k = 2,\dots,n-1$ to integrate by parts the first term and then using $\frac{\partial}{\partial y^k} = \frac{1}{\sigma}\frac{\partial}{\partial x^k}$ one gets

$$\begin{aligned}\partial_m \tau_{i_2\dots i_s} &= \frac{\sigma^s}{s}\beta_{mi_2\dots i_s}(u,v,y^k)\Big|_0^1 - \int_0^1 \frac{\sigma^{s-1}}{s}\frac{\partial}{\partial x^l}\left(\beta_{mi_2\dots i_s}(u,v,y^k)\right)x^l d\sigma\\ &\quad + x^{i_1}\int_0^1 \sigma^{s-1}\frac{\partial}{\partial x^m}\beta_{i_1\dots i_s}(u,v,y^k)dt\\ &= \frac{1}{s}\beta_{mi_2\dots i_s}(u,v,x^k) + x^{i_1}\int_0^1 \sigma^{s-1}\left(\frac{\partial}{\partial x^m}\beta_{i_1\dots i_s}(u,v,y^k) - \frac{1}{s}\frac{\partial}{\partial x^{i_1}}\beta_{mi_2\dots i_s}(u,v,y^k)\right),\end{aligned}$$

so

$$\begin{aligned}\overline{d}\tau &= \frac{1}{(s-1)!}\Bigg[\frac{1}{s}\beta_{mi_2\dots i_s}\overline{d}x^m \wedge \overline{d}x^{i_2} \wedge \dots \wedge \overline{d}x^{i_s}\\ &\quad + x^{i_1}\int_0^1 \sigma^{s-1}\left(\frac{\partial}{\partial x^m}\beta_{i_1\dots i_s}(u,v,y^k) - \frac{1}{s}\frac{\partial}{\partial x^{i_1}}\beta_{mi_2\dots i_s}(u,v,y^k)\right)\overline{d}x^m \wedge \overline{d}x^{i_2} \wedge \dots \wedge \overline{d}x^{i_s}\Bigg]\end{aligned}$$

As $\beta = \frac{1}{s!}\beta_{i_1 \ldots i_s} \overline{d}x^{i_1} \wedge \ldots \overline{d}x^{i_s}$, it follows that (recall Section 2.1.3 for square bracket notation)

$$\begin{aligned} \overline{d}\tau &= \beta \\ &+ x^{i_1} \frac{1}{(s-1)!} \int_0^1 \sigma^{s-1} \left(\partial_{[m}\beta_{|i_1|i_2 \ldots i_s]} - \frac{1}{s} \partial_{i_1}\beta_{[mi_2 \ldots i_s]} \right) (u, v, y^k) \overline{d}x^m \wedge \overline{d}x^{i_2} \wedge \ldots \wedge \overline{d}x^{i_s}. \end{aligned} \tag{4.8}$$

As $\overline{d}\beta = 0$ (i.e., $\partial_{[m}\beta_{i_1 \ldots i_s]} = 0$), using the antisymmetry character of β between any of two fixed indices (for this computation, the generalized Kronecker delta can also be used):

$$\begin{aligned} 0 &= \partial_{[i_1}\beta_{mi_2 \ldots i_s]} = \partial_{i_1}\beta_{[mi_2 \ldots i_s]} - \partial_{[m}\beta_{|i_1|i_2 \ldots i_s]} + \ldots + (-1)^s \partial_{[m}\beta_{i_2 \ldots i_s]i_1} \\ &= \partial_{i_1}\beta_{[mi_2 \ldots i_s]} - s\partial_{[m}\beta_{|i_1|i_2 \ldots i_s]} \end{aligned}$$

so $\partial_{i_1}\beta_{[mi_2 \ldots i_s]} = s\partial_{[m}\beta_{|i_1|i_2 \ldots i_s]}$. Sustituting this in (4.8), the result follows.

If β is v-invariant, choose the $(s-2)$-form Σ to be v-invariant and $f \equiv f(u)$ in (4.7) in order to have v-invariance in τ. ■

Observe that, for this operator too, the exterior derivative of a v-invariant s-form is also v-invariant. Hence, this operator can be naturally defined for sections in $\Gamma(\Omega_{v_0}, T^k_s\overline{\mathcal{M}})$.

4.3.4 The Covariant Derivative $\overline{\nabla}$ on $\overline{\mathcal{M}}$

Definition 4.3.7. *The* covariant derivative $\overline{\nabla}$ *on* $\overline{\mathcal{M}}$ *associated to a Brinkmann decomposition* $\{u, v\}$ *is the map*

$$\begin{aligned} \overline{\nabla}: \quad \mathfrak{X}(\overline{\mathcal{M}}) \times \mathfrak{X}(\overline{\mathcal{M}}) &\longrightarrow \mathfrak{X}(\overline{\mathcal{M}}) \\ (X, Y) &\to \overline{\nabla}_X Y \end{aligned}$$

defined at each point $p \in M$ *by*

$$(\overline{\nabla}_X Y)_p := (\nabla^{\bar{g}}_{\check{X}} \check{Y})_p$$

where $\check{X}$ *and* $\check{Y}$ *are the restrictions of* X, Y *to the leaf* $M_{(u(p),v(p))}$ *and* $\nabla^{\bar{g}}$ *the Levi-Civita connection of the first fundamental form* $\bar{g}$ *on that leaf.*

Clearly, $\overline{\nabla}$ satisfies the formal properties of a symmetric covariant derivative. Observe that, for $X, Y \in \mathfrak{X}(M)$, $\overline{\nabla_X Y} \neq \overline{\nabla}_{\overline{X}}\overline{Y}$ in general (for example, if $X = E_0$ then $\overline{X} = 0$).

Indeed, it is possible to extend the definition of $\overline{\nabla}_X Y$ to any tensor field section $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ in the following manner: let X be any vector field section in $\mathfrak{X}(\overline{\mathcal{M}})$ and define $\overline{\nabla}_X f = X(f)$ for any smooth function f. Then, since $\nabla^{\bar{g}}$ commutes with contractions, so does $\overline{\nabla}$. Therefore, for any $Y \in \mathfrak{X}(\overline{\mathcal{M}})$ and $\omega \in \Lambda(\overline{\mathcal{M}})$:

$$\overline{\nabla}_X (C^1_1(\omega \otimes Y)) = C^1_1(\overline{\nabla}_X(\omega \otimes Y)) = C^1_1(\overline{\nabla}_X \omega \otimes Y) + C^1_1(\omega \otimes \overline{\nabla}_X Y),$$

so

$$\overline{\nabla}_X(\omega(Y)) = (\overline{\nabla}_X\omega)(Y) + \omega(\overline{\nabla}_X Y) \Longrightarrow (\overline{\nabla}_X\omega)(Y) = X(\omega(Y)) - \omega(\overline{\nabla}_X Y).$$

This construction is extended to any tensor field $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ as follows:

$$\begin{aligned}(\overline{\nabla}_X T)(\omega_1,\dots,\omega_k, Y^1,\dots,Y^s) &= X(T(\omega_1,\dots,\omega_k,Y^1,\dots,Y^s)) \\ &-T(\overline{\nabla}_X\omega_1\dots,\omega_k,Y^1,\dots,Y^s) - \dots - T(\omega_1,\dots,\overline{\nabla}_X\omega_k,Y^1,\dots,Y^s) \\ &+T(\omega_1,\dots,\omega_k,\overline{\nabla}_X Y^1,\dots,Y^s) + \dots + T(\omega_1,\dots,\omega_k,Y^1,\dots,\overline{\nabla}_X Y^s),\end{aligned}$$

for any $\omega_1,\dots,\omega_k \in \Lambda(\overline{\mathcal{M}})$ and $Y^1,\dots,Y^s \in \mathfrak{X}(\overline{\mathcal{M}})$.

Therefore, observe that $\overline{\nabla}$ also satisfies the property

$$\overline{\nabla}\overline{g} = 0.$$

The Covariant Derivative $\overline{\nabla}$ and Brinkmann Charts

Let $\{x^\alpha\}$ be a fixed Brinkmann chart. Define the *Christoffel symbols* $\overline{\Gamma}^i_{jk}$ of $\overline{\nabla}$ in the basis $\{\partial_i\}$ associated to the given Brinkmann chart by the relation

$$\overline{\nabla}_{\partial_j}\partial_i = \overline{\Gamma}^k_{ij}\partial_k.$$

From $\overline{\nabla}\overline{g} = 0$ one can derive in a standar manner the formula for the Christoffel symbols in terms of the metric $\overline{g}$:

$$\overline{\Gamma}^i_{jk} = \frac{1}{2}\overline{g}^{il}\left(\partial_j\overline{g}_{lk} + \partial_k\overline{g}_{jl} - \partial_l\overline{g}_{jk}\right)$$

where $(\overline{g}^{ij})$ is the inverse matrix of $(\overline{g}_{ij})$. Therefore, the Christoffel symbols are smooth and v-invariant functions, since the metric $\overline{g}$ does not depend on the coordinate v.

Then, the covariant derivative $\overline{\nabla}$ on $\overline{\mathcal{M}}$ for any $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ can be expressed in the basis $\{\partial_i\}$ of $\overline{\mathcal{M}}$ with cobasis $\{\bar{d}x^i\}$ as:

$$\overline{\nabla}_m T^{i_1\dots i_k}_{j_1\dots j_s} = \partial_m(T^{i_1\dots i_k}_{j_1\dots j_s}) + \sum_{a=1}^{k}\overline{\Gamma}^{i_a}_{lm}T^{i_1\dots i_{(a-1)}l i_{(a+1)}\dots i_k}_{j_1\dots j_s} - \sum_{b=1}^{s}\overline{\Gamma}^{l}_{j_b m}T^{i_1\dots i_k}_{j_1\dots j_{(b-1)}l j_{(b+1)}\dots j_s} \tag{4.9}$$

Again, the derivative of a v-invariant section with respect to $\overline{\nabla}$ is also v-invariant, and this operator can be naturally defined for sections in $\Gamma(\Omega_{v_0}, T^k_s\overline{\mathcal{M}})$. Thus, $\overline{\nabla}T_{\Omega_{v_0}}$ is the restriction of $\overline{\nabla}T_{\overline{\mathcal{M}}}$ to Ω_{v_0}.

4.3.5 The curvature tensor $\overline{\mathcal{R}}$ on $\overline{\mathcal{M}}$

The covariant derivative $\overline{\nabla}$ yields a natural *curvature tensor* $\overline{\mathcal{R}}$ *of the foliation* $\overline{\mathcal{M}}$ defined formally as the usual curvature of $\overline{\nabla}$:

$$\overline{\mathcal{R}}(X,Y)Z = (\overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]})Z \in \mathfrak{X}(\overline{\mathcal{M}}),\ \forall X,Y,Z \in \mathfrak{X}(\overline{\mathcal{M}}). \tag{4.10}$$

From here one can define the *Ricci tensor* $\overline{\mathcal{R}\mathrm{ic}}$ and *scalar curvature* $\overline{\mathcal{S}}$ *of* $\overline{\mathcal{M}}$ in the standar manner. All of them satisfy the standard symmetries corresponding to a curvature tensor.

Observe that, given a Brinkmann chart $\{x^\alpha\}$, from the Ricci identities for $\overline{\nabla}^{\overline{g}}$ on each leaf of $\overline{\mathcal{M}}$ (see equation (2.1)) a *Ricci identity for* $\overline{\nabla}$ in the basis $\{\partial_i\}$ follows automatically for any section $T \in \Gamma(T^k_s\overline{\mathcal{M}})$:

$$(\overline{\nabla}_n\overline{\nabla}_m - \overline{\nabla}_m\overline{\nabla}_n)T^{i_1\ldots i_k}_{j_1\ldots j_s} = \sum_{b=1}^{s}\overline{\mathcal{R}}^l{}_{j_b nm}T^{i_1\ldots i_k}_{j_1\ldots j_{b-1}lj_{b+1}\ldots j_s} - \sum_{a=1}^{k}\overline{\mathcal{R}}^{i_a}{}_{lnm}T^{i_1\ldots i_{a-1}li_{a+1}\ldots i_k}_{j_1\ldots j_s} \quad (4.11)$$

Properties of the foliation associated to $\overline{\mathcal{R}}$

Definition 4.3.8. *Fix a Brinkmann decomposition $\{u, v\}$. The foliation $\overline{\mathcal{M}}$ is called u-*Einstein *if $\overline{\mathcal{R}ic} = \mu\overline{g}$ for some function $\mu \equiv \mu(u)$ (that is, with $d\mu \wedge du = 0$). In particular, when μ is constant, $\overline{\mathcal{M}}$ is called* Einstein, *and when $\mu \equiv 0$ we say that $\overline{\mathcal{M}}$ is* Ricci-flat.

In the case $\overline{\mathcal{R}} = 0$ (respectively $\overline{\nabla}\,\overline{\mathcal{R}} = 0$), the foliation $\overline{\mathcal{M}}$ is said to be flat *(respectively* locally symmetric*).*

Some simple properties follow immediately.

Proposition 4.3.9. *Let (M, g) be a Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$.*

(1) If $\overline{\nabla}^r\,\overline{\mathcal{R}} = 0$ for some $r > 1$, then $\overline{\mathcal{M}}$ is a locally symmetric foliation.

(2) If $\overline{\mathcal{M}}$ is locally symmetric and Ricci-flat, then it is flat.

(3) If $\overline{\mathcal{M}}$ is flat, the Brinkmann decomposition admits a chart $\{u, v, y^i\}$ such that the metric g becomes:

$$g = -2du(dv + Hdu + W_i dy^i) + \delta_{ij}dy^i dy^j.$$

Proof. (1) For each (u, v), the Riemannian result which states that $\overline{\nabla}^r\,\overline{\mathcal{R}} = 0$ implies local symmetry can be applied to each leaf $\overline{M}_{(u,v)}$ (see Section 5.2, in particular Theorem 5.2.3 and Corollary 5.2.4).

(2) Apply Proposition 3.2.15(3) to each leaf of $\overline{\mathcal{M}}$.

(3) We sketch a procedure to obtain the required new coordinates $y^i = y^i(u, x^j)$, for the sake of completeness. Let γ be an integral curve of ∂_u along the hypersurface $\Omega = \{v = 0\}$, and put for small $U \subset \mathbb{R}^{n-2}$, $0 \in U$:

$$\varphi :]-\epsilon, \epsilon[\times U \to M, \qquad (u, w^i) \mapsto \overline{\exp}_{\gamma(u)}(w^i\partial_i|_{\gamma(u)}),$$

where $\overline{\exp}_{\gamma(u)}$ denotes the exponential at $\gamma(u)$ on the leaf $\overline{M}_{(\gamma(u),0)}$ for the (flat) metric $\overline{g}$. The classical Cartan theorem shows that each map $\varphi_u := \varphi(u, \cdot)$ is an affine transformation from U to the leaf. So, regarding $\{w^i\}$ as coordinates in $\mathbb{R}^{n-2}$,

$$\varphi_u^*\overline{g} = h_{ij}(u)dw^i dw^j$$

for some constants $h_{ij}(u)$ on each leaf (which define an Euclidean metric and depend smoothly on u). Now, φ allows to consider $\{u, w^i\}$ as coordinates in Ω, with each coordinate vector field $\partial/\partial w^i$ parallel on each leaf of $\overline{\mathcal{M}}_{(u,0)}$. By using the Gram-Schmidt procedure, an orthonormal basis $\{V_j = B^i_j \frac{\partial}{\partial w^i}\}$ in $T\overline{\mathcal{M}}$ is obtained. Indeed, by construction the transition matrix (B^i_j) depends smoothly only on u and, thus, by Section 4.3.1 $\overline{\nabla} V_j = \overline{d}(B^i_j)\frac{\partial}{\partial w^i} = 0$. Therefore, Proposition 4.3.6 (4) implies that $(V_j)^\flat = dy^j_\Omega$ for some functions $\{y^j_\Omega(u, w^j)\}$ on some open subset of Ω. The required functions $\{y^j\}$ are obtained by extending $\{y^j_\Omega(u, w^j)\}$ to a neighborhood of M in a v-invariant way according to Section 4.3.1. ∎

4.3.6 Connection one-forms and Curvature two-forms

Fix a Brinkmann chart $\{x^\alpha\}$. The *Connection forms* $\overline{\omega}^i_j \in \Lambda(\overline{\mathcal{M}})$ and the corresponding *curvature two-forms* $\overline{\Omega}^i_j \in \Lambda^2(\overline{\mathcal{M}})$ of the foliation $\overline{\mathcal{M}}$ for the given coordinate basis $\{\partial_i\}$ are defined in a standard manner:

$$\overline{\nabla}_X \partial_i = \overline{\omega}^j_i(X)\partial_j, \quad \overline{\mathcal{R}}(X,Y)\partial_j = \overline{\Omega}^i_j(X,Y)\partial_i, \quad \forall X, Y \in \mathfrak{X}(\overline{\mathcal{M}}).$$

Observe that $\overline{\omega}^i_j = \overline{\Gamma}\,^i_{jk}\overline{d}x^k$, where $\overline{\Gamma}\,^i_{jk}$ are the Christoffel symbols of $\overline{\nabla}$ introduced in Section 4.3.4. Then, a simple computation shows that the first and second Cartan's equations for $\overline{\mathcal{M}}$ still hold (recall Section 2.1.6 for the computations, and use $\overline{d}\theta^i = 0$):

$$\begin{aligned} 0 &= \overline{\omega}^i_j \wedge \overline{d}x^j, \\ \overline{\Omega}^i_j &= \overline{d}\,\overline{\omega}^i_j + \overline{\omega}^i_k \wedge \overline{\omega}^k_j\,. \end{aligned} \tag{4.12}$$

Then, $\overline{\nabla}\overline{g} = 0$ yields a formula analogous to (2.13):

$$\overline{d}g_{\alpha\beta} = \overline{\omega}^\sigma_\alpha \overline{g}_{\sigma\beta} + \overline{\omega}^\sigma_\beta \overline{g}_{\alpha\sigma}. \tag{4.13}$$

4.4 Connection and Curvature

In this section the connection and the curvature of a Brinkmann space will be calculated for the partly null frame $\{E_\alpha\}$ given in (4.5), via Cartan techniquess (see Section 2.1.6). In the expressions of ω^α_β and Ω^α_β of a Brinkmann space for a fixed $\{E_\alpha\}$, there appear $\overline{\omega}^i_j$ and $\overline{\Omega}^i_j$ –the connection and the curvature forms of the foliation $\overline{\mathcal{M}}$, respectively–, together with components of some specific tensor fields defined on $\overline{\mathcal{M}}$, underlying the relevance of all the geometric structure already developed on the foliation $\overline{\mathcal{M}}$ in Section 4.3. Fix a Brinkmann chart $\{x^\alpha\}$. Define $h \in \Lambda\overline{\mathcal{M}}$ and $t \in T^0_2(\overline{\mathcal{M}})$ as

$$h = \overline{d}H - \dot{W}, \tag{4.14}$$

$$t = -\frac{1}{2}\left(\dot{\overline{g}} + \overline{d}W\right). \tag{4.15}$$

Observe that the symmetric and skew-symmetric parts of t are precisely $-\dot{\overline{g}}/2$ and $-\overline{d}W/2$, respectively. We emphasize that h and t depend on the Brinkmann chart.

Use Section 2.1.6 to define the connection one-forms $\{\omega^\alpha_\beta\}^{n-1}_{\alpha,\beta=0}$ and the curvature two-forms $\{\Omega^\alpha_\beta\}^{n-1}_{\alpha,\beta=0}$ of a Brinkmann space.

Computation of the connection one-forms $\{\omega^\alpha_\beta\}^{n-1}_{\alpha,\beta=0}$

Observe that in the partly null frame (4.5) the one-form $\theta^0 = du$ is parallel. Applying (4.6) to $g_{ij}(= \overline{g}_{ij} = \mathring{\overline{g}}_{ij})$ it follows that

$$dg_{ij} = \dot{g}_{ij} du + \overset{\circ}{\overbrace{\overline{d}g_{ij}}}. \tag{4.16}$$

Using the properties of the homomorphism $^\circ$ together with (4.13) one obtains:

$$\overset{\circ}{\overbrace{\overline{d}g_{ij}}} = \overset{\circ}{\overbrace{\overline{\omega}^k_i}} g_{kj} + \overset{\circ}{\overbrace{\overline{\omega}^k_j}} g_{ki}.$$

Denote $\overset{\circ}{\overbrace{\overline{\omega}^k_i}}$ as $\mathring{\overline{\omega}}^k_i$. Substituting this expression in (4.16), an expression in terms of the foliation $\overline{\mathcal{M}}$ for dg_{ij} is obtained:

$$dg_{ij} = \dot{g}_{ij} du + \mathring{\overline{\omega}}^k_i g_{kj} + \mathring{\overline{\omega}}^k_j g_{ki}. \tag{4.17}$$

With this information at hand together with (2.13), one can check that, fixed a partly null frame $\{E_\alpha\}$, the connection one-forms of a Brinkmann space in such frame satisfy that:

$$\omega^0_\beta = \omega^\alpha_1 = 0, \qquad \omega^1_0 = \omega^0_1 = 0 \tag{4.18}$$

$$\omega^1_i = \omega^j_0 g_{ij}, \qquad dg_{ij} = \omega^k_i g_{kj} + \omega^k_j g_{ki} \tag{4.19}$$

$$\dot{g}_{ij}\theta^0 = (\omega^k_i - \mathring{\overline{\omega}}^k_i) g_{kj} + (\omega^k_j - \mathring{\overline{\omega}}^k_j) g_{ik}. \tag{4.20}$$

To prove the above, use

$$g = -2\theta^0\theta^1 + g_{ij}\theta^i\theta^j \; (\equiv -\theta^0 \otimes \theta^1 - \theta^1 \otimes \theta^0 + \overline{g}_{ij}\theta^i \otimes \theta^j),$$

also (4.17) and that θ^0 is parallel (or equivalently E_1 is parallel), and take all the possible different values for α, β in (2.13).

Finally, the connection one-forms are computed:

Lemma 4.4.1. *For any Brinkmann chart, the only non-identically vanishing connection one-forms associated to a partly null frame $\{E_\alpha\}$ are:*

$$\omega^1_i = h_i\theta^0 - t_{ij}\theta^j, \qquad \omega^j_0 = g^{ij}\omega^1_i \tag{4.21}$$

$$\omega^i_j = -g^{ik} t_{kj}\theta^0 + \mathring{\overline{\omega}}^i_j. \tag{4.22}$$

where $h_i = (H_{,i} - \dot{W}_i)$ and $t_{ij} = \frac{1}{2}(-\dot{g}_{ij} + W_{i,j} - W_{j,i})$ are the components of the tensors defined in (4.14) *and* (4.15).

Proof. On the one hand, sustituting (4.18) in the first Cartan's equation of structure (4.12) one has

$$d\theta^0 = 0; \quad d\theta^1 = -\omega^1_i \wedge \theta^i; \quad d\theta^i = -\omega^i_0 \wedge \theta^0 - \omega^i_j \wedge \theta^j. \tag{4.23}$$

On the other hand, calculating the exterior derivatives of each θ^α in (4.5) and using (4.14) and (4.15), one also has

$$d\theta^0 = 0; \quad d\theta^1 = h_i\theta^i \wedge \theta^0 + t_{[ij]}\theta^j \wedge \theta^i; \quad d\theta^i = 0. \tag{4.24}$$

Comparing (4.24) with (4.23), one obtains

$$-\omega^1_i \wedge \theta^i \;=\; h_i\theta^i \wedge \theta^0 + t_{[ij]}\theta^j \wedge \theta^i, \tag{4.25}$$

$$\omega^i_0 \wedge \theta^0 + \omega^i_j \wedge \theta^j \;=\; 0. \tag{4.26}$$

Consequently, from (4.25),

$$\omega^1_i = h_i\theta^0 - l_{ij}\theta^j, \tag{4.27}$$

where the functions l_{ij} satisfy $l_{[ij]} = t_{[ij]}$. Using (4.19) and (4.27), from (4.26) one obtains

$$\omega^i_k \wedge \theta^k = g^{ij} l_{jk}\theta^k \wedge \theta^0,$$

hence, for some one-forms $\{Q^i_j\}_{i,j=2}^{n-1} \in \Lambda(M)$ such that

$$Q^i_j \wedge \theta^j = 0, \tag{4.28}$$

it follows that

$$\omega^i_k = -g^{ij} l_{jk}\theta^0 + Q^i_k, \tag{4.29}$$

If one sustitutes this expresion in (4.20), the following arises:

$$l_{(ij)} = t_{(ij)} = -\frac{1}{2}\dot{g}_{ij} \quad \text{so} \quad l_{ij} = t_{ij} \quad \text{and} \quad \overset{\circ}{\overline{\omega}}{}^k_i g_{kj} + \overset{\circ}{\overline{\omega}}{}^k_j g_{ki} = Q^k_i g_{kj} + Q^k_j g_{ki}.$$

Taking the wedge product with θ^i in the last expression, since the first Cartan's equation (4.12) implies $0 = \overset{\circ}{\overline{\omega}}{}^i_j \wedge \theta^j$, and (4.28),(4.29) are satisfied, one concludes that:

$$g_{ki}\overset{\circ}{\overline{\omega}}{}^k_j \wedge \theta^i = g_{ki} Q^k_j \wedge \theta^i \Longrightarrow Q^k_j = \overset{\circ}{\overline{\omega}}{}^k_j. \qquad \blacksquare$$

It should be noted that $\overline{\omega^i_j} = \overline{\omega}^i_j$, but $\overset{\circ}{\overline{\omega}}{}^i_j \neq \omega^i_j$. Then, from (4.21-4.22) it is immediate to obtain the non-vanishing components $\gamma^\alpha_{\beta\lambda}$ of the connection one-forms ω^α_β, defined by $\omega^\alpha_\beta = \gamma^\alpha_{\beta\lambda}\theta^\lambda$, which are:

$$\begin{aligned} \gamma^1_{i0} &= g_{ij}\gamma^j_{00} = h_i; \qquad \gamma^1_{ij} = g_{ik}\gamma^k_{oj} = -t_{ij}; \\ \gamma^i_{j0} &= -g^{ik}t_{kj}; \qquad \gamma^i_{jk} = \overline{\gamma}^i_{jk}. \end{aligned}$$

Recall also the identity $\nabla_{E_\beta} E_\alpha = \gamma^\lambda_{\alpha\beta} E_\lambda$. This together with (4.21) provides the simple formula

$$h(X) = g(\overset{\circ}{X}, \nabla_{E_0} E_0), \quad \forall X \in \mathfrak{X}(\overline{\mathcal{M}}). \tag{4.30}$$

Computation of the Curvature two-forms $\{\Omega^\alpha_\beta\}_{\alpha,\beta=0}^{n-1}$

From Section 2.1.6 one has for the generically non-vanishing Ω^α_β:

$$\begin{aligned}\Omega^1_i &= g_{ij}\Omega^j_0 = d\omega^1_i + \omega^1_j \wedge \omega^j_i,\\ \Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j.\end{aligned}$$

It only remains to substitute the values of $\{\omega^\alpha_\beta\}_{\alpha,\beta=0}^{n-1}$ obtained in Proposition 4.4.1 in the expressions above and use

$$\overline{\nabla}_j h_i = (\overline{\nabla}_{\overline{E}_j} h)(\overline{E}_i) = (\overbrace{\overline{\nabla}_{\overline{E}_j} h}^{\circ})(E_i); \qquad t^i{}_j = \overline{g}^{ir} t_{rj}$$

to obtain that:

Proposition 4.4.2. *For any Brinkmann chart, the non-identically vanishing curvature two-forms associated to a partly null frame* $\{E_\alpha\}$ *are:*

$$\Omega^1_i = (\overline{\nabla}_j h_i + \dot{t}_{ij} + t^k{}_i t_{kj})\theta^j \wedge \theta^0 + \frac{1}{2}(\overline{\nabla}_k t_{ij} - \overline{\nabla}_j t_{ik})\theta^j \wedge \theta^k, \tag{4.31}$$

$$\Omega^i_j = \overline{\Omega}^i_j + (\overline{\nabla}_k t^i{}_j + \dot{\overline{\Gamma}}^i_{jk})\theta^0 \wedge \theta^s. \tag{4.32}$$

Similarly to $\gamma^\alpha_{\beta\lambda}$, the non-vanishing components $R^\alpha{}_{\beta\gamma\delta}$ of the curvature tensor R, defined by $\Omega^\alpha_\beta = \frac{1}{2}R^\alpha{}_{\beta\gamma\delta}\theta^\gamma \wedge \theta^\delta$ can be read off from (4.31-4.32):

$$R^1{}_{i0j} = -(\overline{\nabla}_j h_i + \dot{t}_{ij} + t^k{}_i t_{kj}), \tag{4.33}$$

$$R^1{}_{ijk} = \overline{\nabla}_k t_{ij} - \overline{\nabla}_j t_{ik}, \tag{4.34}$$

$$R^i{}_{jkl} = \overline{\mathcal{R}}^i{}_{jkl}, \tag{4.35}$$

$$R^i{}_{j0k} \;(= -g^{ri}R^1{}_{krj}) = \overline{\nabla}_k t^i{}_j + \dot{\overline{\Gamma}}^i{}_{jk} \tag{4.36}$$

where, in the last expression[2], the dot acts on the functions $\overline{\Gamma}^i{}_{jk}$. From here, the non-vanishing components of the Ricci tensor can be easily computed :

$$\begin{aligned}R_{00} &= R^i{}_{0i0} = -g^{ij}R^1{}_{j0i} = \overline{\nabla}_i h^i + g^{ij}\dot{t}_{ji} + t^{ki}t_{ki},\\ R_{0i} &= \overline{\nabla}_i t^j{}_j - \overline{\nabla}_j t^j{}_i,\\ R_{ij} &= \overline{\mathcal{R}}_{ij},\end{aligned}$$

and the scalar curvature of M turns out to be equal to that of $\overline{\mathcal{M}}$:

$$S = \overline{\mathcal{S}}.$$

Therefore, we have proven the following result:

[2]Observe that in general $R^1{}_{\alpha\beta\mu} = -R_{0\alpha\beta\mu}$. To check that (4.34) and (4.36) are essentially the same, use $\overline{\nabla}_i \dot{g}_{jk} = \dot{\overline{\Gamma}}{}^r{}_{ji} g_{rk} + \dot{\overline{\Gamma}}{}^r{}_{ki} g_{jr}$.

Proposition 4.4.3. *The curvature tensor* $\overline{\mathcal{R}}$ *of the foliation* $\overline{\mathcal{M}}$ *as defined in* (4.10)*, and its associated Ricci tensor* $\overline{\mathcal{R}ic}$ *and scalar curvature* $\overline{\mathcal{S}}$ *satisfy that*

$$\overline{\mathcal{R}} = \overline{R}; \quad \overline{\mathcal{R}ic} = \overline{Ric}; \quad \overline{\mathcal{S}} = \overline{S},$$

where $\overline{R}$*,* $\overline{Ric}$ *and* $\overline{S}$ *are the projections by the homomorphism* $^{-}$ *of the curvature tensor* R*, the Ricci tensor* Ric *and the scalar curvature* S *of the Brinkmann space* (M, g)*, respectively.*

Using this result, from now on there will be no need to distinguish between these objects and we will use only the notation $\overline{R}$, $\overline{\mathrm{Ric}}$ and $\overline{S}$ for them. Observe that, for example, the curvature tensor field $\overline{R} = \overline{\mathcal{R}}$ of the foliation $\overline{\mathcal{M}}$ is related to the curvature tensor field R of a Brinkmann space by means of the following formula:

$$\overline{R}(X,Y)Z = \overline{R(\mathring{X}, \mathring{Y})\mathring{Z}}, \qquad \forall X, Y, Z \in \mathfrak{X}(\overline{\mathcal{M}}).$$

Furthermore, if, say, the first index of t and $\mathring{t}$ is raised, then the same expression $t^i{}_j$ is obtained, as the isomorphisms $\flat$ and $\sharp$ commute with the homomorphism $T \mapsto \mathring{T}$ ($t^i{}_j \equiv g^{i\alpha} t_{\alpha j} = g^{ir} t_{rj}$).

4.5 The operator D_0

A new operator on $\overline{\mathcal{M}}$, the D_0 operator, is introduced in this section. This operator is defined in terms of the covariant derivative along the vector field E_0 in M. This new operator will be very helpful when the first and second covariant derivatives of the curvature (∇R and $\nabla\nabla R$) of a Brinkmann space are to be calculated in Section 6.3.1. The concept of D_0-parallelism is also necessary in Section 4.6 to extend the Classical Eisenhart Theorem from Riemannian manifolds to Brinkmann spaces.

Definition 4.5.1. *The* D_0 *operator associated to any Brinkmann chart is defined as:*

$$\begin{array}{rccc} D_0 : & \Gamma(T^k_s\overline{\mathcal{M}}) & \longrightarrow & \Gamma(T^k_s\overline{\mathcal{M}}) \\ & T & \to & D_0 T = \overline{(\nabla_{E_0}\mathring{T})} \end{array}$$

Consequently, $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ *is said to be* D_0-parallel *if* $D_0 T = 0$.

D_0 measures the variation, projected to $T^k_s\overline{\mathcal{M}}$, of tensor fields $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ under displacements along the direction E_0, which is transverse to the leaves of $\overline{\mathcal{M}}$.

Proposition 4.5.2. D_0 *satisfies the formal properties of a tensor derivation on* $\overline{\mathcal{M}}$*:*

i) $\mathbb{R}$*-linearity:* $D_0(aA + bB) = aD_0A + bD_0B$, $\forall a, b \in \mathbb{R}$, $\forall A, B \in \Gamma(T^k_s\overline{\mathcal{M}})$.

ii) Leibniz rule: $D_0(A \otimes B) = (D_0A) \otimes B + A \otimes (D_0B)$, $\forall A, B \in \Gamma(T^k_s\overline{\mathcal{M}})$.

iii) Commutativity with contractions: $D_0(C^i_j(A)) = C^i_j(D_0A)$, $\forall A \in \Gamma(T^k_s\overline{\mathcal{M}})$*, where* C^i_j *denotes the contraction of the* i^{th} *contravariant slot with the* j^{th} *covariant one.*

Proof. Use that the linear homomorphisms $^-$ and $^\circ$ commute with contractions and tensor products, $\overline{\mathring{T}} = T$ for any $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ and that ∇ is a tensor derivation. Then, if C^i_j is the contraction between two indices in $\{2, \dots, n-1\}$:

i) $D_0(aA + bB) \equiv \overline{\nabla_{E_0}(\overset{\circ}{\overbrace{aA + bB}})} = \overline{\nabla_{E_0}(a\mathring{A} + b\mathring{B})} = \overline{a\nabla_{E_0}\mathring{A} + b\nabla_{E_0}\mathring{B}} = a\overline{\nabla_{E_0}\mathring{A}} + b\overline{\nabla_{E_0}\mathring{B}} = aD_0A + bD_0B.$

ii) $D_0(A \otimes B) \equiv \overline{\nabla_{E_0}(\overset{\circ}{\overbrace{A \otimes B}})} = \overline{\nabla_{E_0}(\mathring{A} \otimes \mathring{B})} = \overline{(\nabla_{E_0}\mathring{A}) \otimes \mathring{B} + \mathring{A} \otimes (\nabla_{E_0}\mathring{B})} = \overline{\nabla_{E_0}\mathring{A}} \otimes \overline{\mathring{B}} + \overline{\mathring{A}} \otimes \overline{\nabla_{E_0}\mathring{B}} = (D_0A) \otimes B + A \otimes (D_0B).$

iii) $D_0(C^i_j(A)) \equiv \overline{\nabla_{E_0}\overset{\circ}{\overbrace{C^i_j(A)}}} = \overline{\nabla_{E_0}C^i_j(\mathring{A})} = \overline{C^i_j(\nabla_{E_0}\mathring{A})} = C^i_j(\overline{\nabla_{E_0}\mathring{A}}) = C^i_j(D_0A).$ ■

Proposition 4.5.3. *For $X \in \mathfrak{X}(\overline{\mathcal{M}})$, the decomposition of $\nabla_{E_0}\mathring{X}$ in $T\mathcal{U} \oplus (T\mathcal{U})^\perp$ is*

$$\nabla_{E_0}\mathring{X} = h(X)E_1 + \overset{\circ}{\overbrace{D_0X}}\,.$$

Proof. Putting $\nabla_{E_0}\mathring{X} = X_1 + X_2$, with $X_1 \in \Gamma(T\mathcal{U})$ and $X_2 \in \Gamma((T\mathcal{U})^\perp)$, the definition of D_0 gives immediately $D_0X = \overline{X_2}$. For X_1, using that $g(\mathring{X}, E_0) = g(\mathring{X}, E_1) = 0$, that E_1 is parallel and formula (4.30) we get:

$$X_1 = -g(\nabla_{E_0}\mathring{X}, E_1)E_0 - g(\nabla_{E_0}\mathring{X}, E_0)E_1 = g(\mathring{X}, \nabla_{E_0}E_1)E_0 + g(\mathring{X}, \nabla_{E_0}E_0)E_1 = h(X)E_1\,.$$ ■

Proposition 4.5.4. *$\overline{g}$ is D_0-parallel, that is to say:*

$$D_0\overline{g} = 0. \tag{4.37}$$

Proof. By definition, $D_0\overline{g} \equiv \overline{\nabla_{E_0}\mathring{\overline{g}}}$ and $g = -\theta^0 \otimes \theta^1 - \theta^1 \otimes \theta^0 + \mathring{\overline{g}}$. Since θ^0 is parallel and $\nabla_{E_0}g = 0$,

$$-\theta^0 \otimes \nabla_{E_0}\theta^1 - \nabla_{E_0}\theta^1 \otimes \theta^0 + \nabla_{E_0}\mathring{\overline{g}} = 0$$

Therefore, as $\overline{(\theta^0 \otimes \nabla_{E_0}\theta^1 + \nabla_{E_0}\theta^1 \otimes \theta^0)} = 0$, we have that $\overline{\nabla_{E_0}\mathring{\overline{g}}} = 0$. ■

If $X = X^i\partial_i \in \mathfrak{X}(\overline{\mathcal{M}})$ and $\omega = \omega_i\overline{d}x^i \in \Lambda(\overline{\mathcal{M}})$ in the bases $\{\partial_i\}, \{\overline{d}x^i\}$ of $T\overline{\mathcal{M}}$ and $T^*\overline{\mathcal{M}}$, respectively, associated to a fixed Brinkmann chart and the partly null frame $\{E_\alpha\}, \{\theta^\alpha\}$, then

$$D_0X = \overline{\nabla_{E_0}\mathring{X}} = \overline{E_0(X^i)E_i + \gamma^\alpha_{j0}X^jE_\alpha} = (E_0(X^i) + \gamma^i_{j0}X^j)\partial_i = (E_0(X^i) - g^{ik}t_{kj}X^j)\partial_i,$$

while

$$D_0\omega = \overline{\nabla_{E_0}\mathring{\omega}} = \overline{E_0(\omega_i)\theta^i - \gamma^j_{\alpha 0}\omega_j\theta^\alpha} = (E_0(\omega_i) - \gamma^j_{i0}\omega_j)\overline{d}x^i = (E_0(\omega_i) + g^{jk}t_{ki}\omega_j)\overline{d}x^i.$$

These formulas can be expressed in the local basis $\{\partial_i\}$ of $T\overline{\mathcal{M}}$ by means of:

$$\begin{aligned}(D_0X)^i &\equiv D_0X^i = (\partial_u - H\partial_v)(X^i) - t^i{}_jX^j,\\ (D_0\omega)_i &\equiv D_0\omega_i = (\partial_u - H\partial_v)(\omega_i) + t^j{}_i\omega_j.\end{aligned}$$

Its generalization to any section $T \in \Gamma(T^k_s\overline{\mathcal{M}})$ is then

$$\begin{aligned}(D_0T)^{i_1\ldots i_k}_{j_1\ldots j_s} &\equiv D_0T^{i_1\ldots i_k}_{j_1\ldots j_s}\\ &= (\partial_u - H\partial_v)(T^{i_1\ldots i_k}_{j_1\ldots j_s}) - \sum_{a=1}^{k} t^{i_a}{}_m T^{i_1\ldots i_{(a-1)}mi_{(a+1)}\ldots i_k}_{j_1\ldots j_s}\\ &\quad + \sum_{b=1}^{s} t^m{}_{j_b} T^{i_1\ldots i_k}_{j_1\ldots j_{(b-1)}mj_{(b+1)}\ldots j_s}. \end{aligned} \tag{4.38}$$

For v-invariant tensor fields the expression D_0T simplifies ($D_0T = \overline{\nabla_{\partial_u}\mathring{T}}$) such that

$$(D_0T)^{i_1\ldots i_k}_{j_1\ldots j_s} = \dot{T}^{i_1\ldots i_k}_{j_1\ldots j_s} - \sum_{a=1}^{k} t^{i_a}{}_m T^{i_1\ldots i_{(a-1)}mi_{(a+1)}\ldots i_k}_{j_1\ldots j_s} + \sum_{b=1}^{s} t^m{}_{j_b} T^{i_1\ldots i_k}_{j_1\ldots j_{(b-1)}mj_{(b+1)}\ldots j_s}. \tag{4.39}$$

Next, we collect some elementary properties of the D_0-parallel transport to be used later. First, let η be an integral curve of E_0 and take a vector field X_η along η which is everywhere tangent to $\overline{\mathcal{M}}$, i.e., $X_\eta \in \Gamma(T_\eta\overline{\mathcal{M}})$. The derivative $D_0(X_\eta)$ of X_η, as well as its D_0-parallelism, makes an obvious sense.

Lemma 4.5.5. *Let $p \in M$, $\vec{v} \in T_p\overline{\mathcal{M}}$ and η the integral curve of E_0 with $\eta(0) = p = (u_p, v_p, x^i_p)$. Then, there exists a unique vector field X_η obtained as the D_0-parallel transport along η such that $X_{\eta(0)} = \vec{v}$.*

Proof. The local existence and uniqueness of V_η comes from the corresponding equations of the D_0-parallel transport, namely:

$$\frac{d}{d\tau}(V^k_\eta(\tau)) - t^k{}_i(\eta(\tau))V^i_\eta(\tau) = 0 \qquad \text{with } V^k_\eta(0) = v^k.$$

A standard reasoning of maximality yields the global result. ∎

Again, a standard reasoning leads to:

Proposition 4.5.6. *The map which sends each $\vec{v} \in T_p\overline{\mathcal{M}}$ to its unique D_0-parallel transport $X_\eta(\tau) \in T_{\eta(\tau)}\overline{\mathcal{M}}$ along the integral curve η of E_0 is a linear isometry from $T_p\overline{\mathcal{M}}$ to $T_{\eta(\tau)}\overline{\mathcal{M}}$.*

Proof. $(\overline{g}(X,Y))|_\eta$ depends only on the parameter u of η. The result follows from (4.37) and the D_0-transport of the vectors. ∎

Our last result yields an extension to all the leaves of any vector field on one leaf.

Proposition 4.5.7. *Let $X_{\overline{M}}$ be a vector field on a leaf $\overline{M}_{(u_0,v_0)}$ of $\overline{\mathcal{M}}$. Then, there exists a unique v-invariant and D_0-parallel vector field $X_{\overline{\mathcal{M}}} \in \mathfrak{X}(\overline{\mathcal{M}})$ which extends $X_{\overline{M}}$.*

Proof. Consider the hypersurface $\Omega_{v_0} = \{v = v_0\}$ and extend $X_{\overline{M}}$ to a vector field $X_{\Omega_{v_0}}$ on Ω_{v_0} by taking each integral curve η of E_0 which starts at some $p \in \overline{M}_{(u_0,v_0)}$, and defining $X_{\Omega_{v_0}} \circ \eta$ as the D_0-parallel transport of X_p. Then, extend $X_{\Omega_{v_0}}$ to a v-invariant vector field according to Section 4.3.1. ■

4.6 Reducibility and a Generalized Eisenhart Theorem

The proof of our main Theorem 1.1 will be carried out by means of some local decompositions of the Brinkmann space. To that end, we need an appropriate notion of reducibility (compare with Section 2.3) and a version of a classical theorem by Eisenhart (Theorem 2.3.6) adapted to Brinkmann spaces.

Definition 4.6.1. *A Brinkmann decomposition $\{u, v\}$ of a Brinkmann space (M, g) is* spatially reducible *if there exists a Brinkmann chart $\{u, v, x^i\}$ around each point and a partition of the indices $I_1 = \{2, \ldots, d+1\}, I_2 = \{d+2, \ldots, n-1\}$ for some $d \in \{1, \ldots, n-3\}$ such that $\overline{g}_{aa'} = 0$ and $\partial_{a'}\overline{g}_{ab} = 0$, where the unprimed indices a, b always belong to the same subset I_m ($m \in \{1, 2\}$) and the primed ones a', b' to the other one.*

For such a Brinkmann chart, we say that $T \in \Gamma(T^r_s\overline{\mathcal{M}})$ is reducible *whenever $T = T^{(1)} + T^{(2)}$, with*

$$T^{(m)} = T^{(m)}{}^{a_1\ldots a_r}_{b_1\ldots b_s}(u, x^c)\partial_{a_1} \otimes \ldots \otimes \partial_{a_r} \otimes \overline{d}x^{b_1} \otimes \ldots \otimes \overline{d}x^{b_s}, \text{ and } a_1, \ldots, a_r, b_1, \ldots, b_s, c \in I_m.$$

We will denote such decomposition as $T = T^{(1)} \oplus T^{(2)}$.

Observe that, if a Brinkmann decomposition is spatially reducible, then the metric $\overline{g}$ on $T\overline{\mathcal{M}}$ is reducible as a tensor field. Besides, spatial reducibility implies the following property, which provides a more intrinsic expression for some of its consequences: for some spatially reducible Brinkmann decomposition $\{u, v\}$ there exist two foliations $\overline{\mathcal{M}}^{(1)}, \overline{\mathcal{M}}^{(2)}$ of M such that $T\overline{\mathcal{M}} = T\overline{\mathcal{M}}^{(1)} \oplus T\overline{\mathcal{M}}^{(2)}$, and this sum is orthogonal with respect to $\overline{g}$. In this case, we write $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}$ and $\overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)}$, according to the notation in Definition 4.6.1. Then, the metric g on M can be writen as:

$$g = -2du(dv + Hdu + \mathring{W}) + \mathring{g}^{(1)} \oplus \mathring{g}^{(2)}$$

(recall that H and W depend on the chosen coordinates $\{x^i\}$). In this context, a tensor field section T is reducible if and only if $T = T^{(1)} + T^{(2)}$ where each $T^{(m)}$ is invariant by the flow of vectors in $\overline{\mathcal{M}}^{(m')}$ ($\mathcal{L}_X T^{(m)} = 0, \forall X \in \Gamma(T\overline{\mathcal{M}}^{(m')})$) and it vanishes when applied on any element of $T\overline{\mathcal{M}}^{(m')}$ and $T^*\overline{\mathcal{M}}^{(m')}$. In particular, this happens for $\overline{g}$. The Riemannian manifold $(\overline{M}, \overline{g})$ can be written as the product of two manifolds which will also be denoted, abusing the notation, by $(\overline{M}^{(1)}, \overline{g}^{(1)})$ and $(\overline{M}^{(2)}, \overline{g}^{(2)})$, each $\overline{M}^{(m)}$ generating $\overline{\mathcal{M}}^{(m)}$ as $\overline{M}$ generated $\overline{\mathcal{M}}$, according to Section 4.2.1. Thus, the equality $\overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)}$ may refer either to the metric decomposition in a leaf $\overline{M}_{(u_0,v_0)}$ or in $\overline{\mathcal{M}}$. The latter depends on u and is v-invariant. Even though this ambiguity is harmless, we will always refer to decompositions in $\overline{\mathcal{M}}$ except if otherwise is specified.

Sometimes, a Brinkmann chart may admit a partition $I_1, \ldots, I_s, s \geq 2$ of the indices $\{2, \ldots, n-1\}$ so that $\bar{g} \in \Gamma(T_2^0\overline{\mathcal{M}})$ satisfies the properties given in Definition 4.6.1 for each $i, j \in I_m$, $k', l' \in I_{m'}$ and $m \neq m'$. In this case, the notation for orthogonal decomposition is naturally extended:

$$\overline{M} = \overline{M}^{(1)} \times \ldots \times \overline{M}^{(s)}, \quad \overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \ldots \times \overline{\mathcal{M}}^{(s)}, \quad \overline{g} = \overline{g}^{(1)} \oplus \ldots \oplus \overline{g}^{(s)},$$

and notions as being Einstein or flat are also extended to each $\overline{M}^{(m)}$, $\overline{\mathcal{M}}^{(m)}$ in a trivial way. However, the following caution must be kept in mind. The given decomposition of $\overline{\mathcal{M}}$ induces also an orthogonal decomposition at each leaf $\overline{M}_{(u,v)}$, in particular at $\overline{M}$. Nevertheless, as $\overline{g}$ is "u-dependent" such a decomposition may be irreducible in the sense of the traditional de Rham's theorem for some leaves, but reducible for other ones. For example, take the metric:

$$\overline{g} = \sum_{i=2}^{d+1} (1 + uF_i(x_2, \ldots, x_{d+1}))(dx^i)^2 + \sum_{i=d+2}^{n-1} (dx^i)^2.$$

The Brinkmann decomposition $\{u, v\}$ is then spatially reducible to $\overline{\mathcal{M}}_1 \times \overline{\mathcal{M}}_2$ where $\overline{\mathcal{M}}_2$ is a flat foliation, and the leaves $\{u = 0, v = v_0\}$ of the foliation $\overline{\mathcal{M}}_1$ are flat Riemannian manifolds. So for $u = 0$ the metric $\overline{g}$ of the foliation $\overline{\mathcal{M}}$ is reduced to a product of $n-2$ spaces of the type $(\mathbb{R}, dx)$, while if $u \neq 0$, the function $F_i(x_2, \ldots, x_{d+1})$ could be chosen to obtain leaves of, say, constant curvature. Indeed, if we take the functions F_i depending also on u, more involved structures on each leaf can be obtained.

In principle, we will not care about this possible "spatial irreducibility" (or Riemannian irreducibility) of the metrics $\overline{g}^{(m)}$ on the leaves. Eventually, though, we will arrive in Chapter 6 at a decomposition of $\overline{g}$ which will induce an irreducible decomposition of all the leaves, independent of u.

4.6.1 Extending Eisenhart Classical Theorem

Our aim now is to prove a version of the Theorem 2.3.6 adapted to the spatial reducibility of Definition 4.6.1 for Brinkmann decompositions. In our generalized version, the reduced metrics $\overline{g}^{(m)}$ will be dependent on u but the λ_m will still be constants, independent of u. Observe that though the proof is quite similar to the one of Theorem 2.3.6, there are some interesting differences between them.

Theorem 4.6.2. *Let (M, g) be a Brinkmann space and fix a Brinkmann chart $\{u, v, x^i\}$. Assume that there exists a symmetric v-invariant, $\overline{\nabla}$-parallel and D_0-parallel section $L \in \Gamma(T_2^0\overline{\mathcal{M}})$*

which is not proportional to $\overline{g}$. Then, the Brinkmann decomposition $\{u, v\}$ is spatially reducible and L is reducible. Furthermore, the decomposition $\{u, v\}$ admits a Brinkmann chart $\{u, v, y^i\}$ such that:

1. $\overline{g} = \overline{g}^{(1)} \oplus \ldots \oplus \overline{g}^{(s)}$ *for some* $s \geq 2$.
2. $L = \sum_{m=1}^{s} \lambda_m \, \overline{g}^{(m)}$ *for some constants* $\lambda_m \in \mathbb{R}$.

Proof. Let p be any point of the chart. We construct an orthonormal basis of eigenvector fields $V_i \in T\overline{\mathcal{M}}$ defined on the hypersurface $\Omega_{v_p} = \{v = v_p\}$ as we describe now: consider the eigenvalue problem of L_p with respect to $\overline{g}_p$ on the vector space $T_p\overline{\mathcal{M}}$, i.e.,

$$L_p(\,.\,, \vec{v}) - \lambda \overline{g}_p(\,.\,, \vec{v}) = 0, \tag{4.40}$$

and take an orthonormal basis $\{\vec{v}_i\}_{i=2}^{n-1}$ of eigenvectors of L_p in $T_p\overline{M}_{(u_p,v_p)}$. Extend this basis to a normal neighborhood $\overline{U}$ of p in the leaf $\overline{M}_{(u_p,v_p)}$ by defining $V_i|_q$ at each $q \in \overline{U}$ as the vector obtained by $\overline{\nabla}$-parallelly transporting $\vec{v}_i$ along the unique geodesic $\gamma_q : [0, 1] \to \overline{U}$ from p to q. Clearly, if $\vec{v}_i$ is a λ-eigenvector, then $V_i|_q$ is an eigenvector of L_q with the same eigenvalue λ, because the one-forms on γ_q defined as $\tau \mapsto L_{\gamma_q(\tau)}(\,.\,, V_{\gamma_q(\tau)})$ and $\tau \mapsto \lambda \overline{g}_{\gamma_q(\tau)}(\,.\,, V_{\gamma_q(\tau)})$ are parallel and coincide at p due to (4.40). Besides, $\{V_i\}$ is an orthonormal basis on $\overline{U}$. Now, obtain the sought basis $\{V_i\}$ on Ω_{v_p} by propagating each $V_i|_q$ in a D_0-parallel manner along the integral curve of E_0 at $q \in \overline{U}$. Since L is D_0-parallel, $\{V_i\}$ is still an orthonormal basis of eigenvector fields of L and the eigenvalues of L are constant on Ω_{v_p}.

From v-invariance, the eigenvalues and the dimension of each eigenspace remain constant all over M. Therefore, if we denote any λ-eigenvector field as $V^{(\lambda)}$, and $\lambda_1, \ldots, \lambda_s$ are the eigenvalues of L with corresponding multiplicities $m_1, \ldots, m_s$, we can reorder the vector fields so that

$$\{V_1^{(\lambda_1)}, \ldots\, V_{m_1}^{(\lambda_1)}, \ldots\, V_1^{(\lambda_s)}, \ldots, V_{m_s}^{(\lambda_s)}\}$$

is an orthonormal basis of $T\overline{\mathcal{M}}$.

Let λ be one of the eigenvalues and let us prove that the distribution S_λ generated by its eigenvectors is involutive. Taking the $\overline{\nabla}_V$ covariant derivative of $L(\cdot\,, V_i^{(\lambda)}) = \lambda\overline{g}(\cdot, V_i^{(\lambda)})$ for any $V \in \Gamma(T\overline{\mathcal{M}})$ and using that L is $\overline{\nabla}$-parallel, we have that $\nabla_V V_i^{(\lambda)}$ lies in S_λ, and so does $[V_i^{(\lambda)}, V_j^{(\lambda)}]$. Analogously, the distribution $S_\lambda^\perp$ which assigns to each point p' the orthogonal of $(S_\lambda)_{p'}$ in $T_{p'}\overline{\mathcal{M}}$ is involutive. Regarding $S_\lambda^\perp$ as a distribution contained in $T\Omega$, there are $m_\lambda + 1$ functionally independent functions on Ω which are solutions of the equation $X(f) = 0, \forall X \in S_\lambda^\perp$. The first of these functions can be chosen as $u|_\Omega$ for all λ. The other functions $y_\lambda^i, i = 1, \ldots m_\lambda$, will complete a cooordinate chart for Ω when the construction is repeated for all the eigenvalues $\lambda = \lambda_j, l = 1, \ldots, s$. From these coordinates $\{u|_\Omega, y_{\lambda_j}^{i_j} : i_j = 1, \ldots m_{\lambda_j}, j = 1, \ldots, s\}$ on Ω we can construct a chart on M by extending the previous functions in a v-invariant way according to Section 4.3.1 thereby including the coordinate v.

To finish the proof, argue similarly to the last part of the proof of Theorem 2.3.6. ∎

Clearly, under the hypotheses of Theorem 4.6.2, the dimension of each eigenspace of L remains constant and it is parallely transported along the integral curves of E_0 and E_1, and particular curves contained in the leaves of $\overline{\mathcal{M}}$. Indeed, $\mathring{L} \in \Gamma(T^0_2 M)$ is also diagonalizable —observe that E_0 and E_1 are associated to the eigenvalue 0—, so its Segré type is $[1, 11 \ldots 1]$ (up to degeneracies).

However, the hypotheses on L are not enough to ensure that $\mathring{L}$ is parallel for (M, g). For example, in a given partly null frame for a fixed Brinkmann chart, $\nabla \mathring{L}_{ij} = (D_0 L_{ij})\theta^0 + (\overline{\nabla}_m L_{ij})\theta^m + (\partial_v L_{ij})dv \in \Lambda(M)$ would be zero under the hypotheses of the Theorem, but $\nabla \mathring{L}_{i0} = L_{im}\omega^m_0 \in \Lambda(M)$ are not necessarily zero.

Chapter 5

General Properties of rth-order Symmetric Spaces

In this chapter we are going to deal with spaces that satisfy the following condition on the curvature tensor field R:

$$\nabla^r R \,(= \nabla \overset{(r)}{\ldots} \nabla R) = 0, \qquad \text{for some } r \geq 1. \tag{5.1}$$

If the curvature tensor R of the manifold satisfies the condition (5.1) above, then the manifold is said to be a *rth-order symmetric* (*rth-symmetric* for short) space. Moreover, we say that (M, g) is a *proper rth-symmetric manifold* for some $r > 1$ if it is rth-symmetric but not $(r-1)$th-symmetric. Observe that when $r = 1$ such spaces are locally symmetric spaces (see Chapter 3).

The study of *rth-order symmetric spaces* is interesting from two different perspectives: first, from the mathematical point of view because they have never been studied before in arbitrary signature (apart from the positive-definite case, see Section 5.2), and second, because of its possible implications in the area of physics (in particular for the Lorentzian case).

In this Chapter we will see that, in the Riemannian case or when admitting generic points, all proper rth-symmetric spaces with $r > 1$ are locally symmetric spaces. After that, we will present some (non-locally symmetric) simple examples of this kind of spaces in the Lorentzian case (see Definition 5.4.1), which will be presented as the generalization of the locally symmetric Lorentzian manifolds given in Section 3.2.4. As already pointed out, they will essentially exhaust all the 2nd-symmetric Lorentzian manifolds (Theorem 1.1).

First, some necessary notions on affine and homothetic vector fields are introduced.

5.1 Affine and Homothetic Vector Fields

For the definition of the Lie derivative of a connection, see Section 2.1.5. If $f \in C^\infty(N)$ is a differentiable function, its second order covariant derivative, to be used later, is given by the following formulae:

$$\nabla^2 f(X,Y) = \mathrm{Hess} f(X,Y) = \frac{1}{2}\mathcal{L}_{\mathrm{grad}(f)} g(X,Y) \tag{5.2}$$

Definition 5.1.1. *Let Z be a vector field on a semi-Riemannian manifold (N,h). Then Z is:*

- affine *if $\mathcal{L}_Z \nabla = 0$ holds, and*
- homothetic *if $\mathcal{L}_Z h = ch$ holds for some constant c.*

Observe that in fact:

Lemma 5.1.2. *Any homothetic vector field Z is affine.*

Proof. Let $\{\varphi_t\}$ be a local flow of Z an homothetic vector field (i.e. $\mathcal{L}_Z h = ch$ holds for some constant c). Thus, $h' = \varphi_t^*(h) = \exp(ct)h$ for each t. Take the manifold N endowed with the metric h'. The map $\varphi_t : (N,h) \longrightarrow (N,h')$ becomes an isometry, therefore preserves connections. But the Christofell symbols in a given chart (and hence the Levi-Civita connections) associated to h and h' are the same. Therefore, $\varphi_t : (N,h) \longrightarrow (N,h)$ preserves the connection on M ($(\varphi_t)^*\nabla = \nabla$), and by the definition of Lie derivation 2.1.6 Z is an affine vector field. ■

Lemma 5.1.3. *If Z is affine, then the following two properties hold:*

(i) $\mathcal{L}_Z T = 0$ *for any curvature concomitant T.*

(ii) $\mathcal{L}_Z(\nabla^r T) = \nabla^r(\mathcal{L}_Z T)$ *for any tensor field T.*

Proof. For (i), since the one-parameter group of diffeomorphisms $\{\varphi_t\}$ associated to Z preserves the connection, it also preserves any tensorial object constructed from the connection. In particular, the curvature tensor field R, the Ricci tensor field Ric, and more generally any curvature concomitant.

For (ii), it is enough to prove the case $r = 1$, which follows from Corollary 2.1.8 and from the definition of affine vector fields. ■

Besides, observe that the condition $\mathcal{L}_Z g = cg$ for an homothetic vector field Z can also be written as

$$g(\nabla_X Z, Y) + g(X, \nabla_Y Z) = cg(X,Y), \qquad \forall X,Y \in \mathfrak{X}(M).$$

Therefore, if Z is a vector field such that $\nabla_X Z = cX$ for all $X \in \mathfrak{X}(M)$, then it is homothetic and satisfies that for any vector field X and one-form ω,

$$\mathcal{L}_Z X(\omega) = \omega(\nabla_Z X - \nabla_X Z) = \omega(\nabla_Z X - cX),$$

$$(\mathcal{L}_Z \omega)(X) = (\nabla_Z \omega)(X) + \omega(\nabla_X Z) = (\nabla_Z \omega + c\omega)(X),$$

so, for any (k,s)-tensor field T

$$\mathcal{L}_Z T = \nabla_Z T + (s-k)cT. \tag{5.3}$$

Lemma 5.1.4. *Let $(\overline{M}, \overline{g})$ be an irreducible Riemannian manifold, T a (k,s)-tensor field on $\overline{M}$ and define $f = \frac{1}{2}\overline{g}(T,T)$ where $\overline{g}$ is extended to tensor fields in the natural way.*

If $\nabla^2 T = 0$ then the vector field $Z = grad(f)$ is affine and $\nabla_X Z = cX$ for some constant c and for all $X \in \mathfrak{X}(M)$.

Proof. Using (1.5) and (5.2) for $f = \frac{1}{2}\overline{g}(T,T)$ one has:

$$\begin{aligned}
\mathrm{Hess}f(Y,X) &= (\nabla^2 f)(Y,X) = \frac{1}{2}(\nabla^2 \overline{g}(T,T))(Y,X) \\
&= \frac{1}{2}(\nabla_X(\nabla_Y \overline{g}(T,T)) - \frac{1}{2}\nabla_{\nabla_X Y}\overline{g}(T,T)) \\
&= \overline{g}(\nabla_X(\nabla_Y T), T) + \overline{g}(\nabla_Y T, \nabla_X T) - \overline{g}(\nabla_{\nabla_X Y} T, T) \\
&= \overline{g}(\nabla^2 T(X,Y), T) + \overline{g}(\nabla_Y T, \nabla_X T) \\
&= \overline{g}(\nabla_X T, \nabla_Y T),
\end{aligned}$$

Therefore, $(\nabla_U \mathrm{Hess}f)(X,Y) = \overline{g}(\nabla^2 T(X,U), \nabla_Y T) + \overline{g}(\nabla_X T, \nabla^2 T(Y,U))$ for any tensor field U. Given that $\nabla^2 T = 0$, it follows that Hess$f$ is a symmetric and parallel $(0,2)$-tensor field, and using the classical Eisenhart Theorem 2.3.6 $\mathrm{Hess}f(X,Y) = c\overline{g}(X,Y)$ follows[1]. Then, last equality in (5.2) implies that $Z = \mathrm{grad}(f)$ is a homothetic vector field, i.e., $\mathcal{L}_Z\overline{g}(X,Y) = 2c\overline{g}(X,Y)$. So, using Lemma 5.1.2, Z is affine. Moreover,

$$\overline{g}(\nabla_X Z, Y) = \mathrm{Hess}f(X,Y) = c\overline{g}(X,Y), \ \ \forall X,Y \Longrightarrow \nabla_X Z = cX, \ \ \forall X \in \mathfrak{X}(M),$$

as required. ■

Even more, we have the following result for rth-symmetric spaces, to be use latter:

Lemma 5.1.5. *If there exists a vector field Z on an rth-symmetric Lorentzian space (M,g) such that $\nabla_X Z = cX$ for all $X \in \mathfrak{X}(M)$, then either the vector field is parallel or the manifold is locally symmetric.*

Proof. As Z is an homothetic vector field, from Lemmas 5.1.2 and 5.1.3 it follows that $\mathcal{L}_Z(\nabla^k R) = 0$ for any $k \geq 0$. Taking $T = \nabla^{r-1}R$ in (5.3) with R the curvature tensor, as the manifold is rth-symmetric, one obtains that

$$c\nabla^{r-1}R = 0,$$

so either Z is parallel or the manifold is $(r-1)$th-symmetric and, by induction, locally symmetric. ■

[1] Another way to prove this is to take $h = \mathrm{Hess}f$ on a point $p \in M$. The tensor $h_{|p}$ is a bilinear form invariant by the group of holonomy in p of the vector space T_pM, so by the Schur Lemma $h_{|p} = c_{|p}g_{|p}$ for some constant $c_{|p}$. But since h is parallel, $\nabla c_{|p} = 0$, so $c_{|p} = c$ is a constant.

5.2 The Riemannian Case

In 1952 Lichnerowicz [56, 57] proved that any compact Riemannian manifold with vanishing $\nabla^r R$ must be locally symmetric ($\nabla R = 0$). The generalization of this result to any tensor T, and not only the curvature tensor, was given later by Nomizu and Ozeki on complete Riemannian manifolds [64]. These results are stated as follows:

Theorem 5.2.1 (Lichnerowicz 1952, [56, 57]). *Let $(\overline{M}, \overline{g})$ be a* **compact** *Riemannian manifold. If the rth covariant derivative $\nabla^r R$ of the curvature tensor field R vanishes for some $r > 1$, then $\nabla R = 0$.*

Theorem 5.2.2 (Nomizu and Ozeki 1962,[64]). *Let $(\overline{M}, \overline{g})$ be a* **complete** *Riemannian manifold and T an arbitrary tensor field on M. If $\nabla^r T = 0$ for some $r > 1$, then $\nabla T = 0$.*

Ten years later Tanno in [87] extended the result by removing the assumption of completeness, but restricting its applicability to a specific type of tensors, called in the literature *curvature concomitants*, i.e., a linear combination of contractions and tensorial products of derivatives of R (see, e.g. [77, 79] and Section 6.2.1 for more information).

Theorem 5.2.3 (Tanno 1972, [87]). *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold and T the curvature tensor R, Ricci tensor Ric or the Weyl tensor C of the manifold. If $\nabla^r T = 0$ for some $r > 1$, then $\nabla T = 0$.*

More generally, if T is a curvature (k, s)-tensor concomitant , then $\nabla^r T = 0$ for some $r > 1$ implies $\nabla T = 0$.

As a consequence of Theorem 5.2.3, we have that:

Corollary 5.2.4. *For $r \geq 2$:*

1. *Any r-th symmetric Riemannian manifold is, in fact, a locally symmetric space.*

2. *There are no proper r-th symmetric Riemannian manifolds.*

> **Remark 5.2.5.** *Since Theorems 5.2 and 5.2.3 for locally Euclidean spaces are trivial, the proof of them without irreducibility is easily simplified to the case where $\overline{M}$ is irreducible and not locally Euclidean by first going to the universal covering of $\overline{M}$ and then using the de Rham decomposition theorem (see Section 2.3).*

Let us first give the proof of Theorem 5.2.3:

Proof of Theorem 5.2.3. It is enough to prove the assertion for $r = 2$. Assume $(\overline{M}, \overline{g})$ is irreducible (use Remark 5.2.5). Take the metrically equivalent $(0, s + k)$-tensor field to T, which will be denoted with the same symbol T. Then, defining f and Z as in Lemma

5.1.4 we have by Lemma 5.1.3 that $\mathcal{L}_Z(\nabla T) = \nabla(\mathcal{L}_Z T) = 0$. Thus, applying last formula in Lemma 5.1.4, it follows that

$$0 = \nabla_Z(\nabla T) + (s+k)c\nabla T = (s+k)c\nabla T.$$

Therefore, $\nabla T = 0$ or $c = 0$. But if $c = 0$, then Hess$f(X,Y) = \overline{g}(\nabla_X T, \nabla_Y T) = 0$ for all $X, Y \in \mathfrak{X}(M)$, for $X = Y$ and, as $\overline{g}$ is positive-definite, we have that $\nabla T = 0$. ■

Proof of Theorem 5.2. If $(\overline{M}, \overline{g})$ were complete, then the vector field Z in the proof above would be a globally defined homothetic vector field. By a theorem of Kobayashi [51] it follows that it is a Killing vector field. Therefore, $\mathcal{L}_Z\overline{g} = 0$ and $c = 0$, for any tensor field T (and not only for the given curvature concomitants), hence Theorem 5.2 is proven. ■

If T is not a curvature concomitant and non-completeness is assumed, the introduction of the concept of *generic points* in a manifold is needed, as far as we know, to state similar results.

5.3 The Semi-Riemannian Case

In 1972 Tanno also proved in his paper [87], and for Riemannian manifolds, that for any tensor field T the vanishing of $\nabla^r T$ implies the vanishing of ∇T around generic points (i.e., points that admit the inverse of R as an endomorphism between the space of two-forms). There, he basically followed the proof of Theorem 5.2.3 making slight modifications. Later in 2008, Senovilla generalized this result to any signature (see [79]), and he proved it in a totally different way. Indeed, Tanno in [87] worked also on the semi-Riemannian case, but, again, he based his proof on that of Theorem 5.2.3, which restricts his study substantially.

In this section, the definition of generic points is given (see also [25]), and the two results by Tanno (in Riemannian and semi-Riemannian manifolds) are proven. Then, the general result by Senovilla is proven, and its consequence on r-th symmetric semi-Riemannian spaces is stated.

5.3.1 Definition of Generic Points

Let (N,h) be a semi-Riemannian manifold. Then, think of its curvature tensor field R as an endomorphism between the space of two-forms (see, for example, [66]), i.e., characterized on simple 2-forms as:

$$\begin{aligned} R : \Lambda^2 N &\longrightarrow \Lambda^2 N \\ \omega \wedge \tau &\rightarrow 2R(\omega^\sharp, \tau^\sharp, \cdot, \cdot) \end{aligned}$$

Obviously, it can be extended to any two-form by linearity. By the anti-symmetric character of R in its first and second pair of slots, the endomorphism is well-defined. Observe that, in abstract index form, R is identified with $R^{\alpha\beta}{}_{\lambda\mu}$ and for $\Omega \in \Lambda^2 N$, $R(\Omega)$ with $R^{\alpha\beta}{}_{\lambda\mu}\Omega_{\alpha\beta}$.

Then,

Definition 5.3.1. *A point p in a semi-Riemannian manifold (N, h) is* generic *if $R_{|p}$ the Riemannian tensor R at p is non-singular, i.e., if the endomorphism R on the space of two-forms at p is non-singular.*

Let $p \in N$ be a generic point. Then, there exists the inverse endomorphism $R_{|p}{}^{-1}$ of $R_{|p}$ –denoted in abstract index form as $(R_{|p}{}^{-1})^{\alpha\beta}{}_{\lambda\mu}$– at the point p. In fact, by the smoothness of R, the inverse R^{-1} is defined in some neighborhood U of p. This endomorphism satisfies:

$$R_{|q}{}^{\alpha\beta}{}_{\rho\sigma}(R_{|q}{}^{-1})^{\rho\sigma}{}_{\lambda\mu} = \frac{1}{2}\delta^{\alpha\beta}_{\lambda\mu} := \frac{1}{2}(\delta^{\alpha}_{\lambda}\delta^{\beta}_{\mu} - \delta^{\alpha}_{\mu}\delta^{\beta}_{\lambda}), \quad \forall q \in U \tag{5.4}$$

where $\delta^{\alpha\beta}_{\lambda\mu}$ is the generalized Kronecker delta (see Section 2.1.3).

5.3.2 Tanno's Results for semi-Riemannian Manifolds

The first result of Tanno is stated as follows:

Theorem 5.3.2 (Tanno 1972, [87]). *Let $(\overline{M}, \overline{g})$ be a Riemannian manifold and T any tensor field on $\overline{M}$. Around generic points, the vanishing of its second covariant derivative implies the vanishing of the first covariant derivative.*

Proof. We can assume that $(\overline{M}, \overline{g})$ is irreducible by Remark 5.2.5. Then, following the proof of Theorem 5.2.3, define $f = \frac{1}{2}\overline{g}(T, T)$ and prove that $Z = \nabla f$ is affine and $\nabla^2 f$ is parallel (i.e. $\nabla^2 Z = 0$). Then, by formula (2.9):

$$0 = R(Z, \cdot).$$

Evaluate at any point q of a neighborhood U of p where R^{-1} is defined and apply R^{-1} to the equation above, to get

$$Z_q{}^{\flat} \wedge \vec{x}\,{}^{\flat} = 0, \quad \forall q \in U \text{ and } \forall \vec{x} \in T_q\overline{\mathcal{M}}.$$

Therefore, $Z_{|U} = 0$ and $\text{Hess} f(X, Y)_{|U} = \overline{g}(\nabla_X T, \nabla_Y T)_{|U} = 0$ for all X, Y, for $X = Y$ and, as $\overline{g}$ is positive-definite, we have that $\nabla T_{|U} = 0$. ■

The result for semi-Riemannian manifolds makes use of Theorem 2.3.12, and it is stated as follows:

Theorem 5.3.3. *Let (N, h) be an irreducible semi-Riemannian manifold with index q and assume that any of these two cases is satisfied:*

- *dim $N = 2$ or odd,*
- *dim $N \geq 4$ is even and $n - q \neq q$.*

Then, around generic points, $\nabla^2 T = 0$ implies that ∇T is null (i.e., $h(\nabla T, \nabla T) = 0$) and $h(T,T)$ is constant.

Proof. Following the same steps as in the proof of Theorem 5.3.2 (use Theorem 2.3.12), one arrives at $Z_{|U} = \nabla f_{|U} = 0$ in some neighborhood U of a generic point p, with $f = \frac{1}{2}h(T,T)$. Therefore, $(h(T,T))_{|U}$ is constant and, as $\mathrm{Hess} f_{|U} = (h(\nabla T, \nabla T))_{|U} = 0$,the tensor field ∇T restricted to U is null. ■

5.3.3 Senovilla's General Result

This result is stated as follows:

Theorem 5.3.4 (Senovilla 2008, [79])**.** *Let (N,h) be a semi-Riemannian manifold and T any tensor field on N. Around generic points, the vanishing of its second covariant derivative implies the vanishing of the first covariant derivative.*

Proof. Tensors in this proof will be denoted in abstract index form. Let p be a generic point and T a tensor field with vanishing second covariant derivative (i.e., $\nabla_\lambda \nabla_\mu T^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s} = 0$). The Ricci identity in abstract index form (2.1) applied to $Q = T$ and $Q = \nabla T$ provides, respectively,

$$0 = \sum_{i=1}^{s} R^{\rho}{}_{\beta_i \lambda\mu} T^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_{i-1}\rho\beta_{i+1}\ldots\beta_s} - \sum_{i=1}^{r} R^{\alpha_i}{}_{\rho\lambda\mu} T^{\alpha_1 \ldots \alpha_{i-1}\rho\alpha_{i+1}\ldots\alpha_r}_{\beta_1\ldots\beta_s}, \tag{5.5}$$

$$0 = R^{\rho}{}_{\nu\lambda\mu}\nabla_\rho T^{\alpha_1 \ldots \alpha_r}_{\beta_1\ldots\beta_s} + \sum_{i=1}^{s} R^{\rho}{}_{\beta_i\lambda\mu}\nabla_\nu T^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_{i-1}\rho\beta_{i+1}\ldots\beta_s} - \sum_{i=1}^{r} R^{\alpha_i}{}_{\rho\lambda\mu}\nabla_\nu T^{\alpha_1 \ldots \alpha_{i-1}\rho\alpha_{i+1}\ldots\alpha_r}_{\beta_1\ldots\beta_s}. \tag{5.6}$$

Evaluating (5.5) and (5.6) at any point q in the neighborhood U of p where R^{-1} is defined, contracting them with the inverse $(R^{-1})^{\lambda\mu}{}_{\rho\sigma}$ and using (5.4) and the identity $g_{\alpha\rho}\delta^{\rho}_{\beta} = g_{\alpha\beta}$, one obtains, respectively,

$$\begin{aligned} 0 = {} & \sum_{i=1}^{s} \left(g_{\beta_i\sigma} T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_{i-1}\rho\beta_{i+1}\ldots\beta_s} - g_{\beta_i\rho} T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_{i-1}\sigma\beta_{i+1}\ldots\beta_s} \right) \\ & - \sum_{i=1}^{r} \left(\delta^{\alpha_i}_{\rho} g_{\rho\sigma} - \delta^{\alpha_i}_{\sigma} g_{\rho\rho} \right) T^{\alpha_1\ldots\alpha_{i-1}\rho\alpha_{i+1}\ldots\alpha_r}_{\beta_1\ldots\beta_s}, \\ 0 = {} & g_{\nu\sigma}\nabla_\rho T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} - g_{\nu\rho}\nabla_\sigma T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} + \sum_{i=1}^{s}\left(g_{\beta_i\sigma}\nabla_\nu T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_{i-1}\rho\beta_{i+1}\ldots\beta_s} - g_{\beta_i\rho}\nabla_\nu T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_{i-1}\sigma\beta_{i+1}\ldots\beta_s}\right) \\ & - \sum_{i=1}^{r} \left(\delta^{\alpha_i}_{\rho} g_{\rho\sigma} - \delta^{\alpha_i}_{\sigma} g_{\rho\rho} \right) \nabla_\nu T^{\alpha_1\ldots\alpha_{i-1}\rho\alpha_{i+1}\ldots\alpha_r}_{\beta_1\ldots\beta_s}. \end{aligned}$$

Then, covariantly differentiating the first and substracting the second, one gets

$$g_{\nu\sigma}\nabla_\rho T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} - g_{\nu\rho}\nabla_\sigma T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = 0,$$

and contracting ν and ρ one proves that $\nabla_\sigma T^{\alpha_1\ldots\alpha_r}_{\beta_1\ldots\beta_s} = 0$. ■

Therefore, the following holds for rth-symmetric manifolds.

Corollary 5.3.5. *For any $r \geq 2$ if a (connected) semi-Riemannian manifold (N, h) contains a generic point, then $\nabla^2 T = 0$ implies $\nabla T = 0$ for any tensor field T. In particular, if the manifold is rth-symmetric, then it is locally symmetric.*

Proof. Let p be a generic point. By Theorem 5.3.4, there exists a local chart (U, φ) such that $p \in U$ and $(\nabla T)_{|U} = 0$. Therefore, $\nabla^2 T = 0$ implies $\nabla T = 0$ (since ∇T at any point q is obtained by parallel transporting it along any curve from p). ■

Furthermore:

Theorem 5.3.6 (Theorem 2.1 in [79]). *All semi-symmetric spaces (and thus, 2nd-symmetric spaces) are of constant curvature at generic points.*

5.4 Cahen-Wallach Spaces of higher order

The simple examples of rth-symmetric ($\nabla^r R = 0$) Lorentzian manifolds we introduce below will become the archetypes for these spaces at least for $r = 2$. In fact, this family is a generalization of the d-dimensional Cahen-Wallach spaces $CW^d(A)$ introduced in Theorem 3.2.16, the latter being the archetypes for the locally symmetric ($\nabla R = 0$) Lorentzian spaces.

For $r \geq 1$ and $d > 2$, let A be $(d-2) \times (d-2)$- dimensional matrix whose components are polynomials on u of degree less or equal than $r - 1$, i.e.,

$$A \equiv A(u) = A^{(r-1)} u^{r-1} + \ldots + A^{(1)} u + A^{(0)} \tag{5.7}$$

where each $A^{(l)}$ is a constant symmetric $(d-2) \times (d-2)$ matrix for all $l \in \{0, \ldots, r-1\}$.

Definition 5.4.1. *The d-dimensional* generalized Cahen-Wallach space of order r *with $r \geq 1$, denoted by $CW_r^d(A)$ where $A = (A_{ij})$ is a matrix of type* (5.7)*, is the space $(\mathbb{R}^d, g_A)$ with*

$$g_A = -2du \left(dv + A_{ij} x^i x^j du \right) + \delta_{ij} dx^i dx^j, \tag{5.8}$$

where δ_{ij} is the Kronecker delta.

When $r = 0$, then $CW_0^d(A)$ is defined as the flat space. A generalized Cahen-Wallach space of order r with $r \geq 1$ is proper *if $A^{(r-1)} \neq (0)$, that is, if at least one of the polynomials of A has degree $r - 1$.*

Observe that $CW_1^d(A) = CW^d(A)$. We denote *the family of d-dimensional generalized Cahen-Wallach spaces of order r* as CW_r^d.

Proposition 5.4.2. *Any proper generalized Cahen-Wallach space of arbitrary order r with $r \geq 1$ satisfies that:*

(i) it is a Brinkmann space, therefore, it contains a parallel lightlike vector field,

(ii) it is analytic and (geodesically) complete, and,

(iii) it is semi-symmetric and proper rth-symmetric.

Proof. (i) is obvious; just compare the metric (5.8) with the expression for the Brinkmann metric (4.1) in a fixed Brinkmann chart.

For (ii): the analyticity is also obvious from the expression of the metric, and the completeness follows from [18, Proposition 3.5], where a more general type of plane waves was treated. We add the proof for this specific type of plane waves, for completeness. The geodesics of the spacetimes under consideration are:

$$\gamma(u) = \left(u, \sum_i \dot{x}^i x^i + au + b, x^i(u) \right),$$

where a and b are arbitrary constants and the functions $x^i(u)$ satisfy the diferential equations $\ddot{x} = Ax$ written in a matrix form. By the smoothness of A, there exist smooth solutions $x^i(u)$ for the equation $\ddot{x} = Ax$, for all values of the parameter u. Consequently v can also be calculated for all values of u.

The properties in (iii) follow by computing the derivatives of the curvature tensor in the partly null frame $\{E_\alpha\} = \{\partial_u - H\partial_v, \partial_v, \partial_i - W_i\partial_v\} = \{\partial_u - A_{ij}x^i x^j \partial_v, \partial_v, \partial_i\}$ associated to a fixed Brinkmann chart (see Section 4.5). The only non-vanishing components of $\nabla^l R$ for $l \in \{0, \dots, r-1\}$ are

$$\nabla_0 \overset{(l)}{\dots} \nabla_0 \, R^1{}_{i0j} = \frac{d^l A_{ij}}{du^l} = \sum_{k=l}^{r-1} \frac{k!}{(k-l)!} A_{ij}^{(k)} u^{k-l} \tag{5.9}$$

and for $l \geq r$

$$\nabla^l R = 0. \tag{5.10}$$

The conditions of semi-symmetry in a given (arbitrary) frame $\{V^\alpha\}$ of TM are expressed as $\nabla_{[\alpha}\nabla_{\beta]} R_{\lambda\mu\nu\rho} = 0$. So (5.9) leads to semi-symmetry and (5.10) to rth-symmetry. ■

The following lemma will be used to reduce the global version of Theorem 1.1 to the local one (see Theorem 6.3.2).

Proposition 5.4.3. *Let (M, g) be a complete simply-connected Lorentzian manifold which is locally isometric to the product of some generalized Cahen-Wallach space $CW_r^d(A)$ with a simply connected Riemannian symmetric space. Then, (M, g) is in fact globally isometric to such a single product.*

Proof. By assumption, (M, g) is locally isometric to an analytical manifold due to Propositions 3.2.3 and 5.4.2 and, thus, it is analytical too. The result follows from the fact that, for any two complete simply-connected analytic semi-Riemannian manifolds (N, h) and (N', h'), every isometry defined between connected open subsets of N and N' can be uniquely extended to an isometry for the entire N and N' (see, for example [52, Cor. 6.4 Ch. VI] for the Riemannian case, and [52, Th. 6.1 Ch. VI], [65, Cor. 7.29] for its generalization to the semi-Riemannian one). ■

Chapter 6

Lorentzian Second-order Symmetric Spaces

In this chapter we are going to deal with the particular case $r = 2$ of Lorentzian rth-symmetric spaces. First, we present some of their characterizations, and state a specific property of these spaces, due to Senovilla [79]: the existence of a parallel lightlike vector field. Therefore, any 2nd-symmetric space is a Brinkmann space (see page 53 for the definition of a Brinkmann space). This allows us to solve the main problem treated in this thesis: the complete classification of Lorentzian proper 2nd-symmetric spaces.

6.1 Characterizations

In the proper 2nd-symmetric spaces some lightlike directions are remarkable. For that reason we recall first the notion of null sectional curvature.

Definition 6.1.1 (Harris 1985, [43]). *Let (N, h) be a semi-Riemannian manifold and $p \in N$.*

Let Π_p be a lightlike plane in T_pN and $\vec{w}, \vec{x} \in \Pi_p$ a lightlike and a spacelike vector respectively that span Π_p (necessarily $h(\vec{w}, \vec{x}) = 0$, otherwise Π_p is timelike).The null sectional curvature $K_{\vec{w}}(\Pi_p)$ of Π_p associated to $\vec{w}$ *is defined as:*

$$K_{\vec{w}}(\Pi_p) = \frac{R_p(\vec{x}, \vec{w}, \vec{x}, \vec{w})}{h_p(\vec{x}, \vec{x})}.$$

Notice that the value of this curvature depends on the choice of $\vec{w}$, but its sign is independent of this choice. It is well-known that the curvature is constant (at each point and, thus, if $n \geq 3$ at all points) if and only if all the null sectional curvatures vanish (see [65, Proposition 8.28]). Besides, the curvature on non-degenerate planes determines the curvature on any plane by the following limit process: let $\vec{z}, \vec{u} \in T_pN$ that span a degenerate (lightlike) plane. Then, there exist two sequences of vectors $\{\vec{z}_k\}$ and $\{\vec{u}_k\}$ which converge to $\vec{z}$ and $\vec{u}$ respectively, and such that the planes Π_k spanned by $\vec{z}_k$ and

$\vec{u}_k$ are non-degenerate. Therefore:

$$R_p(\vec{z}, \vec{u}, \vec{z}, \vec{u}) = \lim_{k\to\infty} R_p(\vec{z}_k, \vec{u}_k, \vec{z}_k, \vec{u}_k).$$

We introduce the following basic notion on parallelism of tensor fields, necessary in our subsequent proofs:

Lemma 6.1.2. *Let T be a (k, s)-tensor field on a semi-Riemannian manifold (N, h).*

Then T is parallel if and only if for any curve α and any parallelly propagated vector fields $X_1, \ldots, X_s$ and one-forms $\omega^1, \ldots, \omega^{k-1}$ along α, the vector field

$$T^* = T(\omega^1, \ldots, \omega^{k-1}, \cdot, X_1, \ldots, X_s)$$

is a parallel vector field along α .

Proof. Use that

$$\nabla_{\gamma'} T^* = (\nabla_{\gamma'} T)(\omega^1, \ldots, \omega^r, X_1, \ldots, X_{s-1}, \cdot). \tag{6.1}$$

■

Now, let (N, h) be a semi-Riemannian manifold, p any point in N and a tangent plane $\Pi_p \subset T_pN$. For any geodesic γ such that $p = \gamma(0)$, consider the parallelly transported plane $\tau \mapsto \Pi_\gamma(\tau) \subset T_{\gamma(\tau)}N$. Using that the scalar product between parallelly propagated vector fields is constant and that the sectional curvature $K(\Pi_p)$ on non-degenerate planes Π_p determines the full curvature tensor, one easily derives the next Lemma.

Lemma 6.1.3. *The following three conditions are equivalent in a semi-Riemannian manifold (N, h):*

- *(i) For any non-degenerate tangent plane $\Pi_p \subset T_pN$, its parallel transport Π_γ along any geodesic γ satisfies that $\frac{d}{d\tau}(K(\Pi_\gamma))$ remains constant along γ.*
- *(ii) For any parallelly propagated vector fields X, Y, Z along any geodesic γ, the vector field $(\nabla_{\gamma'} R)(X, Y)Z$ is itself parallelly propagated along γ.*
- *(iii) $\nabla_X \nabla_Y R := \nabla_X(\nabla_Y R) - \nabla_{\nabla_X Y} R$ is skew-symmetric in $X, Y \in \mathfrak{X}(N)$.*

Moreover, if these conditions hold, then the following property follows:

- *(S) if Π is a lightlike plane with radical spanned by $\vec{v} \in T_pN$, and V and Π_γ are the parallel transports of $\vec{v}$ and Π along any geodesic γ, respectively, then $\frac{d}{d\tau}(K_V(\Pi_\gamma))$ remains constant along γ, where $K_V(\Pi_\gamma)$ denotes the null sectional curvature of Π_γ along γ.*

Proof. To prove (i)$\Longleftrightarrow$ (ii), differentiate the formula $R(X,Y,Z,W) = h(R(X,Y)Z,W)$ along a geodesic γ with X,Y,Z,W parallel vector fields along γ, and obtain

$$\frac{d}{d\tau}(R(X,Y,Z,W)) = h(\nabla_{\gamma'}(R(X,Y)Z),W) = h((\nabla_{\gamma'}R)(X,Y)Z,W)$$

Then, using this equation, (i) follows easily from (ii). For the converse, reason as in Proposition 3.1.2.

For (ii)$\Longrightarrow$(iii), let $p \in N$, $\vec{v},\vec{w},\vec{x},\vec{y},\vec{z} \in T_pN$ and γ the geodesic with initial values $(\gamma(0),\gamma'(0)) = (p,\vec{v}+\vec{w})$. Let X,Y,Z be the vector fields obtained after parallel transporting $\vec{x},\vec{y},\vec{z}$ along γ respectively. Using (ii) and (6.1) for $T = \nabla R$, one has that (recall notation $\nabla_X\nabla_Y R := \nabla_X(\nabla_Y R) - \nabla_{\nabla_X Y}R$ to be used only in this proof):

$$0 = \nabla_{\gamma'}(\nabla_{\gamma'}(R(X,Y)Z) = (\nabla_{\gamma'}\nabla_{\gamma'}R)(X,Y)Z.$$

Evaluating at $\gamma(0)$, one gets

$$0 = (\nabla_{\vec{v}+\vec{w}}\nabla_{\vec{v}+\vec{w}}R)(\vec{x},\vec{y})\vec{z} = (\nabla_{\vec{v}}\nabla_{\vec{v}}R)(\vec{x},\vec{y})\vec{z}+(\nabla_{\vec{w}}\nabla_{\vec{w}}R)(\vec{x},\vec{y})\vec{z}+(\nabla_{\vec{v}}\nabla_{\vec{w}}R+\nabla_{\vec{w}}\nabla_{\vec{v}}R)(\vec{x},\vec{y})\vec{z}.$$

Let γ_1 and γ_2 be two geodesics with initial values $(\gamma_1(0),\gamma_1'(0)) = (p,\vec{v})$ and $(\gamma_2(0),\gamma_2'(0)) = (p,\vec{w})$. Reasoning as above, one can prove that

$$(\nabla_{\vec{v}}\nabla_{\vec{v}}R)(\vec{x},\vec{y})\vec{z} = 0 = (\nabla_{\vec{w}}\nabla_{\vec{w}}R)(\vec{x},\vec{y})\vec{z}.$$

So we conclude that

$$(\nabla_{\vec{v}}\nabla_{\vec{w}}R + \nabla_{\vec{w}}\nabla_{\vec{v}}R)(\vec{x},\vec{y})\vec{z} = 0 \ \ \forall\vec{v},\vec{w},\vec{x},\vec{y},\vec{z} \in T_pN.$$

In other words,

$$(\nabla_{\vec{v}}\nabla_{\vec{w}}R + \nabla_{\vec{w}}\nabla_{\vec{v}}R) = 0, \ \ \forall\vec{v},\vec{w} \in T_pN.$$

For the converse, take $X = Y = \gamma'$ for any geodesic curve γ. By skew-symmetry and $\nabla_{\gamma'}\gamma' = 0$, one obtains that the tensor $\nabla_{\gamma'}R$ is parallel along γ, so (ii) follows.

The statement (S) follows easily from the condition (ii) by the limit procces described in page 87. ∎

This result leads to the sought characterization of 2nd-symmetry.

Proposition 6.1.4. *The following statements are equivalent for a semi-Riemannian manifold* (N,h)*:*

(i) $\nabla\nabla R = 0$*, i.e.,* (N,h) *is a 2nd-symmetric space.*

(ii) If V,X,Y,Z *are parallelly propagated vector fields along any curve* α*, then* $(\nabla_V R)(X,Y)Z$ *is itself parallelly propagated along the curve.*

(iii) (N, h) *is semi-symmetric (i.e., the curvature tensor fulfills* $R(X,Y)R = 0$ *for all vector fields* X, Y*) and satisfies the equivalent conditions in Lemma 6.1.3 for any geodesic* γ*.*

Proof. As (ii) characterizes when the tensor ∇R is parallel (see Proposition 6.1.2), this condition is equivalent to (i). Also, both conditions imply (iii) trivially. For the converse, notice that, by semi-symmetry, $\nabla\nabla R(\cdot\,; Y, X) = \nabla_X(\nabla_Y R) - \nabla_{\nabla_X Y} R$ is symmetric in X, Y, since

$$R(X,Y)R := \nabla_X(\nabla_Y R) - \nabla_Y(\nabla_X R) - \nabla_{[X,Y]}R = \nabla\nabla R(\cdot\,; Y, X) - \nabla\nabla R(\cdot\,; X, Y) = 0,$$

and, *a fortiori*, it vanishes by applying the condition (iii) of Lemma 6.1.3. ∎

6.2 Existence of the Parallel Lightlike Vector Field

A pioneering study of Lorentzian 2nd-symmetric spaces was developed by Senovilla in [79]. There, he found a fundamental property which has been crucial in the research presented in the next Section, and is collected in here in the following theorem.

Theorem 6.2.1 (Senovilla 2008, [79])**.** *Any proper 2nd-symmetric Lorentzian space admits a parallel lightlike vector field* K*.*

The proof of this theorem was given in terms of causal tensors and super-energy tensors in [79]. In this section, we present a simplified proof of the theorem, also based on the original one [79]. One can also find an alternative proof involving holonomy groups in [2]. Observe that from the previous theorem one can deduce the uniqueness of the vector field K:

Corollary 6.2.2. *Let* (M, g) *be a proper (connected) 2nd-symmetric Lorentzian space. Then,* $\nabla R \neq 0$ *holds everywhere, and* (M, g) *admits a unique parallel lightlike direction.*

Proof. Obviously, if $(\nabla R)_p \neq 0$ at some $p \in M$, the (parallel tensor) ∇R cannot vanish at any point. To prove the existence of the parallel lightlike direction recall that, as we have already mentioned, around each point p there exists a parallel lightlike vector K [79, Theorem 4.2]. Moreover, there cannot exist a second such K' that is independent of K at p, as otherwise a parallel timelike vector field T could be constructed as a linear combination of K and K'; hence, the metric would split around p as a product $-dt^2 \oplus g_R$, with $T = \nabla t$ and g_R a Riemannian metric, which should be locally symmetric, in contradiction with $(\nabla R)_p \neq 0$. Thus, the corresponding parallel lightlike directions locally generated by K and K' must agree, so that they match in a single global one. ∎

For the proof of Theorem 6.2.1, we will always assume that the manifold is simply connected. Then, some technical results, identities and properties of concomitants on 2nd-symmetric spaces are needed.

6.2.1 Curvature Concomitants in a 2nd-symmetric space

For notational convenience we set $\nabla^0 T = T$ for any tensor field T.

Definition 6.2.3. *A* curvature tensorial concomitant *is any tensor field constructed as a linear combination of contractions and tensorial products of the metric and derivatives* $\nabla^s R$ *with finite* s.

Definition 6.2.4 ([34, 68, 71, 83] and references therein)**.** *A* curvature scalar invariant *is a curvature* $(0,0)$*-tensorial concomitant.*

Definition 6.2.5 ([79, 77])**.** *The* degree *of a curvature concomitant is the maximun power of the Riemann tensor used in the concomitant, i.e., the maximum number of tensorial products of* R *involved in one single summand of the concomitant.*

The order *is the maximum number of covariant derivatives involved in one single summand of the concomitant.*

Definition 6.2.6. *A curvature tensorial concomitant is* homogeneous with respect to the degree *if it has the same degree of* R *in all its terms.*

A curvature tensorial concomitant is homogeneous with respect to the order *if it has the same order of covariant derivatives in all its terms.*

A curvature tensorial concomitant is homogeneous *when it is homogeneous with respect to the degree and the order.*

Example 6.2.7. *(1) The scalar curvature* S *and the norm of* R *given by* $R^{\alpha\beta\lambda\mu}R_{\alpha\beta\lambda\mu}$ *are scalar invariants of order zero and degree one and two respectively.*

(2) $R_{\alpha\beta\lambda\mu}R^{\alpha\beta\lambda}{}_{\nu} + \nabla_\rho R_{\mu\gamma}\nabla^\rho R_\nu{}^\gamma$ *is a curvature* $(0,2)$*-tensor concomitant of same order and degree equal to two, homogeneous with respect to degree but not with respect to order.*

Then:

Lemma 6.2.8 ([79],Lemma 3.2)**.** *Any curvature one-form concomitant* ω *in a Lorentzian manifold* (M,g) *which is parallel must be necessarily lightlike or zero.*

Proof. If it is not lightlike or does not vanish, by Example 2.3.3 the metric is non-degenerately reducible to a flat extension $M_1 \times M_2$ of a manifold M_2 of dimension $n-1$

$$g = c\omega\otimes\omega + (g - c\omega\otimes\omega).$$

But this is not possible because the curvature one-form concomitant ω lies in the flat submanifold $(M_1, c\omega\otimes\omega)$, so as it is constructed in terms of the Riemann tensor, it must vanish. ■

A *null two-form* F is defined as a simple form (i.e., a wedge product) such that $g(F,F) = 0$, with g the natural extension of the metric to tensor fields (see [7]). Since any simple two-form can be expressed as the wedge product of two mutually orthogonal one-forms, it follows that for a null two-form there exist a lightlike vector $\vec{k}$ and a spacelike one $\vec{p}$ orthogonal to $\vec{k}$ such that

$$F = \vec{k} \wedge \vec{p}.$$

Then,

Lemma 6.2.9 ([79], Lemma 3.3)**.** *Let (M,g) be a 2nd-symmetric Lorentzian manifold with dimension $n > 2$. Then, any parallel curvature two-form concomitant F is a null two-form (possibly zero), and if it is non identically zero there exists a parallel lightlike vector field.*

Proof. Define the $(0,2)$-symmetric tensor L in abstract index notation by $L_{\alpha\beta} = F_{\alpha\rho}F_{\beta}{}^{\rho}$, which is parallel by assumption. Then, by Proposition 2.3.8 three possiblities arise: either L vanishes, it is proportional to the metric, or it is not. Define for two arbitrary vector fields $X, \widetilde{X}$ the vector fields $Y = F(X,\cdot)^\sharp$ and $\widetilde{Y} = F(\widetilde{X},\cdot)^\sharp$, so that $Y_\alpha = F_{\beta\alpha}X^\beta$ and $\widetilde{Y}_\alpha = F_{\beta\alpha}\widetilde{X}^\beta$. Then,

$$g(Y,\widetilde{Y}) = L(X,\widetilde{X}),$$

$$g(Y,Y) = L(X,X) \qquad g(\widetilde{Y},\widetilde{Y}) = L(\widetilde{X},\widetilde{X}),$$

$$g(Y,X) = F(X,X) = 0, \qquad g(\widetilde{Y},\widetilde{X}) = F(\widetilde{X},\widetilde{X}) = 0.$$

If $L = 0$ it follows that $Y, \widetilde{Y}$ are orthogonal and lightlike vector fields, so they must be proportional. Therefore, there exists a lightlike vector field K such that

$$F(X,\cdot) = \lambda(X)K^\sharp, \forall X.$$

Since F is antisymmetric, the same arguments stand with $Y = F(\cdot,X)$ and $\widetilde{Y} = F(\cdot,\widetilde{X})$ to prove that there exists a lightlike vector field $\widetilde{K}$ such that

$$F(\cdot,X) = \eta(X)\widetilde{K}^\sharp, \forall X.$$

Thus, $F = K^\sharp \wedge \widetilde{K}^\sharp$, which implies via $L = 0$ that $g(\widetilde{K},K) = 0$, so $\widetilde{K}$ and K are proportional lightlike vectors, and since F is antisymmetric, $F = 0$.

If $L = \lambda g \neq 0$, take $X = K$ a lightlike vector field so that Y is a lightlike vector field orthogonal to K. Thus,

$$F(K,\cdot) = \eta(K)K^\sharp, \forall K.$$

Take $\widetilde{X} = \widetilde{K}$ a lightlike vector field such that $g(K,\widetilde{K}) = -1$. Then

$$L(K,\widetilde{K}) = -\eta(K)\eta(\widetilde{K}) = -\lambda \qquad \text{for each pair } K, \widetilde{K}.$$

Hence, $\eta(K)$ must have the same sign for every lightlike vector field K, which is only possible if either $\lambda > 0$ or there are only two independent lightlike vector fields in the manifold. Therefore, $\lambda > 0$ unless $n = 2$, which is not the case. Take now X a timelike vector field. Then, Y is also a timelike vector field but orthogonal to X, which is not possible.

Consequently, if $L_{\alpha\beta} \neq 0$, then $L_{\alpha\beta}$ is a parallel $(0, 2)$-symmetric tensor not proportional to the metric. By standard considerations on the descomposition of two-forms in simple forms one has that either $L_{\alpha\beta}$ admits a timelike eigenvector (and hence it is of Segre type $[1, 1 \ldots 1]$) or $F_{\alpha\beta}$ has a unique null eigenvector, in which case $L_{\alpha\beta}$ is of Segre type $[21 \ldots 1]$. If F has a unique null eigenvector $\vec{k}$, it can be decomposed as a sum of a null two-form F_1 and another two-form F_2, orthogonal to the null plane π spanned by F_1 and without components along $\vec{k}^{\flat}$. Then, $L = \lambda\vec{k}^{\flat} \otimes \vec{k}^{\flat} + L_2$ with L_2 also orthogonal to π and without components along $\vec{k}^{\flat}$, therefore orthogonal to a timelike vector. Thus, the tensor field defined as $L_{2,\alpha\rho}L_{2,\beta}{}^{\rho}$ is a parallel $(0, 2)$-symmetric tensor not proportional to the metric with a timelike eigenvector associated to the zero eigenvalue, and the metric is non-degenerately reducible, unless $F_2 = 0$.

Therefore, if $F_{\alpha\beta}$ is not null, by Proposition 2.3.8 the manifold is non-degenerately reducible to a product of irreducible 2nd-symmetric spaces. As the two-form concomitant decomposes as the sum of the respective reduced two-forms, in each irreducible part either $L_{\alpha\beta} = 0$ and thus $F_{\alpha\beta} = 0$ too, or the dimension of the reduced manifold is two. But the 2-dimensional 2nd-symmetric spaces are the constant curvature ones, and thus, being $F_{\alpha\beta}$ parallel and a curvature concomitant, it must vanish also in these parts.

Finally, if $F_{\alpha\beta}$ is lightlike and simple, then $F_{\alpha\beta} = 2k_{[\alpha}p_{\beta]}$ with $\vec{k}$ lightlike and $\vec{p}$ spacelike and orthogonal to $\vec{k}$. Take $\vec{p}$ unitary by re-scaling $\vec{k}$ if necessary. Then, $L_{\alpha\beta} = k_\alpha k_\beta$, and as $L_{\alpha\beta}$ is parallel, so it is $\vec{k}$. ∎

Observe that in a 2nd-symmetric space necessarily the degree of a non-zero curvature scalar invariant must be greater or equal than its order, as the components involving derivatives of the Riemman tensor must not be of order higher than one. Besides, all the curvature tensorial concomitants with order equal to degree are parallel.

Proposition 6.2.10 ([79], Proposition 3.1)**.** *In a 2nd-symmetric Lorentzian space either there exists a parallel lightlike vector field, or all scalar curvature invariants of order m and degree up to $m + 2$ are constant.*

Therefore, all curvature invariants of any order and degree formed as functions of the homogeneous ones of order $m \geq 0$ and degree up to $m + 2$ are constant.

Proof. Let I be a scalar invariant of order m, so the degree must be greater or equal to m. If the degree is m one has that $\nabla I = 0$ and I is constant.

If the degree is $m + 1$, $\nabla_\alpha \nabla_\beta I = 0$ and $\omega_\alpha = \nabla_\alpha I$ is a parallel one-form concomitant so by Lemma 6.2.8 it is lightlike (and ω^α is a parallel lightlike vector field) or zero. If the latter, I is constant.

Finally, if the degree is $m+2$, $\nabla_\alpha \nabla_\beta \nabla_\lambda I = 0$ and $L_{\alpha\beta} = \nabla_\alpha \nabla_\beta I$ is a parallel symmetric $(0, 2)$-tensor field. By Proposition 2.3.8 there exist three posibilities: either there exists a parallel lightlike vector field, $L_{\alpha\beta}$ is proportional to the metric, or the manifold is non-degenerately reducible. In the last case, if we decompose the manifold in a direct product of irreducible manifolds locally (which are 2nd-symmetric spaces), then I will be a function of the corresponding scalar invariants of each product space and in one of the reduced parts $L_{\alpha\beta}$ will be proportional to the metric. Therefore, we can restrict ourselves to the second case: $L_{\alpha\beta}(= \nabla_\alpha\nabla_\beta I) = cg_{\alpha\beta}$. Then, defining the vector field $Z^\alpha = \nabla^\alpha I$ we have that $\nabla_\alpha Z_\beta = cg_{\alpha\beta}$ and by Lemma 5.1.5 either $c = 0$ or the (reduced) manifold is locally symmetric. In both cases $L_{\alpha\beta} = 0$, so that $\nabla_\alpha\nabla_\beta I = 0$. This condition coincides with the case when the degree is $m + 1$, and the result follows. ■

Corollary 6.2.11 ([79], Corollary 3.1)**.** *In a 2nd-symmetric Lorentzian space either there exists a parallel lightlike vector field, or the following statements hold:*

(i) All curvature one-form concomitants of order m and degree up to $m + 1$ are zero.

(ii) All curvature scalar invariants with order equal to degree vanish.

(iii) All curvature (0,2)-tensorial concomitants with order equal to degree vanish.

Proof. Let I_α be a curvature one-form concomitant of order m, so the degree must be greater or equal to m. If the degree is m, as the space is 2nd-symmetric I_α must be parallel and hence, by Lemma 6.2.8 it must be zero or lightlike. If the degree is $m + 1$, then $\nabla_\beta I_\alpha$ is parallel so that $F_{\alpha\beta} = \nabla_{[\alpha} I_{\beta]}$ is a parallel curvature two-form concomitant. By Lemma 6.2.9 either there is a parallel lightlike vector field or $F_{\alpha\beta} = 0$. In the latter, I_α becomes (locally) an exact one-form and there exists a function f such that $I_\alpha = \nabla_\alpha f$. Then, $\nabla_\alpha\nabla_\beta f$ is a parallel symmetric $(0, 2)$-tensor field. Thus, reasoning as in the last case of the proof of Proposition 6.2.10 it follows that $\nabla_\alpha\nabla_\beta f = 0$, so I_α is a parallel curvature one-form concomitant. Hence, by Lemma 6.2.8 it must be zero or lightlike.

Now, take a scalar invariant I with order and degree equal to m. Since the manifold is 2nd-symmetric, each R must go with a ∇. Thus, one can always construct a curvature one-form concomitant of order $m - 1$ and degree m such that $I = \nabla_\mu I^\mu$. By (i), unless there exists a parallel lightlike vector field, I_α is zero, so I is zero too.

Finally, let $I_{\alpha\beta}$ be a curvature (0,2)-tensorial concomitant with order equal to degree, so that $\nabla_\lambda I_{\alpha\beta} = 0$. By point (ii) its trace $I^\alpha{}_\alpha$ vanishes, and thus, if $I_{\alpha\beta}$ is not zero its totally symmetric part $I_{(\alpha\beta)}$ is not proportional to the metric. Again using the beginning of the proof of Proposition 6.2.10 $I_{(\alpha\beta)} = 0$. Therefore, $I_{\alpha\beta} = I_{[\alpha\beta]}$ is a parallel curvature two-form concomitant and by Lemma 6.2.9 either there is a parallel lightlike vector field or $I_{[\alpha\beta]} = 0$. ■

6.2.2 Proof of Theorem 6.2.1

Using the Ricci Identity in abstract index form (2.1) in a 2nd-symmetric space, $\nabla\nabla R = 0$ and $\nabla\nabla\nabla R = 0$ yield, respectively,

$$
\begin{aligned}
0 &= R^\rho{}_{\beta_1\lambda\mu} R^\alpha{}_{\rho\beta_2\beta_3} + R^\rho{}_{\beta_2\lambda\mu} R^\alpha{}_{\beta_1\rho\beta_3} + R^\rho{}_{\beta_3\lambda\mu} R^\alpha{}_{\beta_1\beta_2\rho} - R^\alpha{}_{\rho\lambda\mu} R^\rho{}_{\beta_1\beta_2\beta_3}, \\
0 &= R^\rho{}_{\nu\lambda\mu} \nabla_\rho R^\alpha{}_{\beta_1\beta_2\beta_3} + R^\rho{}_{\beta_1\lambda\mu} \nabla_\nu R^\alpha{}_{\rho\beta_2\beta_3} + R^\rho{}_{\beta_2\lambda\mu} \nabla_\nu R^\alpha{}_{\beta_1\rho\beta_3} + R^\rho{}_{\beta_3\lambda\mu} \nabla_\nu R^\alpha{}_{\beta_1\beta_2\rho} \\
&\quad - R^\alpha{}_{\rho\lambda\mu} \nabla_\nu R^\rho{}_{\beta_1\beta_2\beta_3}.
\end{aligned}
\tag{6.2}
$$

Then, covariantly differentiating the first and substracting the second, one gets

$$\begin{aligned} 0 &= \nabla_\nu R^\rho{}_{\beta_1\lambda\mu} R^\alpha{}_{\rho\beta_2\beta_3} + \nabla_\nu R^\rho{}_{\beta_2\lambda\mu} R^\alpha{}_{\beta_1\rho\beta_3} + \nabla_\nu R^\rho{}_{\beta_3\lambda\mu} R^\alpha{}_{\beta_1\beta_2\rho} \\ &\quad -\nabla_\nu R^\alpha{}_{\rho\lambda\mu} R^\rho{}_{\beta_1\beta_2\beta_3} - R^\rho{}_{\nu\lambda\mu} \nabla_\rho R^\alpha{}_{\beta_1\beta_2\beta_3}. \end{aligned} \tag{6.3}$$

Besides, the covariant derivative of (6.3) substracted with (6.2) yields

$$\nabla_{(\gamma} R^\rho{}_{\tau)\lambda\mu} \nabla_\rho R^\alpha{}_{\beta\nu\delta} = 0. \tag{6.4}$$

Thus, at any point $p \in M$, the tensor $\nabla_\alpha R_{\beta\lambda\mu\nu}$ satisfies the properties in Corollary 2.1.5, so there exists a lightlike vector $\vec{l}$ such that $l^\alpha \nabla_\alpha R_{\beta\lambda\mu\nu} = 0$ and $l^\beta \nabla_\alpha R_{\beta\lambda\mu\nu} = 0$. Using the symmetry properties of the tensor, one gets that actually $\nabla_\alpha R_{\beta\lambda\mu\nu}$ is totally orthogonal to the lightlike vector.

If $\vec{k}$ is another lightlike vector such that $g(\vec{l},\vec{k}) = -1$, the orthogonal splitting of $\nabla_\alpha R_{\beta\lambda\mu\nu}$ with respect to the plane span$\{\vec{l},\vec{k}\}$ leads to a decomposition involving a sum of tensorial products between tensors orthogonal to span$\{\vec{l},\vec{k}\}$ and $\vec{l}$, as follows:

$$\begin{aligned} \nabla_\alpha R_{\beta\lambda\mu\nu} &= A_{\alpha\beta\lambda\mu\nu} + l_\alpha B_{\beta\lambda\mu\nu} + l_{[\beta} C_{\lambda]\mu\nu\alpha} + l_{[\mu} C_{\nu]\beta\lambda\alpha} + l_\alpha l_{[\beta} D_{\lambda]\mu\nu} + l_\alpha l_{[\mu} D_{\nu]\beta\lambda} \\ &\quad + l_{[\beta} E_{\lambda]\alpha[\mu} l_{\nu]} + l_\alpha l_{[\beta} H_{\lambda][\mu} l_{\nu]} \end{aligned}$$

where $A_{\alpha\beta\lambda\mu\nu}, B_{\beta\lambda\mu\nu}, C_{\lambda\mu\nu\alpha}, D_{\lambda\mu\nu}, E_{\lambda\alpha\mu}$ and $H_{\lambda\mu}$ are totally orthogonal to $\vec{l}$ and $\vec{k}$, apart from other symmetry properties inherited from the Riemann tensor. Then, notice that

$$\nabla_\alpha R_{\beta\lambda\mu\nu} \nabla^\alpha R^{\beta\lambda\mu\nu}$$

is a curvature scalar invariant of order equal to degree, and

$$\nabla_\rho R_{\beta\lambda\mu\nu} \nabla_\sigma R^{\beta\lambda\mu\nu} \qquad \text{and} \qquad \nabla_\alpha R_{\rho\lambda\mu\nu} \nabla^\alpha R_\sigma{}^{\lambda\mu\nu}$$

are curvature (0,2)-tensorial concomitants with order equal to degree. So by Corollary 6.2.11, either there exists a parallel lightlike vector field, or they all vanish.

Assume that there does not exist a parallel lightlike vector field. Then,

$$\begin{aligned} \nabla_\alpha R_{\beta\lambda\mu\nu} \nabla^\alpha R^{\beta\lambda\mu\nu} &= A_{\alpha\beta\lambda\mu\nu} A^{\alpha\beta\lambda\mu\nu} = 0, \\ k^\rho k^\sigma (\nabla_\rho R_{\beta\lambda\mu\nu} \nabla_\sigma R^{\beta\lambda\mu\nu}) &= B_{\beta\lambda\mu\nu} B^{\beta\lambda\mu\nu} = 0, \\ k^\rho k^\sigma (\nabla_\alpha R_{\rho\lambda\mu\nu} \nabla^\alpha R_\sigma{}^{\lambda\mu\nu}) &= \frac{1}{4} C_{\lambda\mu\nu} C^{\lambda\mu\nu} = 0, \end{aligned}$$

and since these square of the norm of the tensors A, B, C are computed with respect to a positive definite scalar product in span$\{\vec{l},\vec{k}\}^\perp$, they vanish and

$$\nabla_\alpha R_{\beta\lambda\mu\nu} = l_\alpha l_{[\beta} D_{\lambda]\mu\nu} + l_\alpha l_{[\mu} D_{\nu]\beta\lambda} + l_{[\beta} E_{\lambda]\alpha[\mu} l_{\nu]} + l_\alpha l_{[\beta} H_{\lambda][\mu} l_{\nu]}$$

From $(\nabla_{(\gamma} R_{\tau)}{}^\rho{}_{\lambda\mu} \nabla_\rho R_{\alpha\beta\nu\delta}) k^\tau k^\mu k^\alpha k^\delta = 0$ and from $\nabla_{[\alpha} R_{\beta\lambda]\mu\nu} k^\alpha k^\nu = 0$ one gets, respectively,

$$E_\beta{}^\rho{}_\nu (E_{\rho\gamma\lambda} - 2D_{\lambda\gamma\rho}) = 0, \qquad D_{\mu\beta\lambda} = E_{[\beta\lambda]\mu}$$

so that

$$E_\beta{}^\rho{}_\nu(2E_{\rho\gamma\lambda} - E_{\gamma\rho\lambda}) = 0.$$

Then, from the symmetries of $\nabla_\alpha R_{\beta\lambda\mu\nu}$ it is easy to check that $E_{(\alpha|\beta|\lambda)} = E_{\lambda\beta\alpha}$. Therefore, by the equation above, $E_\beta{}^\rho{}_\nu E_{\gamma\rho\lambda} = E_\beta{}^\rho{}_\nu E_{\gamma\lambda\rho} = E_\beta{}^\rho{}_\nu E_{\rho\gamma\lambda}$ and thus, $E_\beta{}^\rho{}_\nu E_{\gamma\rho\lambda} = 0$. In particular $E^{\beta\rho\nu}E_{\beta\rho\nu} = 0$ and $E_{\alpha\beta\lambda}$ must vanish. So finally

$$\nabla_\alpha R_{\beta\lambda\mu\nu} = l_\alpha l_{[\beta} H_{\lambda][\mu} l_{\nu]}.$$

By 2nd-symmetry, the $(0,6)$-tensorial concomitant

$$T_{\alpha\beta\lambda\mu\nu\eta} = \nabla_\alpha R_{\lambda\rho\nu\sigma}\nabla_\beta R_\mu{}^\rho{}_\eta{}^\sigma = (H^{\rho\sigma}H_{\rho\sigma})l_\alpha l_\beta l_\lambda l_\mu l_\nu l_\eta$$

is parallel. Therefore, whenever $H^{\rho\sigma}H_{\rho\sigma}$ and thus H does not vanish (hence the manifold is not locally symmetric in that region), one can re-scale $\vec{l}$ to another lightlike vector field $\vec{l}'$ so that

$$T_{\alpha\beta\lambda\mu\nu\eta} = l'_\alpha l'_\beta l'_\lambda l'_\mu l'_\nu l'_\eta$$

in the region, and $\vec{l}'$ is the required parallel and lightlike vector field.

■

6.3 Structure Theorem

In this section we are going to present the path we have followed in order to prove the local version of Theorem 1.1, which we rewrite here for the convenience of the reader:

Theorem 6.3.1 (Local structure theorem). *An n-dimensional proper 2nd-symmetric Lorentzian space (M,g) is locally isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$ where (M_2,g_2) is a non-flat Riemannian symmetric space and (M_1,g_1) is a proper generalized Cahen-Wallach space of order 2.*

The proof of this theorem is done in four steps. In the first step (Section 6.3.1), using that any proper 2nd-symmetric space must be a Brinkmann space (Proposition 6.2.1), the first and second covariant derivatives of the curvature tensor –the tensor fields ∇R and $\nabla\nabla R$– for a Brinkmann space are calculated, in a fixed partly null frame. In the second step (Section 6.3.2), a fundamental reduction of the equations is made. In the third step (Section 6.3.3), we prove that the solution of the problem is equivalent to solving the equations of 2nd-symmetry on two simpler spaces (Proposition 6.3.20). To arrive at this conclusion, one must prove that there exists a Brinkmann decomposition $\{u,v\}$ of a Brinkmann space (M,g) such that $\overline{g}, H$ and W are simultaneously reducible (as defined in Section 4.6), so that the metric takes the form:

$$g = -2du(dv + (H^{(1)} + H^{(2)})du + \mathring{W}^{(1)} + \mathring{W}^{(2)}) + \mathring{\overline{g}}^{(1)} \oplus \mathring{\overline{g}}^{(2)}.$$

Finally, it only remains to solve the equations on each simpler space, which is done in Section 6.3.4.

The global version of the Structure theorem 6.3.1 states the following:

Theorem 6.3.2 (Global structure theorem)**.** *Let (M,g) be a geodesically complete and simply connected proper 2nd-symmetric Lorentzian space. Then, (M,g) is globally isometric to a direct product as the one given in Theorem 6.3.1.*

Proof. Observe that by Corollary 6.2.2, (M,g) admits a unique parallel lightlike vector field, to which a unique Cahen-Wallach space of order two is associated. In fact, this is the Cahen-Wallach space of Theorem 6.3.1. Therefore, applying Lemma 5.4.3 we obtain the result. ∎

6.3.1 Equations of Second-Symmetry

From Proposition 6.2.1 one knows that any proper 2nd-symmetric space must be a Brinkmann space. Therefore, the first and second covariant derivatives of the curvature for a Brinkmann space will be calculated here in a fixed partly null frame $\{E_\alpha\}$, in order to obtain the equations of 2nd-symmetry and solve them.

First derivative of the curvature: ∇R

Let (M,g) be a Brinkmann space with a fixed Brinkmann chart, and $\{E_\alpha\}$ its associated partly null frame as in (4.5). As E_1 is parallel, the curvature tensor will be determined on each Brinkmann chart by its value on quadruples of vectors tangent to $\overline{\mathcal{M}}$, plus some extra tensors which take care of the remaining components (partly along the θ^1 or E_0 directions).

The motivation of the next definition is that the tensors A and B, together with $\overline{R}$, will collect all the information of the curvature tensor R of the manifold.

Definition 6.3.3. *For any Brinkmann chart and its associated partly null frame $\{E_\alpha\}$ we define $A \in \Gamma(T^0_2\overline{\mathcal{M}})$ and $B \in \Gamma(T^0_3\overline{\mathcal{M}})$ as $A := \theta^1(R(E_0,\cdot))$ and $B := \theta^1(R)$, that is to say*

$$A(X,Y) = \theta^1(R(E_0,\mathring{Y})\mathring{X}), \qquad B(X,Y,Z) = \theta^1(R(\mathring{Y},\mathring{Z})\mathring{X}) \qquad \forall X,Y,Z \in \mathfrak{X}(\overline{\mathcal{M}}).$$

From the symmetries of the curvature tensor it is obvious that A is symmetric

$$A(X,Y) = A(Y,X) \qquad \forall X,Y, \in \mathfrak{X}(\overline{\mathcal{M}})$$

and that B is skew-symmetric in its last two slots

$$B(X,Y,Z) = -B(X,Z,Y)$$

and it satisfies a cyclic identity

$$B(X,Y,Z) + B(Y,Z,X) + B(Z,X,Y) = 0, \quad \forall X,Y,Z \in \mathfrak{X}(\overline{\mathcal{M}}).$$

Observe that, in the given basis $\{\partial_i\}$ of $T\overline{\mathcal{M}}$,

$$A_{ij} = A(\partial_i,\partial_j) = \theta^1(R(E_0,E_j)E_i) = R^1{}_{i0j},$$

$$B_{ijk} = B(\partial_i, \partial_j, \partial_k) = \theta^1(R(E_j, E_k)E_i) = R^1{}_{ijk}.$$

Then, the previous properties can be expressed using this notation as

$$A_{ij} = A_{(ij)}, \; B_{ijk} = B_{i[jk]}, \text{ and } B_{[ijk]} = 0.$$

In what follows, and for the sake of brevity, we will resort to using index notation in many cases, which is sufficient to illustrate these properties and reveals itself as very helpful in the required complicated calculations for 2nd-symmetry. As a starting example, note that we additionally have for instance

$$B_{ij}{}^k (= g^{kr} B_{ijr}) = R^k{}_{j0i}.$$

With these definitions we can encode all the information of the covariant derivative of the curvature tensor R of the manifold in the following way:

Lemma 6.3.4. *If $\mathcal{S}[T]$ gives the symmetric part of any covariant section $T \in \Gamma(T^0_s\overline{\mathcal{M}})$ in its last two slots, and $C_{ij} = g_{ik}C^k_j$ denotes the contraction of the i^{th} and j^{th} covariant indices (via the metric $\bar{g}^{-1}$), the projection of ∇R on $T\overline{\mathcal{M}}$ by the linear homomorphism $^-$ gives rise to the following formulae:*

$$\begin{aligned}
\overline{\theta^1(\nabla_{E_0} R(E_0, \cdot))} &= D_0 A + 2\mathcal{S}\left[C^1_3(h^\sharp \otimes B)\right] \\
\overline{\theta^1(\nabla R(E_0, \cdot))} &= \overline{\nabla} A - 2\mathcal{S}\left[C_{15}(t \otimes B)\right] \\
\overline{\theta^1(\nabla_{E_0} R)} &= D_0 B + h(\overline{R}) \\
\overline{\theta^1(\nabla R)} &= \overline{\nabla} B - C^1_1(t \otimes \overline{R}) \\
\overline{\nabla_{E_0} R} &= D_0 \overline{R} \\
\overline{\nabla R} &= \overline{\nabla}\, \overline{R}
\end{aligned}$$

There are no more equations involving derivatives of R (up to equivalent tensor fields by the symmetries of the tensor).

Proof. A direct computation of ∇R in the partly null frame $\{E_\alpha\}$ provides the formulae. Starting with the general formula (2.12) with $T = R$ given by

$$\nabla_\alpha R^\sigma{}_{\beta\lambda\mu} = E_\alpha(R^\sigma{}_{\beta\lambda\mu}) + \gamma^\sigma_{\rho\alpha} R^\rho{}_{\beta\lambda\mu} - \gamma^\rho_{\beta\alpha} R^\sigma{}_{\rho\lambda\mu} - \gamma^\rho_{\lambda\alpha} R^\sigma{}_{\beta\rho\mu} - \gamma^\rho_{\mu\alpha} R^\sigma{}_{\beta\lambda\rho} \tag{6.5}$$

one easily obtains, substituting appropriately, that,

$$\begin{aligned}
\nabla_0 R^1{}_{i0j} &= D_0 A_{ij} + 2h^k B_{(ij)k}, \\
\nabla_s R^1{}_{i0j} &= \overline{\nabla}_s A_{ij} - 2t^k{}_s B_{(ij)k}, \\
\nabla_0 R^1{}_{ijk} &= D_0 B_{ijk} + h_r \overline{R}^r{}_{ijk}, \\
\nabla_s R^1{}_{ijk} &= \overline{\nabla}_s B_{ijk} - t_{rs} \overline{R}^r{}_{ijk}, \\
\nabla_0 R^i{}_{jkl} &= D_0 \overline{R}^i{}_{jkl}, \\
\nabla_s R^i{}_{jkl} &= \overline{\nabla}_s \overline{R}^i{}_{jkl},
\end{aligned}$$

from which the formulae follow. Indeed, formula (6.5) gives no more possibilities for non-vanishing components of ∇R in this frame. ■

Second derivative of the curvature: $\nabla\nabla R$

Next, we give names to the left-hand sides of the relations above (except for the last one), thereby defining five tensor fields on $\overline{\mathcal{M}}$ which will allow for simpler expressions when computating $\nabla^2 R$. Again, our purpose is to define vector fields on $T\overline{\mathcal{M}}$ whose components correspond to the non-vanishing components of ∇R in a fixed partly null frame.

Definition 6.3.5. *For any Brinkmann chart and its associated partly null frame* $\{E_\alpha\}$ *we define* $\widetilde{A} \in \Gamma(T_2^0\overline{\mathcal{M}})$, $\widehat{A}, \widetilde{B} \in \Gamma(T_3^0\overline{\mathcal{M}})$, $\widehat{B} \in \Gamma(T_4^0\overline{\mathcal{M}})$ *and* $\widetilde{R} \in \Gamma(T_3^1\overline{\mathcal{M}})$, *for all* $X, Y, Z, V \in \mathfrak{X}(\overline{\mathcal{M}})$ *by:*

$$\begin{aligned}
\widetilde{A}(X,Y) &= \theta^1\left((\nabla_{E_0}R)(E_0,\mathring{Y})\mathring{X}\right),\\
\widehat{A}(X,Y,Z) &= \theta^1\left((\nabla_{\mathring{X}}R)(E_0,\mathring{Z})\mathring{Y}\right),\\
\widetilde{B}(X,Y,Z) &= \theta^1\left((\nabla_{E_0}R)(\mathring{Y},\mathring{Z})\mathring{X}\right),\\
\widehat{B}(X,Y,Z,V) &= \theta^1\left((\nabla_{\mathring{X}}R)(\mathring{Z},\mathring{V})\mathring{Y}\right),\\
\widetilde{R}(X,Y)Z &= \overline{\nabla_{E_0}R(\mathring{X},\mathring{Y})\mathring{Z}}.
\end{aligned}$$

Observe that, therefore, one has $\widetilde{R} = D_0\overline{R}$, $\widetilde{B} = D_0 B + h(\overline{R})$, etcetera in agreement with the formulae in Lemma 6.3.4.

Again, in the given basis $\{\partial_i\}$,

$$\begin{aligned}
\widetilde{A}_{ij} &= \widetilde{A}(\partial_i,\partial_j) = \nabla_0 R^1{}_{i0j}, & \widehat{A}_{sij} &= \widehat{A}(\partial_s,\partial_i,\partial_j) = \nabla_s R^1{}_{i0j},\\
\widetilde{B}_{ijk} &= \widetilde{B}(\partial_i,\partial_j,\partial_k) = \nabla_0 R^1{}_{ijk}, & \widehat{B}_{sijk} &= \widehat{B}(\partial_s,\partial_i,\partial_j,\partial_k) = \nabla_s R^1{}_{ijk},\\
\widetilde{R}^i{}_{jkl} &= \overline{d}x^i\left(\widetilde{R}(\partial_k,\partial_l)\partial_j\right) = \nabla_0 R^i{}_{jkl}.
\end{aligned}$$

Besides, from the symmetries of the curvature tensor it is obvious that

(1) $\widetilde{A}$ and $\widehat{A}$ are symmetric in the last two indices:

$$\widetilde{A}_{ij} = \widetilde{A}_{(ij)}; \quad \widehat{A}_{sij} = \widehat{A}_{s(ij)}.$$

(2) $\widetilde{B}, \widehat{B}, \widetilde{R}$ are skew-symmetric in their last two indices:

$$\widetilde{B}_{ijk} = \widetilde{B}_{i[jk]}; \quad \widehat{B}_{sijk} = \widehat{B}_{si[jk]}; \quad \widetilde{R}_{sijk} = \widetilde{R}_{si[jk]}.$$

(3) $\widetilde{B}, \widehat{B}, \widetilde{R}$ satisfy a cyclic identity:

$$\widetilde{R}_{i[jkl]} = 0; \quad \widetilde{B}_{[ijk]} = 0; \quad \widehat{B}_{s[ijk]} = 0.$$

(4) $\widetilde{R}$ also satisfies that

$$\widetilde{R}_{ijkl} = -\widetilde{R}_{jikl} \quad \text{and} \quad \widetilde{R}_{ijkl} = \widetilde{R}_{klij},$$

so that it has all the symmetries of a Riemann tensor.

One can prove that, in addition,

$$\widetilde{B}_{ij}{}^{k} = \nabla_0 R^{k}{}_{j0i} \quad \text{and} \quad \widehat{B}_{sij}{}^{k} = \nabla_s R^{k}{}_{j0i}.$$

We also point out the following basic relations:

$$\widetilde{R}_{ijkl} = -2\widehat{B}_{[ij]kl} \tag{6.6}$$

$$\widetilde{B}_{kij} = 2\widehat{A}_{[ij]k} \tag{6.7}$$

$$\widehat{B}_{[i|l|jk]} = 0 \qquad (\Longrightarrow \widehat{B}_{[i|l|j]k} = -\frac{1}{2}\widehat{B}_{klij})$$

They follow by direct application of the second Bianchi identity $\nabla_{[\alpha}R_{\beta\sigma]\rho\mu} = 0$ in the partly null frame $\{E_\alpha\}$: the first two relations follow by taking $\{\alpha, \beta, \sigma\} = \{0, i, j\}$ and the last one with $\{\alpha, \beta, \sigma\} = \{i, j, k\}$.

Lemma 6.3.6. *Using the notation in Definition 6.3.5, the projection of $\nabla\nabla R$ on $T\overline{\mathcal{M}}$ by the linear homomorphism $^{-}$ gives rise to the following formulae:*

$$\overline{\nabla\nabla R} = \overline{\nabla}\,\overline{\nabla}\,\overline{R}, \tag{6.8a}$$

$$\overline{\nabla_{E_0}\nabla R} = D_0\overline{\nabla}\,\overline{R}, \tag{6.8b}$$

$$\overline{\nabla\nabla_{E_0} R} = \overline{\nabla}\,\widetilde{R} + C_{13}(t \otimes \overline{\nabla}\,\overline{R}), \tag{6.8c}$$

$$\overline{\nabla_{E_0}\nabla_{E_0} R} = D_0\widetilde{R} - C_1^1(h^\sharp \otimes \overline{\nabla}\,\overline{R}), \tag{6.8d}$$

$$\overline{\theta^1(\nabla\nabla R)} = \overline{\nabla}\widehat{B} - C_1^1(t \otimes \overline{\nabla}\,\overline{R}), \tag{6.8e}$$

$$\overline{\theta^1(\nabla_{E_0}\nabla_s R)} = D_0\widehat{B} + C_1^1(h \otimes \overline{\nabla}\,\overline{R}), \tag{6.8f}$$

$$\overline{\theta^1(\nabla\nabla_{E_0} R)} = \overline{\nabla}\widetilde{B} - C_1^1(t \otimes \widetilde{R}) + C_{13}(t \otimes \widehat{B}), \tag{6.8g}$$

$$\overline{\theta^1(\nabla_{E_0}\nabla_{E_0} R)} = D_0\widetilde{B} + C_1^1(h \otimes \widetilde{R}) - C_1^1(h^\sharp \otimes \widehat{B}), \tag{6.8h}$$

$$\overline{\theta^1(\nabla\nabla R(E_0, \cdot))} = \overline{\nabla}\widehat{A} - 2\mathcal{S}[C_{16}(t \otimes \widehat{B})], \tag{6.8i}$$

$$\overline{\theta^1(\nabla_{E_0}\nabla R(E_0, \cdot))} = D_0\widehat{A} + 2\mathcal{S}[C_4^1(h^\sharp \otimes \widehat{B})], \tag{6.8j}$$

$$\overline{\theta^1(\nabla\nabla_{E_0} R(E_0, \cdot))} = \overline{\nabla}\widetilde{A} - 2\mathcal{S}[C_{15}(t \otimes \widetilde{B})] + C_{13}(t \otimes \widehat{A}), \tag{6.8k}$$

$$\overline{\theta^1(\nabla_{E_0}\nabla_{E_0} R(E_0, \cdot))} = D_0\widetilde{A} + 2\mathcal{S}[C_3^1(h^\sharp \otimes \widetilde{B})] - C_1^1(h^\sharp \otimes \widehat{A}). \tag{6.8l}$$

There are no more equations of this type involving second derivatives of R (up to equivalent tensor fields by the symmetries of the tensor).

Proof. Another direct computation of $\nabla\nabla R$ in the partly null frame $\{E_\alpha\}$, using the following formula, that comes from (2.12) with $T = \nabla R$

$$\begin{aligned}\nabla_\alpha\nabla_\nu R^{\sigma}{}_{\beta\lambda\mu} &= E_\alpha(\nabla_\nu R^{\sigma}{}_{\beta\lambda\mu}) + \gamma^{\sigma}_{\rho\alpha}\nabla_\nu R^{\rho}{}_{\beta\lambda\mu} - \gamma^{\rho}_{\nu\alpha}\nabla_\rho R^{\sigma}{}_{\beta\lambda\mu} - \gamma^{\rho}_{\beta\alpha}\nabla_\nu R^{\sigma}{}_{\rho\lambda\mu} \\ &\quad -\gamma^{\rho}_{\lambda\alpha}\nabla_\nu R^{\sigma}{}_{\beta\rho\mu} - \gamma^{\rho}_{\mu\alpha}\nabla_\nu R^{\sigma}{}_{\beta\lambda\rho}\end{aligned}$$

leads to the following expressions:

$$\nabla_m \nabla_s R^i{}_{jkl} = \overline{\nabla}_m \overline{\nabla}_s \overline{R}^i{}_{jkl}, \tag{6.9a}$$
$$\nabla_0 \nabla_s R^i{}_{jkl} = D_0 \overline{\nabla}_s \overline{R}^i{}_{jkl}, \tag{6.9b}$$
$$\nabla_s \nabla_0 R^i{}_{jkl} = \overline{\nabla}_s \widetilde{R}^i{}_{jkl} + t^r{}_s \overline{\nabla}_r \overline{R}^i{}_{jkl}, \tag{6.9c}$$
$$\nabla_0 \nabla_0 R^i{}_{jkl} = D_0 \widetilde{R}^i{}_{jkl} - h^r \overline{\nabla}_r \overline{R}^i{}_{jkl}, \tag{6.9d}$$
$$\nabla_n \nabla_s R^1{}_{ijk} = \overline{\nabla}_n \widehat{B}_{sijk} - t_{rn} \overline{\nabla}_s \overline{R}^r{}_{ijk}, \tag{6.9e}$$
$$\nabla_0 \nabla_s R^1{}_{ijk} = D_0 \widehat{B}_{sijk} + h_r \overline{\nabla}_s \overline{R}^r{}_{ijk}, \tag{6.9f}$$
$$\nabla_s \nabla_0 R^1{}_{ijk} = \overline{\nabla}_s \widetilde{B}_{ijk} - t^r{}_s (\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \tag{6.9g}$$
$$\nabla_0 \nabla_0 R^1{}_{ijk} = D_0 \widetilde{B}_{ijk} + h^r (\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \tag{6.9h}$$
$$\nabla_k \nabla_s R^1{}_{i0j} = \overline{\nabla}_k \widehat{A}_{sij} - 2t^r{}_k \widehat{B}_{s(ij)r}, \tag{6.9i}$$
$$\nabla_0 \nabla_s R^1{}_{i0j} = D_0 \widehat{A}_{sij} + 2h^r \widehat{B}_{s(ij)r}, \tag{6.9j}$$
$$\nabla_k \nabla_0 R^1{}_{i0j} = \overline{\nabla}_k \widetilde{A}_{ij} - t^r{}_k (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \tag{6.9k}$$
$$\nabla_0 \nabla_0 R^1{}_{i0j} = D_0 \widetilde{A}_{ij} + h^r (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}). \tag{6.9l}$$

which are the only possibly non-vanishing components of $\nabla\nabla R$ in the frame $\{E_\alpha\}$, and correspond respectively to the ones given in (6.8). ■

We are now equipped with almost all the elements that will allow us to solve the problem of 2nd-symmetry, as this is equivalent to solving the equations given by setting all the expressions in (6.8) (or equivalently (6.9)) equal to zero.

Equations for 2nd-symmetric Brinkmann spaces

Sumarizing, given any Brinkmann chart, $\nabla\nabla R = 0$ leads to the following set of equations via Lemma 6.3.6:

$$\overline{\nabla}_m \overline{\nabla}_s \overline{R}^i{}_{jkl} = 0, \qquad \overline{\nabla}_s \widetilde{R}^i{}_{jkl} + t^r{}_s \overline{\nabla}_r \overline{R}^i{}_{jkl} = 0, \tag{6.10a}$$
$$D_0 \overline{\nabla}_s \overline{R}^i{}_{jkl} = 0, \qquad D_0 \widetilde{R}^i{}_{jkl} - h^r \overline{\nabla}_r \overline{R}^i{}_{jkl} = 0 \tag{6.10b}$$

$$\overline{\nabla}_n \widehat{B}_{sijk} - t_{rn} \overline{\nabla}_s \overline{R}^{\,r}{}_{ijk} = 0, \qquad \overline{\nabla}_s \widetilde{B}_{ijk} - t^r{}_s (\widetilde{R}_{rijk} - \widehat{B}_{rijk}) = 0, \tag{6.10c}$$
$$D_0 \widehat{B}_{sijk} + h_r \overline{\nabla}_s \overline{R}^{\,r}{}_{ijk} = 0, \qquad D_0 \widetilde{B}_{ijk} + h^r (\widetilde{R}_{rijk} - \widehat{B}_{rijk}) = 0, \tag{6.10d}$$

$$\overline{\nabla}_k \widehat{A}_{sij} - 2t^r{}_k \widehat{B}_{s(ij)r} = 0, \qquad \overline{\nabla}_k \widetilde{A}_{ij} - t^r{}_k (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}) = 0, \tag{6.10e}$$
$$D_0 \widehat{A}_{sij} + 2h^r \widehat{B}_{s(ij)r} = 0, \qquad D_0 \widetilde{A}_{ij} + h^r (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}) = 0 \tag{6.10f}$$

together with (Lemma 6.3.4 and Definition 6.3.5)

$$\widetilde{R}^i{}_{jkl}(=\nabla_0 R^1{}_{ijk}) = D_0 \overline{R}^i{}_{jkl} \tag{6.11a}$$
$$\widehat{B}_{sijk}(=\nabla_s R^1{}_{ijk}) = \overline{\nabla}_s B_{ijk} - t_{rs}\overline{R}^r{}_{ijk}, \tag{6.11b}$$
$$\widetilde{B}_{ijk}(=\nabla_0 R^1{}_{ijk}) = D_0 B_{ijk} + h_r \overline{R}^r{}_{ijk}, \tag{6.11c}$$
$$\widehat{A}_{sij}(=\nabla_s R^1{}_{i0j}) = \overline{\nabla}_s A_{ij} - 2t^k{}_s B_{(ij)k}, \tag{6.11d}$$
$$\widetilde{A}_{ij}(=\nabla_0 R^1{}_{i0j}) = D_0 A_{ij} + 2h^k B_{(ij)k}, \tag{6.11e}$$

and (equations (4.33-4.35), $\overline{R} = \overline{\mathcal{R}}$ in Proposition 4.4.3 and Definition 6.3.3)

$$B_{ijk}(=R^1{}_{ijk}) = \overline{\nabla}_k t_{ij} - \overline{\nabla}_j t_{ik}, \tag{6.12a}$$
$$A_{ij}(=R^1{}_{i0j}) = t_{ir}t^r{}_j - \overline{\nabla}_j h_i - D_0 t_{ij}, \tag{6.12b}$$
$$\overline{R}^i{}_{jkl} = R^i{}_{jkl}. \tag{6.12c}$$

6.3.2 Reduction of the equations

The first reduction of the equations involves the intrinsic geometry of $\overline{\mathcal{M}}$. Observe that the first in (6.10a) together with Proposition 4.3.9 (any rth-symmetric foliation $\overline{\mathcal{M}}$ is locally symmetric) imply $\overline{\nabla}\,\overline{R} = 0$. Therefore:

Proposition 6.3.7. *Let (M, g) be a Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$. Then, if (M, g) is a 2nd-symmetric manifold, it follows that the foliation $\overline{\mathcal{M}}$ is locally symmetric, i.e., $\overline{\nabla}\,\overline{R} = 0$.*

The other fundamental simplifications of 2nd-symmetric Brinkmann spaces (that is, of all proper 2nd-symmetric spaces and interesting in its own right), that are going to be proven in this section can be summarized in the following Theorem:

Theorem 6.3.8. *Let (M, g) be a Brinkmann space with a fixed Brinkmann decomposition $\{u, v\}$. Then, if (M, g) is a 2nd-symmetric manifold, it follows that:*

(a) for any associated Brinkmann chart, the tensor fields $\widehat{B}$, $\widetilde{R}$, $\widehat{A}$, $\widetilde{B}$ in Definition 6.3.5 vanish.

(b) $\widetilde{A}$ is a tensor field on $\overline{\mathcal{M}}$, independent of the chosen Brinkmann chart, which is D_0- and $\overline{\nabla}$-parallel.

(c) The scalar curvature S of the manifold is constant.

(d) (M, g) is proper 2nd-symmetric if and only if $\widetilde{A} \neq 0$.

To prove (a) in Theorem 6.3.8 we need the couple of technical results about tensors with particular properties on a real vector space $\mathcal{V}$ of finite dimension which were proven in Section 2.1.4.

Proof of (a) **in Theorem 6.3.8**

From the first reduction stated in Proposition 6.3.7 the equations (6.10) become:

$$\overline{\nabla}_s \overline{R}^i{}_{jkl} = 0, \qquad (6.13a)$$
$$\overline{\nabla}_s \widetilde{R}^i{}_{jkl} = 0, \qquad D_0 \widetilde{R}^i{}_{jkl} = 0, \qquad (6.13b)$$
$$\overline{\nabla}_n \widehat{B}_{sijk} = 0, \qquad D_0 \widehat{B}_{sijk} = 0, \qquad (6.13c)$$
$$\overline{\nabla}_s \widetilde{B}_{ijk} = t^r{}_s(\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \qquad D_0 \widetilde{B}_{ijk} = -h^r(\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \qquad (6.13d)$$
$$\overline{\nabla}_k \widehat{A}_{sij} = 2t^r{}_k \widehat{B}_{s(ij)r}, \qquad D_0 \widehat{A}_{sij} = -2h^r \widehat{B}_{s(ij)r}, \qquad (6.13e)$$
$$\overline{\nabla}_k \widetilde{A}_{ij} = t^r{}_k(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \qquad D_0 \widetilde{A}_{ij} = -h^r(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}). \qquad (6.13f)$$

This second reduction is not straighforward. All the possible integrability equations for the equations (6.13) have to be calculated. As a matter of fact, the equations of semi-symmetry happen to be some of the integrability conditions for the equations of 2nd-symmetry. To calculate these equations, one must use the expressions for the commutativity of $\overline{\nabla}$ and D_0 for v-invariant sections (given in the Proposition below) plus the formula for the Ricci identity for $\overline{\nabla}$ (given in (4.11)). Their D_0- and $\overline{\nabla}$-derivates are also used.

Proposition 6.3.9. *Given a fixed Brinkmann chart $\{u, v, x^i\}$, the commutation of D_0 with $\overline{\nabla}$ for any $F \in \Gamma(T^1_1\overline{\mathcal{M}})$ (trivially extendable to arbitrary $T \in \Gamma(T^k_s\overline{\mathcal{M}})$) is:*

$$(\overline{\nabla}_k D_0 - D_0\overline{\nabla}_k)F^i{}_j = (H_{,k})(\partial_v F^i{}_j) + F^i{}_m B_{kj}{}^m - F^m{}_j B_{km}{}^i - t^m{}_k \overline{\nabla}_m F^i{}_j.$$

and, when F is v-invariant this simplifies to:

$$(\overline{\nabla}_k D_0 - D_0\overline{\nabla}_k)F^i{}_j = F^i{}_m B_{kj}{}^m - F^m{}_j B_{km}{}^i - t^m{}_k \overline{\nabla}_m F^i{}_j\,. \qquad (6.14)$$

Proof. Using (4.9) and (4.38) one gets

$$\begin{aligned}
\overline{\nabla}_k(D_0 F^i{}_j) &= \overline{\nabla}_k(\partial_u F^i{}_j - H\partial_v F^i{}_j - t^i{}_m F^m{}_j + t^m{}_j F^i{}_m),\\
D_0(\overline{\nabla}_k F^i{}_j) &= D_0(\partial_k F^i{}_j + \overline{\Gamma}^i_{mk} F^m{}_j - \overline{\Gamma}^m_{jk} F^i{}_m).
\end{aligned}$$

The result arises as follows. Expand these two expressions taking into account $\partial_v \overline{\Gamma}^i_{jk} = 0$:

$$\begin{aligned}
\overline{\nabla}_k(D_0 F^i{}_j) &= \partial_k\partial_u F^i{}_j + \overline{\Gamma}^i_{rk}\partial_u F^r{}_j - \overline{\Gamma}^r_{jk}\partial_u F^i{}_r\\
&\quad -(H_{,k})(\partial_v F^i{}_j) - H(\partial_k\partial_v F^i{}_j + \overline{\Gamma}^i_{rk}\partial_v F^r{}_j - \overline{\Gamma}^r_{jk}\partial_v F^i{}_r)\\
&\quad -\partial_k(t^i{}_r F^r{}_j) - \overline{\Gamma}^i_{sk}(t^s{}_r F^r{}_j) + \overline{\Gamma}^s_{jk}(t^i{}_r F^r{}_s)\\
&\quad +\partial_k(t^r{}_j F^i{}_r) + \overline{\Gamma}^i_{sk}(t^r{}_j F^s{}_r) - \overline{\Gamma}^s_{jk}(t^r{}_s F^i{}_r),
\end{aligned}$$

and

$$\begin{aligned}
D_0(\overline{\nabla}_k F^i{}_j) &= D_0(\partial_k F^i{}_j + \overline{\Gamma}^i_{rk} F^r{}_j - \overline{\Gamma}^r_{jk} F^i{}_r) \\
&= \partial_u \partial_k F^i{}_j - H\partial_v \partial_k F^i{}_j - t^i{}_r \partial_k F^r{}_j + t^r{}_k \partial_r F^i{}_j + t^r{}_j \partial_k F^i{}_r \\
&\quad \partial_u(\overline{\Gamma}^i_{rk} F^r{}_j) - H\overline{\Gamma}^i_{rk}\partial_v F^r{}_j - t^i{}_s \overline{\Gamma}^s_{rk} F^r{}_j + t^s{}_k \overline{\Gamma}^i_{rs} F^r{}_j + t^s{}_j \overline{\Gamma}^i_{rk} F^r{}_s \\
&\quad -\partial_u(\overline{\Gamma}^r_{jk} F^i{}_r) + H\overline{\Gamma}^r_{jk}\partial_v F^i{}_r + t^i{}_s \overline{\Gamma}^r_{jk} F^s{}_r - t^s{}_k \overline{\Gamma}^r_{js} F^i{}_r - t^s{}_j \overline{\Gamma}^r_{sk} F^i{}_r.
\end{aligned}$$

Therefore,

$$\begin{aligned}
(\overline{\nabla}_k D_0 - D_0 \overline{\nabla}_k) F^i{}_j &= (H_{,k})(\partial_v F^i{}_j) - t^r_k(\partial_r F^i{}_j + \overline{\Gamma}^i_{sr} F^s{}_j - \overline{\Gamma}^s_{jr} F^i{}_s) \\
&\quad + F^r{}_j(-\partial_k t^i{}_r - \overline{\Gamma}^i_{sk} t^s{}_r + \overline{\Gamma}^s_{rk} t^i{}_s - \dot{\overline{\Gamma}}{}^i_{rk}) \\
&\quad + F^i{}_r(\partial_k t^r{}_j + \overline{\Gamma}^r_{sk} t^s{}_j - \overline{\Gamma}^s_{jk} t^r{}_s + \dot{\overline{\Gamma}}{}^r_{jk}) \\
&= (H_{,k})(\partial_v F^i{}_j) - t^r{}_k (\overline{\nabla}_r F)^i{}_j - F^r{}_j(\overline{\nabla}_k t^i{}_r + \dot{\overline{\Gamma}}{}^i_{rk}) \\
&\quad + F^i{}_r(\overline{\nabla}_k t^r{}_j + \dot{\overline{\Gamma}}{}^r_{jk}),
\end{aligned}$$

Finally, using formula (4.36), one gets the desired expression. ■

With these new equations at hand, the proof is done in two steps. First, the vanishing of $\widehat{B}$ and $\widetilde{R}$ is proven, and as a consequence of this, $\widehat{A}$ and $\widetilde{B}$ vanish as well:

Lemma 6.3.10. *If (M, g) is a 2nd-symmetric Brinkmann space with a fixed Brinkmann chart $\{u, v, x^i\}$, the sections $\widehat{B}$ and $\widetilde{R}$ on $\overline{\mathcal{M}}$ vanish.*

Proof. By 2nd-symmetry, the integrability conditions associated to the first in (6.13c) and in (6.13e) read, respectively:

$$\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widehat{B}_{ijkl} = 0 \Longrightarrow \overline{R}^{\,r}{}_{snm}\widehat{B}_{rijk} + \overline{R}^{\,r}{}_{inm}\widehat{B}_{srjk} + \overline{R}^{\,r}{}_{jnm}\widehat{B}_{sirk} + \overline{R}^{\,r}{}_{knm}\widehat{B}_{sijr} = 0, \tag{6.15}$$

$$\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widehat{A}_{sij} = B^r{}_{mn}\widehat{B}_{s(ij)r} \Longrightarrow \overline{R}^{\,r}{}_{snm}\widehat{A}_{rij} + \overline{R}^{\,r}{}_{inm}\widehat{A}_{srj} + \overline{R}^{\,r}{}_{jnm}\widehat{A}_{sir} = 2B^r{}_{nm}\widehat{B}_{s(ij)r}. \tag{6.16}$$

In both cases we used (4.11), and for the last case we also used the first in (6.13c) and in (6.13e), as well as (6.12a). If we differentiate (6.16) with respect to $\overline{\nabla}_k$, using the same information as before plus (6.11b),(6.13a) and (6.15), we obtain

$$\widehat{B}_{s(ij)}{}^r \widehat{B}_{lrnm} = 0.$$

Therefore, $\widehat{B}$ satisfies all the hypotheses in Corollary 2.1.3 at each leaf of $\overline{\mathcal{M}}$ (see Definition 6.3.5 and below), so that $\widehat{B} = 0$ and, by the Bianchi identity (6.6), $\widetilde{R} = 0$ too. ■

This actually implies, due to (6.11a), (6.13a), (6.13d) and (6.13e), that the three sections $\overline{R}$, $\widetilde{B}$ and $\widehat{A}$ are $\overline{\nabla}$-parallel and D_0-parallel. In fact, the equations in (6.11) and (6.13) become, after simplification via Proposition 6.3.10:

$$\overline{\nabla}_s \overline{R}^i{}_{jkl} = 0, \tag{6.17a}$$

$$\overline{\nabla}_s \widetilde{B}_{ijk} = 0, \qquad D_0 \widetilde{B}_{ijk} = 0, \tag{6.17b}$$

$$\overline{\nabla}_k \widehat{A}_{sij} = 0, \qquad D_0 \widehat{A}_{sij} = 0, \tag{6.17c}$$

$$\overline{\nabla}_k \widetilde{A}_{ij} = t^r{}_k (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \qquad D_0 \widetilde{A}_{ij} = -h^r (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}). \tag{6.17d}$$

together with:

$$\widetilde{R}^i{}_{jkl} (= D_0 \overline{R}^i{}_{jkl}) = 0, \tag{6.18a}$$

$$\widehat{B}_{sijk} (= \overline{\nabla}_s B_{ijk} - t_{rs} \overline{R}^r{}_{ijk}) = 0, \tag{6.18b}$$

$$\widetilde{B}_{ijk} = D_0 B_{ijk} + h_r \overline{R}^r{}_{ijk}, \tag{6.18c}$$

$$\widehat{A}_{sij} = \overline{\nabla}_s A_{ij} - 2t^k{}_s B_{(ij)k}, \tag{6.18d}$$

$$\widetilde{A}_{ij} = D_0 A_{ij} + 2h^k B_{(ij)k}, \tag{6.18e}$$

The integrability equations associated to first in (6.17b) and (6.17d), plus the ones associated to (6.17a), (6.18b) and (6.18d) are written, on using (6.12a) for the expressions containing $\overline{\nabla} t$ and the first in (6.17c) for expressions containing $\overline{\nabla}\widehat{A}$, as:

$$2\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widetilde{B}_{ijk} = 0, \qquad 2\overline{\nabla}_{[n}\overline{\nabla}_{m]}\widetilde{A}_{ij} = B^r{}_{mn}(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \qquad \overline{\nabla}_{[n}\overline{\nabla}_{m]}\overline{R}_{ijkl} = 0,$$

$$2\overline{\nabla}_{[n}\overline{\nabla}_{m]}B_{ijk} = B^r{}_{mn}\overline{R}_{r(ij)k}, \qquad \overline{\nabla}_{[n}\overline{\nabla}_{m]}A_{ij} = B^r{}_{mn}B_{(ij)r},$$

which via (4.11) provide

$$\overline{R}^r{}_{inm}\widetilde{B}_{rjk} + \overline{R}^r{}_{jnm}\widetilde{B}_{irk} + \overline{R}^r{}_{knm}\widetilde{B}_{ijr} = 0, \tag{6.19}$$

$$\overline{R}^r{}_{inm}\widetilde{A}_{rj} + \overline{R}^r{}_{jnm}\widetilde{A}_{ir} = B^r{}_{nm}(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \tag{6.20}$$

$$\overline{R}^r{}_{snm}\overline{R}_{rijk} + \overline{R}^r{}_{inm}\overline{R}_{srjk} + \overline{R}^r{}_{jnm}\overline{R}_{sirk} + \overline{R}^r{}_{knm}\overline{R}_{sijr} = 0, \tag{6.21}$$

$$\overline{R}^r{}_{inm}B_{rjk} + \overline{R}^r{}_{jnm}B_{irk} + \overline{R}^r{}_{knm}B_{ijr} = B^r{}_{nm}\overline{R}_{rijk}, \tag{6.22}$$

$$\overline{R}^r{}_{inm}A_{rj} + \overline{R}^r{}_{jnm}A_{ir} = 2B^r{}_{nm}B_{(ij)r}. \tag{6.23}$$

Besides, the integrability conditions derived from (6.14) applied to $\widehat{A}$, B and $\widetilde{A}$ become

$$B_{ms}{}^r\widehat{A}_{rij} + B_{mi}{}^r\widehat{A}_{irs} + B_{mj}{}^r\widehat{A}_{ijr} = 0, \tag{6.24}$$

$$B_{rjk}B_{mi}{}^r + B_{irk}B_{mj}{}^r + B_{ijr}B_{mk}{}^r = \overline{R}^r{}_{ijk}A_{rm}, \tag{6.25}$$

$$B_{mi}{}^r\widetilde{A}_{rj} + B_{mj}{}^r\widetilde{A}_{ir} = (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij})A^r{}_m. \tag{6.26}$$

where we have used that $\widetilde{B}$ and $\widehat{A}$ are D_0- and $\overline{\nabla}$-parallel (equations (6.17b) and (6.17c)) together with (6.12b), (6.17d) for $\widetilde{A}$ and (6.18b) for $\overline{\nabla}B$.

Using these integrability equations we can prove the following further improvement of Proposition 6.3.10, which then completes the proof of (a) in Theorem 6.3.8.

Proposition 6.3.11. *Under the hypotheses of Proposition 6.3.10 the sections $\widetilde{B}$ and $\widehat{A}$ vanish.*

Proof. We use the results of Proposition 6.3.10 throughout the proof to simplify the formulas to be combined in the calculations. Start by applying D_0 to (6.22), and use (6.18a), (6.18c), (6.19) and (6.21) in order to get

$$\widetilde{B}^r{}_{mk}\overline{R}_{sijr} = 0. \tag{6.27}$$

Apply $\overline{\nabla}_n$ to (6.25) and use (6.17a), (6.18b), (6.18d) and (6.22) to obtain

$$\widehat{A}^r{}_{mk}\overline{R}_{sijr} = 0. \tag{6.28}$$

Applying D_0 to (6.20) and using (6.16), the second in (6.17c), in (6.17b) and in (6.17d), (6.18a), (6.18c) and (6.19) we get

$$\widetilde{B}^r{}_{mk}(2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}) = 0. \tag{6.29}$$

However, if we apply D_0 to (6.23) and use (6.18a), (6.18c), (6.18e), (6.20) and (6.22) we derive $2\widetilde{B}^r{}_{mk}B_{(ij)r} + B^r{}_{mk}\widehat{A}_{rij} = 0$. By applying D_0 once more to this last expression and using the second in (6.17b) and in (6.17c), (6.18c), (6.27) and (6.28) we also get

$$\widetilde{B}^r{}_{mk}(2\widetilde{B}_{(ij)r} + \widehat{A}_{rij}) = 0. \tag{6.30}$$

In conclusion, comparing (6.29) and (6.30), we have that

$$\widetilde{B}^r{}_{mk}\widetilde{B}_{(ij)r} = 0$$

so that $\widetilde{B}$ satisfies all the hypotheses in Proposition 2.1.2 for each leaf of $\overline{\mathcal{M}}$, and thus $\widetilde{B} = 0$. By the identity (6.7), we also get $\widehat{A}_{[ij]k} = 0$, i.e., $\widehat{A}_{ijk} = \widehat{A}_{(ij)k}$. D_0-differentiating (6.26), using (6.24) and putting $\widetilde{B} = 0$ in second in (6.17d), in (6.18c) and in (6.20), we get $\widehat{A}_{nrk}\widehat{A}^r{}_{ij} = 0$. Contracting all the indices and using that $\widehat{A}$ is symmetric in the first two, we arrive at $0 = \widehat{A}_{nrk}\widehat{A}^{rnk} = \widehat{A}_{rnk}\widehat{A}^{rnk}$, i.e., $\overline{g}(\widehat{A}, \widehat{A}) = 0$ with $\overline{g}$ a positive-definite metric, so $\widehat{A} = 0$. ∎

Consequently, the equations of 2nd-symmetry reduce to:

$$\overline{\nabla}_k\widetilde{A}_{ij} = 0, \tag{6.31a}$$
$$D_0\widetilde{A}_{ij} = 0, \tag{6.31b}$$
$$\overline{\nabla}_s\overline{R}^i{}_{jkl} = 0, \tag{6.31c}$$

together with

$$\widetilde{R}^i{}_{jkl} = D_0\overline{R}^i{}_{jkl} = 0, \tag{6.32a}$$
$$\widehat{B}_{sijk} = \overline{\nabla}_sB_{ijk} - t_{rs}\overline{R}^r{}_{ijk} = 0, \tag{6.32b}$$
$$\widetilde{B}_{ijk} = D_0B_{ijk} + h_r\overline{R}^r{}_{ijk} = 0, \tag{6.32c}$$
$$\widehat{A}_{sij} = \overline{\nabla}_sA_{ij} - 2t^k{}_sB_{(ij)k} = 0, \tag{6.32d}$$
$$\widetilde{A}_{ij} = D_0A_{ij} + 2h^kB_{(ij)k}, \tag{6.32e}$$

and

$$B_{ijk} = \overline{\nabla}_k t_{ij} - \overline{\nabla}_j t_{ik}, \tag{6.33a}$$
$$A_{ij} = t_{ir} t^r{}_j - \overline{\nabla}_j h_i - D_0 t_{ij}, \tag{6.33b}$$
$$\overline{R}^i{}_{jkl} = R^i{}_{jkl}. \tag{6.33c}$$

Remark 6.3.12. All the equations written in this section, be them 2nd-symmetry ones or their integrability conditions, are satisfied for any Brinkmann chart: if we perform a new decomposition of type (4.2), the aforementioned equations must be satisfied in the new Brinkmann chart too.

We are ready to finish the proof of Theorem 6.3.8.

Proof of Theorem 6.3.8. Recall that the proof of (a) is already done above. To prove (b), make a general change of the partly null frame $\{E_\alpha\}$ associated to a transformation of the type (4.2) –observe that the matrices of change of partly null frames $\{E_\alpha\}$ and $\{E'_\alpha\}$ are given by $\theta'^\alpha(E_\alpha)$ and its inverse $\theta^\alpha(E'_\alpha)$– to obtain

$$\widetilde{A}'_{ij} = \nabla_0 R'^1{}_{i0j} = [\theta^\alpha(E'_0)][\theta'^1(E_\beta)][\theta^\lambda(E'_i)][\theta^\mu(E'_0)][\theta^\nu(E'_j)]\nabla_\alpha R^\beta{}_{\lambda\mu\nu}.$$

so necessarily there appear terms of the type $\nabla_0 R^1{}_{i0j} = \widetilde{A}_{ij}$; $\nabla_s R^1{}_{i0k} = \widehat{A}_{sij}$; $\nabla_0 R^1{}_{ijk} = \widetilde{B}_{ijk}$; $\nabla_s R^1{}_{ijk} = \widehat{B}_{sijk}$;$\nabla_0 R^i{}_{jkl} = \widetilde{R}^i{}_{ijkl}$; $\nabla_s R^i{}_{ijkl} = \overline{\nabla}_s \overline{R}^i{}_{ijkl}$. But by equations (6.31) and (6.32), only the terms with $\widetilde{A}_{ij}$ remain. Therefore, since $\theta^0(E'_0) = 1 = \theta'^1(E_1)$ and $\theta^i(E'_j) = \frac{\partial x^i}{\partial x'^j}$:

$$\widetilde{A}'_{ij} = \frac{\partial x^k}{\partial x'^i}\frac{\partial x^s}{\partial x'^j}\widetilde{A}_{ks},$$

and $\widetilde{A}$ behaves as a tensor field on $\overline{\mathcal{M}}$ for the given Brinkmann decomposition $\{u, v\}$.

For (c), given that $S = \overline{S}$ (Proposition 4.4.3) and due to $\overline{\nabla}\ \overline{R} = 0$ (see (6.31c)), the function $\overline{S}$ depends only on u. From (a) we know that $\widetilde{R} = 0$ and using (6.32a) it follows that $0 = D_0\overline{S}$, so that $S = \overline{S}$ is constant, as required.

To prove (d), observe that, by (a), the equations (6.32) together with Lemma 6.3.4 and Definition 6.3.5 assert that the only non-zero components of ∇R in any partly null frame associated to a Brinkmann chart $\{u, v, x^i\}$ are $\nabla_0 R^1{}_{i0j}(= \widetilde{A}_{ij})$. ■

6.3.3 Transformation into two independent Lorentzian problems

The fact that $\overline{g}$, $\overline{H}$ and $\overline{W}$ are simultaneously reducible is shown to be just a consequence of the fact that $\overline{g}$, $\overline{Ric}$ and $\widetilde{A}$ are simultaneously reducible. The extended Eisenhart Theorem (Theorem 4.6.2) is applied systematically in this section. Therefore, bear in mind Definitions 4.3.8 and 4.6.1 (and all Section 4.6) for a better understanding of the proofs.

Reducibility of $\overline{g}$, $\overline{Ric}$ and $\widetilde{A}$

In the following proposition the way $\overline{g}$ and $\overline{Ric}$ are reduced simultaneously is proven, via the extended Eisenhart theorem. Roughly speaking, the zero eigenvalue of $\overline{Ric}$ will be associated to the flat part of $\overline{g}$. To prove this we will apply Proposition 4.3.9(2), which says that a Ricci-flat foliation $\overline{\mathcal{M}}$ is, in fact, flat.

Proposition 6.3.13. *Let (M, g) be a 2nd-symmetric Brinkmann space. Then, any Brinkmann decomposition $\{u, v\}$ is spatially reducible*

$$\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}, \qquad \overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)}$$

where $\overline{\mathcal{M}}^{(1)}$ is a d-dimensional flat foliation and $\overline{\mathcal{M}}^{(2)}$ is a d'-dimensional locally symmetric foliation with $d, d' \geq 0$ and $d + d' = n - 2$. Furthermore, $\overline{\mathcal{M}}^{(2)}$ is itself the product of s_2 locally symmetric (non-Ricci flat) Einstein foliations:

$$\overline{\mathcal{M}}^{(2)} = \overline{\mathcal{M}}^{(2,1)} \times \ldots \times \overline{\mathcal{M}}^{(2,s_2)} \qquad \overline{g}^{(2)} = \overline{g}^{(2,1)} \oplus \ldots \oplus \overline{g}^{(2,s_2)}.$$

Proof. From Proposition 6.3.7, $\overline{\mathcal{M}}$ is a locally symmetric foliation. Indeed, $\overline{R}$ is $\overline{\nabla}$- and D_0-parallel by (6.31) and (6.32a), so it follows that $D_0\overline{Ric} = 0$ and $\overline{\nabla}\ \overline{Ric} = 0$. Then, if the foliation is u-Einstein (i.e., $Ric = \lambda(u)\overline{g}$), the result is trivial (put $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)}$, $\overline{g} = \overline{g}^{(1)}$ for the Ricci-flat case and use Proposition 4.3.9(2) to get flatness in $\overline{g}^{(1)}$; otherwise, from $D_0\overline{Ric} = 0$ the foliation is Einstein -remember $D_0\overline{g} = 0$- and $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(2)}$, $\overline{g} = \overline{g}^{(2)}$). If it is not u-Einstein, apply the extended Eisenhart Theorem (Theorem 4.6.2) to $L = \overline{Ric}$ and, if $\overline{Ric}$ happens to have a vanishing eigenvalue, choose $\overline{\mathcal{M}}^{(1)}$ as the corresponding Ricci-flat part which, by virtue of Proposition 4.3.9(2), is actually flat. ∎

Therefore, under the conditions of Proposition 6.3.13 with $d, d' > 0$, the partition of the indices corresponds to the reducibility of $\overline{g}$ according to Definition 4.6.1, where I_1 yields the indices of the flat foliation and I_2 the indices of the locally symmetric one. The associated curvature tensors satisfy $\overline{R}^{(1)} = 0$ and $\overline{R}^{(2)} \neq 0$ (so that $\overline{R} \equiv \overline{R}^{(2)}$), and $\overline{\nabla}\ \overline{R}^{(2)} = 0$ (because $\overline{\mathcal{M}}^{(2)}$ is locally symmetric). Besides, by the last consequence in Theorem 4.6.2

$$\overline{Ric} = \sum_{m=1}^{s_2} \mu_m \overline{g}^{(2,m)}, \qquad \mu_m \in \mathbb{R} - \{0\}$$

and the flat coordinates $\{x^2, \ldots, x^{d+1}\}$ correspond to the zero eigenvalue $\mu_0 = 0$.

Convention 6.3.14. From now on:

(1) When dealing with a proper 2nd-symmetric Brinkmann manifold and using Proposition 6.3.13, we will restrict ourselves to spatially reducible Brinkmann decompositions $\{u, v\}$ so that $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}$ as in the proposition. This includes the limit case when $\overline{\mathcal{M}}$ is Einstein, so that either $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)}$ ($d' = 0$) or $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(2)}$ ($d = 0$).

(2) The indices $a, b, c, \ldots$ will run from 2 to $d+1$ and the indices $a', b', c', \ldots$ will run from $d+2$ to $n-1$.

(3) We will denote by μ^* any of the *non-zero* eigenvalues of $\overline{Ric}$.

Next, our aim is to prove that $\widetilde{A}$ is reducible and it admits a decomposition similar to that of $\overline{Ric}$. In fact, we prove that $\overline{Ric}, \widetilde{A}$ and $\overline{g}$ are simultaneously reducible, where, if $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}$ and $\overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)}$ with $(\overline{\mathcal{M}}^{(1)}, \overline{g}^{(1)})$ flat foliation and $(\overline{\mathcal{M}}^{(2)}, \overline{g}^{(2)})$ locally symmetric, the non-trivial part of $\widetilde{A}$ will lie on $\overline{\mathcal{M}}^{(1)}$, whereas the non-trivial part of $\overline{Ric}$ lies on $\overline{\mathcal{M}}^{(2)}$, as we have already shown in Proposition 6.3.13.

Remember that, at this stage, we know that for a proper 2nd-symmetric space, $\widetilde{A}$ behaves as a tensor field on $\overline{\mathcal{M}}$ (Theorem 6.3.8), so that it is possible to apply the extended Eisenhart theorem to it and then make a change of Brinkmann coordinates without losing the obtained expression for $\widetilde{A}$.

Proposition 6.3.15. *Choose any spatially-reducible Brinkmann decomposition of a proper 2nd-symmetric Brinkmann space. Then, $\widetilde{A}$ is reducible as $\widetilde{A}^{(1)} \oplus \widetilde{A}^{(2)}$ with $\widetilde{A}^{(2)} = 0$. In addition, there exists a Brinkmann chart $\{u, v, x^i\}$ such that $\overline{g}^{(1)} = \delta_{ab} dx^a dx^b$ and the matrix $(\widetilde{A}^{(1)}_{ij})$ is constant and diagonal.*

Proof. Applying D_0 to (6.25) and using (6.32c), (6.32e) and Proposition 6.3.11 together with the fact that $\overline{R}$ is D_0-parallel (equation (6.32a)) one derives

$$\overline{R}^r{}_{ijk} \widetilde{A}_{rm} = 0$$

from where we get $\overline{R}^r{}_i \widetilde{A}_{rj} = 0$ for all i, j. However, $\widetilde{A} \neq 0$ by (d) in Theorem 6.3.8 so that, in the Brinkmann chart associated to the spatial reduction of Proposition 6.3.13, if we set $i \in I_2$, i.e., $i = b'$, we get that $\mu^* \widetilde{A}_{b'j} = 0$, that is, $\widetilde{A}_{b'j} = 0$ for all j and

$$\widetilde{A} = \widetilde{A}_{bc}(u, x^i) \overline{d}x^b \overline{d}x^c.$$

Now, from the $\overline{\nabla}$-parallelism of $\widetilde{A}$ ((6.31a)) we have $0 = \overline{\nabla}_{a'} \widetilde{A}_{bc} = \partial_{a'} \widetilde{A}_{bc}$ (last equality because $\gamma^d_{ba'} = 0$), that is, $\widetilde{A}$ is reducible with $\widetilde{A}^{(2)} = 0$ and satisfies the hypotheses of Theorem 4.6.2. Moreover, the coordinate vector fields $\{\partial_{d+2}, \ldots, \partial_{n-1}\}$ span a subspace S of the eigenspace S_0 associated to the zero eigenvalue of $\widetilde{A}$. Therefore, one can follow the steps of the proof of Theorem 4.6.2, but working just on $S^\perp$, $(X(x^a) = 0$, for all $X \in S)$ to obtain

- $\overline{g}^{(1)} = \overline{g}^{(1,1)} \oplus \ldots \oplus \overline{g}^{(1,s_1)}$.
- $\widetilde{A} = \sum_{m=1}^{s_1} \lambda_m \overline{g}^{(1,m)}$.

As $\overline{g}^{(1)}$ is a flat metric, applying Proposition 4.3.9(3) to each of the mutually orthogonal flat metrics $\overline{g}^{(1,m)}$, a further change of coordinates in the flat block yields $\overline{g}^{(1)} = \delta_{ab} dx'^a dx'^b$ (for example, use the proof of that proposition on the restriction to the block of coordinates associated to each $\overline{\mathcal{M}}^{(1,m)}$). In conclusion, in this Brinkmann chart the matrix of the tensor field $\widetilde{A}$ is a diagonal matrix of constants. ■

Remark 6.3.16. Observe that the Brinkmann chart obtained in Proposition 6.3.15:

(1) maintains the Brinkmann decomposition $\{u, v\}$ of Proposition 6.3.13. As the tensor fields $\overline{Ric}$ and $\widetilde{A}$ (see (d) in Theorem 6.3.8) depend only on the decomposition, the conclusions obtained for them are independent on the remainder of coordinates of the Brinkmann chart.

(2) is such that $\widetilde{A}$ and $\overline{Ric}$ are orthogonal and they may vanish simultaneously on a subundle of $\overline{\mathcal{M}}^{(1)}$.

(3) has a flat metric $\overline{g}^{(1)} = \delta_{ab} dx^a dx^b$ so that $\overline{\nabla}_a = \partial_a$. Moreover, $\overline{\nabla}_{a'} T^a_b = \partial_{a'} T^a_b$.

Reducibility of H and W

In order to prove that H and W in the Brinkmann metric (4.1) reduce for a fixed Brinkmann chart, some previous work must be done. Indeed, we first need to prove that the sections h and t (as defined in (4.14) and (4.15)) are reducible.

Lemma 6.3.17. *For a Brinkmann chart $\{u, v, x^i\}$ as in Proposition 6.3.15, the sections B and A in Definition 6.3.3 satisfy that:*

$$B_{aba'} = 0; \quad B_{a'ab} = 0; \quad B_{a'b'a} = 0; \quad B_{aa'b'} = 0; \quad A_{aa'} = 0.$$

Proof. The commutation property (6.14) applied to the $\overline{\nabla}$- and D_0-parallel section $\overline{R}$ (see equations (6.31) and (6.32a)) provides the following integrability condition

$$B_{ms}{}^r \overline{R}_{rijk} + B_{mi}{}^r \overline{R}_{srjk} + B_{mj}{}^r \overline{R}_{sirk} + B_{mk}{}^r \overline{R}_{sijr} = 0.$$

(i) Contracting here i and k and

- taking $m = a$, $s = b$, $j = a'$, one gets: $B_{ab}{}^r \overline{R}_{ra'} = B_{aba'}\, \mu^* = 0$ that is, $B_{aba'} = 0$,
- taking $m = a'$, $s = b'$, $j = a$, one gets: $B_{a'a}{}^r \overline{R}_{b'r} = B_{a'ab'}\, \mu^* = 0$ that is, $B_{a'b'a} = 0$.

(ii) Using now $B_{[ijk]} = 0$, and $B_{i[jk]} = 0$, (i) above and:

- taking $i = a'$, $j = b'$ and $k = a$, one has that $B_{aa'b'} = 0$,
- taking $i = a'$, $j = a$ and $k = b$, one has that $B_{a'ab} = 0$.

(iii) Contracting i and s in the equation (6.23) and taking $j = a, m = a'$, one derives $\overline{R}_{ra'}A^r{}_a = 2B_{(ia)}{}^r B_{ra'}{}^i$. Using here (i) and (ii) above one arrives at $\mu^* A_{a'a} = 0 \Longrightarrow A_{aa'} = 0$. ■

Proposition 6.3.18. *For a Brinkmann chart $\{u, v, x^i\}$ as in Proposition 6.3.15, the sections t and h are reducible.*

Proof. We have to prove that

$$t_{aa'} = 0; \quad t_{a'a} = 0; \quad \partial_{a'} t_{ab} = 0; \quad \partial_a t_{a'b'} = 0; \quad \partial_a h_{a'} = 0; \quad \partial_{a'} h_a = 0.$$

The fact that $\overline{R}$ is D_0-parallel can be rewritten using the expression of D_0 in components (4.39) as

$$\dot{\overline{R}}_{ijkl} + t^r{}_i \overline{R}_{rjkl} + t^r{}_j \overline{R}_{irkl} + t^r{}_k \overline{R}_{ijrl} + t^r{}_l \overline{R}_{ijkr} = 0.$$

Taking $i = a$ one gets that $t^r{}_a \overline{R}_{rjkl} = 0$, so that contracting j and l and taking $k = a'$ one gets

$$0 = t_{a'a} = -t_{aa'}. \tag{6.34}$$

By (6.33a) and Lemma 6.3.17 it follows that $0 = B_{aba'} = \overline{\nabla}_{a'} t_{ab} - \overline{\nabla}_b t_{aa'}$, which implies by Remark 6.3.16(2) and (6.34) that $\partial_{a'} t_{ab} = 0$. Analogously, $0 = B_{a'b'a} = \overline{\nabla}_a t_{a'b'} - \overline{\nabla}_{b'} t_{a'a}$ implies $\partial_a t_{a'b'} = 0$. Lemma 6.3.17 and (6.33b) yield $0 = A_{aa'} = t_{ar} t^r{}_{a'} - \overline{\nabla}_{a'} h_a - D_0 t_{aa'}$, so sustituting (6.34) and using (4.39) and Remark 6.3.16(2) again one obtains $\partial_a h_{a'} = \partial_{a'} h_a = 0$. ■

Theorem 6.3.19. *For a proper 2nd-symmetric Brinkmann space, there exists a spatially reducible Brinkmann decomposition $\{u', v'\}$ such the function H' and the one-form $W' = W'_i dx'^i$ in*

$$g = -2du'(dv' + H'(u', x'^k)du' + W'_i(u', x'^k)dx'^i) + g'_{ij}(u', x'^k)dx'^i dx'^j, \;\; k = 2, \dots, n-1.$$

are reducible in the associated Brinkmann chart $\{u', v', x'^i\}$. This chart is related to that obtained in Proposition 6.3.15 by $\{u', v', x'^i\} = \{u, v + f(u, x^j), x^i\}$ for some function f.

Proof. We must prove that there exists a Brinkmann chart $\{u', v', x'^i\}$ such that

$$H(u', x'^i) = H^{(1)}(u', x'^a) + H^{(2)}(u', x'^{a'}),$$

$$W_a = W_a(u', x'^b)(= W_a^{(1)}(u', x'^b)), \quad W_{a'} = W_{a'}(u', x'^{b'})(= W_{a'}^{(2)}(u', x'^{b'})).$$

Let $\{u, v, x^i\}$ be the Brinkmann chart obtained in Proposition 6.3.15, so that by Proposition 6.3.18, the sections t and h are reducible.

Simplification of W: $\partial_{a'} t_{ab} = 0$ implies that $W_{a,b} - W_{b,a}$ depends only on the coordinates $\{u, x^a\}$, hence there exists a function $f_1(u, x^i)$ such that $W_a(u, x^j) = f_{1,a}(u, x^j) + w_a(u, x^b)$. Analogously, $\partial_a t_{a'b'} = 0$ implies the existence of a function $f_2(u, x^i)$ such that $W_{a'}(u, x^i) = f_{2,a'}(u, x^i) + w_{a'}(u, x^{b'})$. Then, (6.34) tells us that $f_{1,a,a'} = f_{2,a',a}$ so that $f_1(u, x^i) - f_2(u, x^i) = F(u, x^a) + G(u, x^{a'})$ for some functions F and G which can be absorbed into $w_a(u, x^b)$ and $w_{a'}(u, x^{b'})$, respectively. Consequently, there exists a function $f(u, x^j)$ such that

$$W_a(u, x^b) = f_{,a}(u, x^j) + w_a(u, x^b), \quad W_{a'}(u, x^{b'}) = f_{,a'}(u, x^j) + w_{a'}(u, x^{b'}). \tag{6.35}$$

Simplification of H: using $h = \overline{d}H - \dot{W}$ (equation (4.14)), $\partial_a h_{a'} = \partial_{a'} h_a = 0$ provide $H_{,a}(u, x^i) = h_a(u, x^b) + \dot{W}_a(u, x^i)$ and $H_{,a'}(u, x^i) = h_{a'}(u, x^{b'}) + \dot{W}_{a'}(u, x^i)$, which become after use of (6.35), $H_{,a}(u, x^i) = h_a(u, x^b) + \dot{f}_{,a}(u, x^i) + \dot{w}_a(u, x^b)$ and $H_{,a'}(u, x^i) = h_{a'}(u, x^{b'}) + \dot{f}_{,a'}(u, x^i) + \dot{w}_{a'}(u, x^{b'})$, from where it is easy to deduce that

$$H(u, x^i) = \dot{f}(u, x^i) + H^{(1)}(u, x^a) + H^{(2)}(u, x^{a'})$$

for some functions $H^{(1)}, H^{(2)}$.

By choosing now the new Brinkmann decomposition defined by $v' = v + f(u, x^i)$, the thesis follows on using (see (4.3) and (4.4)):

$$\begin{aligned} H &= H' + \dot{f}, \\ W_i &= W'_j \frac{\partial x'^j}{\partial x^i} + f_{,i}. \end{aligned}$$

■

The building blocks of the metric g

Our aim now is to prove that, in fact, the 2nd-symmetry induces the existence of two simpler 2nd-symmetric Brinkmann spaces associated to the original Brinkmann space (M, g). Until now we know that, for a (proper) 2nd-symmetric Brinkmann space, there exists a Brinkmann chart $\{u, v, x^i\}$ and a partition of the indices $I_1 = \{2, \ldots, d+1\}$, $I_2 = \{d+2, \ldots, n-1\}$ for some $d \in \{0, \ldots, n-2\}$ (recall Convention 6.3.14), such that $\overline{g}$, H and W are simultaneously reducible in the sense given in Definition 4.6.1. Therefore, in particular, in the given chart the Brinkmann decomposition $\{u, v\}$ is spatially reducible. Recall from Section 4.6 that if $\{u, v\}$ is spatially reducible, there exist two foliations $\overline{\mathcal{M}}^{(1)}, \overline{\mathcal{M}}^{(2)}$ with associated leaves $(\overline{M}^{(1)}, \overline{g}^{(1)}), (\overline{M}^{(2)}, \overline{g}^{(2)})$ such that $\overline{\mathcal{M}} = \overline{\mathcal{M}}^{(1)} \times \overline{\mathcal{M}}^{(2)}$, $\overline{M} = \overline{M}^{(1)} \times \overline{M}^{(2)}$ and $\overline{g} = \overline{g}^{(1)} \oplus \overline{g}^{(2)}$. Hence, the metric g can be written as

$$g = -2du(dv + (H^{(1)} + H^{(2)})du + \mathring{W}^{(1)} + \mathring{W}^{(2)}) + \mathring{\overline{g}}^{(1)} \oplus \mathring{\overline{g}}^{(2)},$$

so that to any 2nd-symmetric Brinkmann space we can associate a pair of lower-dimensional Brinkmann spaces $(M^{[m]}, g^{[m]}), m \in \{1, 2\}$ by $M^{[m]} = \mathbb{R}^2 \times \overline{M}^{(m)}$ and

$$g^{[m]} = -2du(dv + H^{(m)}du + W^{(m)}) + \mathring{\overline{g}}^{(m)}.$$

Of course, such a pair of spaces may be non-unique. $(M^{[m]}, g^{[m]})$ are the *building blocks* of any proper 2nd-symmetric Lorentzian manifold, and they are extremely helpful in the resolution of the equations for 2nd-symmetry because they actually simplify to the equations corresponding to each of the two simpler Brinkmann spaces $(M^{[m]}, g^{[m]}), m \in \{1, 2\}$ as the next proposition proves.

Proposition 6.3.20. *The pair $(M^{[m]}, g^{[m]})$ of Brinkmann spaces associated to any 2nd-symmetric Brinkmann space (M, g) as given above are, themselves, 2nd-symmetric.*

Proof. Note first of all that $\overline{R}, h$ and t are reducible, so that A, B (see equations (6.33a) and (6.33b)) and their $\overline{\nabla}$- and D_0-derivations (equations (6.32b-6.32e)) are reducible too. Therefore, for $m \in \{1, 2\}$, the reduced sections $\overline{R}^{(m)}$, $h^{(m)}$, $t^{(m)}$, $A^{(m)}$, $B^{(m)}$ correspond to the geometrical objects for the Brinkmann spaces $(M^{[m]}, g^{[m]})$ as defined in (4.10), (4.14), (4.15) and Definition 6.3.3 respectively. It is then straightforward to check that, in the Brinkmann chart of Theorem 6.3.19, the equations of 2nd-symmetry for (M, g) are exactly identical with the combination of the equations of 2nd-symmetry for the two building blocks $(M^{[m]}, g^{[m]})$. ■

In conclusion, applying Proposition 6.3.20 to the Brinkmann decompositions $\{u', v'\}$ of Theorem 6.3.19, we can reorganize the equations of 2nd-symmetry in two simpler sets:

- The equations associated to the Brinkmann space $(M^{[1]}, g^{[1]})$ with coordinates $\{u, v, x^a\}$, such that: $\overline{g}^{(1)} = \delta_{ab} dx^a dx^b$, $\overline{R}^{(1)} = 0$ (ergo $\overline{\text{Ric}}^{(1)} = 0$) and $\widetilde{A}^{(1)} = \sum_{l=1}^{s_1} \lambda_l \overline{g}^{(1,l)}$ with some $\lambda_l \neq 0$.
- Those associated to $(M^{[2]}, g^{[2]})$ with coordinates $\{u, v, x^{a'}\}$, such that: $\overline{g}^{(2)} = \overline{g}^{(2,1)} \oplus \ldots \oplus \overline{g}^{(2,s_2)}$, $\overline{\nabla}\, \overline{R}^{(2,l)} = 0$ but $\overline{R}^{(2,l)} \neq 0$, $\overline{\text{Ric}}^{(2)} = \sum_{l=1}^{s_2} \mu_l \overline{g}^{(2,l)}$ with each $\mu_l \neq 0$ and $\widetilde{A}^{(2)} = 0$.

6.3.4 Proof of the Structure Theorem

We will start the computations in the Brinkmann chart $\{u', v', x'^i\}$ of Theorem 6.3.19 but dropping the primes for the sake of clarity in the notation.

Proof. We consider first the equations for $(M^{[2]}, g^{[2]})$. Since $\widetilde{A}^{(2)} = 0$, (d) in Theorem 6.3.8 informs us that this is a locally symmetric Lorentzian manifold with a parallel lightlike vector field. Therefore, Theorem 3.2.16 implies that it is locally isometric to the product of a symmetric (non-flat) Riemannian space with $\mathbb{L}^2 = (\mathbb{R}^2, -2dudv)$. In particular, up to a change of coordinates of type

$$u' = u, \quad v' = v + F(u, x^{a'}); \quad y^{a'} = y^{a'}(u, x^{b'}) \tag{6.36}$$

$(M^{[2]}, g^{[2]})$ has $H^{(2)} = 0$, $W^{(2)} = 0$ and $\dot{\overline{g}}^{(2)} = 0$.

Consider now the 2nd-symmetry equations associated to $(M^{[1]}, g^{[1]})$. Using $\overline{R}^{(1)} = 0$ in (6.23) we have that $B_{(ab)c} B^c{}_{de} = 0$, hence Proposition 2.1.2 leads us to

$$B^{(1)} = 0. \tag{6.37}$$

From (6.33a) follows that $\overline{\nabla}_c t_{ab} - \overline{\nabla}_b t_{ac} = 0$, which together with the definition of t (4.15) and Remark 6.3.16(2) provides $\partial_c t_{ab} - \partial_b t_{ac} = \partial_a t_{cb} = 0$. We conclude that $t^{(1)}$ is $\overline{\nabla}$-parallel. We have two consequences of this fact:

- $W_{a,b} - W_{b,a} = 2t_{ab}(u)$ depend only on u. Consequently, $W^{(1)}$ can be written as

$$W_a(u, x^b) = f_{,a}(u, x^b) + t_{ac}(u)x^c$$

  for some function $f(u, x^b)$.

- Using (6.32d) and by Lemma 6.3.17 and (6.37) we have that $\partial_c A_{ab} = 0$, which from (6.33b) and Proposition 6.3.18 implies that $\partial_c \partial_b h_a = 0$, that is, $h^{(1)}$ takes the form

$$h_a(u, x^b) = \Lambda_{ac}(u)x^c + B_a(u)$$

  for some functions Λ_{ac} and B_a. From the definition of h (4.14) we then have

$$\begin{aligned} H_{,a}(u, x^b) &= \Lambda_{ac}(u)x^c + B_a(u) + \dot{W}_a(u, x^b) \\ &= \Lambda_{ac}(u)x^c + B_a(u) + \dot{f}_{,a}(u, x^b) + \dot{t}_{ac}(u)x^c. \end{aligned}$$

  Observe that the symmetry of $H_{,a,b}$ implies that $\Lambda_{ab} + \dot{t}_{ab} = \Lambda_{ba} + \dot{t}_{ba} = \Lambda_{(ab)}$ where the antisymmetric character of $\dot{t}$ has been used in the last equality.

In conclusion, the metric for $M^{[1]}$ becomes

$$g^{[1]} = -2du\left(dv + H^{(1)}(u, x^a)du + \mathring{W}^{(1)}\right) + \delta_{ab}dx^a dx^b \tag{6.38}$$

with

$$\begin{aligned} W^{(1)} &= W_a(u, x^b)\overline{d}x^a = \left[f_{,a}(u, x^b) + t_{ac}(u)x^c\right]\overline{d}x^a; \\ H^{(1)}(u, x^a) &= \dot{f}(u, x^a) + \frac{1}{2}\Lambda_{(bc)}(u)x^b x^c + B_c(u)x^c + C(u). \end{aligned}$$

Next, we use the following claim, to be proven later.

Claim 6.3.21. *For $(M^{[1]}, g^{[1]})$, there exists a change of Brinkmann chart of type*

$$u' = u, \quad v' = v + \chi(u, x^a), \quad y^a = R^a_b(u)x^b + D^a(u), \tag{6.39}$$

such that the metric becomes

$$g^{[1]} = -2du'\left(dv' + H'(u', y^a)du'\right) + \delta_{ab}dy^a dy^b. \tag{6.40}$$

where $H'(u', y^a) = -A_{ab}(u')y^a y^b$, A_{ab} being the components of the section A in Definition 6.3.3 associated to the Brinkmann chart $\{u', v', y^i\}$.

Combining the change of coordinates in this claim and the one given in (6.36), there exists a change of coordinates in the entire Brinkmann space (M, g) of type

$$\begin{aligned} u' &= u, & v' &= v + F(u, x^{a'}) + \chi(u, x^a), \\ y^{a'} &= y^{a'}(u, x^{b'}), & y^a &= R^a_b(u)x^b + D^a(u) \end{aligned}$$

such that the metric of (M,g) becomes

$$g = -2du'\,(dv' + H'(u',y^a)du') + \delta_{ab}dy^a dy^b + \overline{g}_{a'b'}(y^{c'})dy^{a'}dy^{b'}$$

where $H'(u',y^a) = -A_{ab}(u')y^a y^b$. To end the proof of Theorem 6.3.1, note that $\widetilde{A} = \dot{A}$ and therefore $D_0\widetilde{A} = 0$ (equation (6.31b)) gives $\ddot{A}_{ab}(u) = 0$. Of course, we need $\dot{A}_{a_0b_0}(u) \neq 0$ for some a_0, b_0 in order for the manifold not to be locally symmetric, due to (d) in Theorem 6.3.8. ■

Let us now proof the claim:

Proof of the claim. In order to find the required χ, D^a and R^a_b, put

$$H'(u',y^a) = -A_{ab}(u')y^a y^b,$$

substitute (6.39) in (6.40) and require that the obtained expression equals (6.38). Then, the following equations arise:

$$\begin{aligned}
&\dot{\chi} = H^{(1)} - H' + \frac{1}{2}\delta_{ab}(\dot{R}^a_c x^c + \dot{D}^a)(\dot{R}^b_d x^d + \dot{D}^b),\\
&\chi_{,a} = W_a + \frac{1}{2}\delta_{bc}\left((\dot{R}^b_d x^d + \dot{D}^b)R^c_a + (\dot{R}^c_d x^d + \dot{D}^c)R^b_a\right),\\
&\delta_{ab}R^a_c R^b_d = \delta_{cd}.
\end{aligned} \tag{6.41}$$

Here, the known data are $t_{ab}(u)$, $B_c(u)$ and $\Lambda_{(ab)}(u)$, while the unknowns are $R^a_b(u)$, $D^b(u)$ and $\chi(u,x^a)$. The integrability conditions (given by the cross derivatives) of the first two expressions yield

$$t_{cd} = \frac{1}{2}\delta_{ab}(\dot{R}^a_c R^b_d - R^b_c \dot{R}^a_d), \tag{6.42}$$

$$\Lambda_{(cd)} = -2A_{be}R^b_c R^e_d - \frac{1}{2}\delta_{ab}(R^a_c \ddot{R}^b_d + R^b_c \ddot{R}^a_d), \tag{6.43}$$

$$B_c = -2A_{be}R^b_c D^e + \delta_{ab}\ddot{D}^a R^b_c. \tag{6.44}$$

To prove that these equations have solutions, proceed as follows. Define $R^c_b(u)$ as a solution of the ODE

$$\dot{R}^c_b = -\delta^{dc}(R^{-1})^a_d\, t_{ab}$$

for the given $t_{ab}(u)$. Then, equation (6.42) is automatically satisfied. Concerning (6.41), note that, for such a solution, the derivative of $\delta_{cd}R^c_a R^d_b$ vanishes due to the anti-symmetry of $t_{ab}(u)$. Therefore, by imposing as initial condition that $R^b_a(0)$ is any orthogonal matrix, the necessary condition (6.41) holds for all u (that is, $R^a_b(u)$ is a curve of rotation matrices, which can be chosen by using $d(d-1)/2$ free parameters codified in $R^b_a(0)$). Once $R^b_c(u)$ is determined, equation (6.43) fixes $A_{ab}(u)$ and, using this, the existence of $D^b(u)$ (which will depend on two new constant vectors, that is, on $2d$ new parameters) and, a posteriori, of χ, is ensured by (6.44) plus, again, standard results in differential equations. ■

As (M_1, g_1) in Theorem 6.3.1 is a proper Cahen-Wallach space of order 2, let O^i_j be the matrix of rotations that diagonalizes the symmetric matrix $A^{(1)}$ in (5.7). Then, performing a change of coordinates of type $u' = u - u_0$ and $y^i = O^i_j x^j$ we can write $A^{(1)}$ as a diagonal matrix and cancel one of the elements of the symmetric matrix $A^{(0)}$. In conclusion, the number of essential parameters of a proper 2nd-symmetric Lorentzian manifold with fixed $K = -\partial_v$ is given by $d - 1 + d(d+1)/2$ (observe that $A^{(1)}$ is a $d \times d$ square matrix). If K is not fixed, by a change of coordinates of type $u' = \varepsilon u - u_0$ with $v' = v/\varepsilon$ for some constant $\varepsilon \neq 0$ and $y^i = O^i_j x^j$, the same simplifications can be achieved and furthermore one of the non-zero eigenvalues of $A^{(1)}$ can be set to ± 1, in which case the number of essential parameters is one less.

Conclusions and Open Problems

Starting from the fact that a Lorentzian proper second-order symmetric manifold must have a parallel lightlike vector field, and, therefore, it is a Brinkmann space, we have proven that the implicit symmetries in the equations of 2nd-symmetry turn out to be sufficient to solve them and actually to find their general solution explicitly. In conclusion, the generalized Cahen-Wallach family of order two given in Subsection 5.4 (as a generalization of the Lorentzian locally symmetric Cahen-Wallach spaces) turn out to be the non-trivial part of the proper 2nd-symmetric spaces. Incidentally, by making a detailed local and global geometrical study of the Brinkmann spaces, different technical tools for some classes of partial differential equations, which may have interest in its own right, have been developed.

The interest in these spaces is stressed by the approach based on holonomy groups carried out by Alekseevsky and Galaev in [2, 35]. So, we outline now some open areas of research and possible future work:

- The study of Lorentzian proper rth-symmetric spaces with $r \geq 3$ and, in general, with metrics of index greater than 1 and any $r \geq 2$. In these cases, our approach is not directly applicable and, in fact, it is not clear how these generalizations would affect even to our starting point ([79, Theorem 4.2]).

  Nevertheless, as the Cahen-Wallach family of order one and two of Lorentzian manifolds given in Subsection 5.4 turn out to be the non-trivial part of the locally and proper 2nd-symmetric spaces respectively, one can expect that this will hold for higher orders, and we pose:

  Open problem. *Is it true that an n-dimensional* proper *rth-symmetric Lorentzian space (M, g) is locally isometric to a direct product $(M_1 \times M_2, g_1 \oplus g_2)$, where (M_2, g_2) is a non-flat Riemannian symmetric space and (M_1, g_1) is a proper generalized Cahen-Wallach space of order r, for all $r > 2$? .*

  The product spaces described above are proper rth-symmetric. The point would be to determine if locally there are more than those. Hence, possible future work in this issue could be:

  – To exclude the existence of a parallel lightlike vector field. To this end, it would be interesting to search for a 3rd-order symmetric manifold in a Lorentzian manifold with a recurrent lightlike vector field. These type of manifolds have been studied in the literature for example in [37, 36].

– To prove that Lorentzian rth-symmetric spaces are semi-symmetric ones. Observe that a proper generalized Cahen-Wallach space of order r is semi-symmetric (Proposition 5.4.2). Thus, this could be another approximation to rth-symmetric spaces. Although this property would not limit the class of spaces as much as the existence of a parallel lightlike vector field does, it would help by adding integrability equations to the set of r-th symmetric equations.

- Recall that the obtained proper 2nd-symmetric spaces share the symmetries of plane waves (for explicit expressions including the somewhat exotic Kerr-Schild symmetries, see [24]). So, an interesting question may be to find connections between the group of symmetries inherent a priori to rth-symmetry, and the actual symmetries of the eventually obtained proper rth-symmetric spaces.

- The non-simply connected case and, in particular, the existence of compact quotients of plane waves, becomes also a natural problem in this setting[1].

[1]We acknowledge Professor A. Zeghib, from ENS Lyon, for discussions stressing the importance of this question

Appendix A

Lie groups, Lie algebras and G-invariant metrics

Lie Groups

Let us start with the definition of a Lie group:

Definition A.1. *A group $(G, \cdot)$ is a* Lie group *provided that:*

(i) G is a smooth manifold.

(ii) The map $G \times G \longrightarrow G$ which assigns to (g_1, g_2) the element $g_1 \cdot {g_2}^{-1}$ is smooth.

G, as a group, has the identity element e. The connected component of G containing e will be denoted as G_0 and called *the identity component of* G. In fact, G_0 is a connected topological group, i.e., it is a group with a topology so that the map $(g_1, g_2) \longrightarrow g_1 \cdot {g_2}^{-1}$ is a continuous function. Besides, any subgroup H of a Lie group G which is also a submanifold is a Lie group by itself, since the map in Definition A.1(ii) restricted to $H \times H$ is smooth from $H \times H$ to H. In particular, if H is a closed subgroup of G, then H is a Lie group.

The following theorem presents a simple way to build smooth manifolds out of Lie groups:

Theorem A.2 ([90], page 120)**.** *If H is a closed subgroup of a Lie group G, there is a unique way to endow $G/H = \{gH : g \in G\}$ with the structure of a smooth manifold so that $\pi : G \longrightarrow G/H$ is a submersion, i.e.:*

- *π is smooth,*
- *$d\pi_p$ is onto for all p in M.*

Now we are going to present a simple way to build smooth manifolds out of a Lie group acting transitively on the manifolds. This construction is an essential part of the Lie description of semi-Riemannian homogeneous (and symmetric) spaces.

Definition A.3. *Let N be a smooth manifold. A Lie group G is called a* Lie transformation group on N*, if there exists a smooth mapping $\varphi : G \times N \longrightarrow N$ defined as $\varphi(g,p) = gp$ which satisfies*

$$\begin{aligned} (g_1 \cdot g_2)p &= g_1 \cdot (g_2 p), \qquad \forall g_1, g_2 \in G \text{ and } \forall p \in N, \\ ep &= p, \qquad \forall p \in N. \end{aligned}$$

In this case, φ is called a left action of G on N*.*

Defining the diffeomorphism $\varphi_g \equiv \varphi_{|\{g\}\times N}$, the inverse of φ_g is given by $\varphi_{g^{-1}}$, and the action φ is said to be *transitive* provided that for any $p, q \in N$ there exists an element g in G such that $gp = q$.

Definition A.4. *If G is a Lie transformation group and $p \in N$, the set*

$$H \equiv \varphi_p^{-1}(p) = \{g \in G : gp = p\},$$

with φ_p being the smooth function $\varphi_{|G\times\{p\}}$, is a closed subgroup of G (hence a Lie group) called the isotropy group of G at p*.*

Theorem A.5 ([90], page 123)**.** *Let G be a transitive Lie transformation group on a manifold N, fix $p \in N$ and let H its isotropy group. Then, the mapping $G/H \longrightarrow N$ defined by $gH \to gp$ is a diffeomorphism.*

Observe that the assignation $gH \to gp$ is well-defined, since

$$g_1 H = g_2 H \Longrightarrow (g_2)^{-1} g_1 H = H \Longrightarrow (g_2)^{-1} g_1 p = p \Longrightarrow g_1 p = g_2 p.$$

In order to deal with semi-Riemannian structures on the quotient space $N = G/H$, we need first to introduce the Lie algebra $\mathfrak{g}$ associated to a Lie group G.

Lie algebra of a Lie Group

Let first see some notions about abstract Lie algebras. A vector space over a field K with characteristic zero (i.e., for all $n \in \mathbb{Z} - \{0\}$ and all $k \in K^*$, $nk \neq 0$) is said to be a *Lie algebra over K* if there exists an operator $\mathfrak{a} \times \mathfrak{a} \longrightarrow \mathfrak{a}$ which maps (X,Y) into $[X,Y]$ such that:

(i) $[\cdot\,,\cdot]$ is bilinear.

(ii) $[X,X] = 0, \forall X \in \mathfrak{a}$.

(iii) It satisfies the Jacobi identity: $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \; \forall X,Y,Z \in \mathfrak{a}$.

The operator $[\cdot\,,\cdot\,]$ is called *a Lie bracket operator*. For example, $\mathfrak{X}(N)$ the set of all C^∞ vector fields of a manifold N is an infinite dimensional Lie algebra over $\mathbb{R}$ with the usual Lie bracket operator $[X,Y] = XY - YX$ for all X, Y in $\mathfrak{X}(N)$.

Let now $\mathfrak{b}$ and $\mathfrak{c}$ be two vector subspaces of $\mathfrak{a}$, and denote

$$[\mathfrak{b},\mathfrak{c}] = \{[X,Y] : X \in \mathfrak{b}, Y \in \mathfrak{c}\}$$

Then, the vector subspace $\mathfrak{b}$ of $\mathfrak{a}$ is a *subalgebra of* $\mathfrak{a}$ if $[\mathfrak{b},\mathfrak{b}] \subseteq \mathfrak{b}$, and an *ideal of* $\mathfrak{a}$ if $[\mathfrak{b},\mathfrak{a}] \subseteq \mathfrak{b}$.

A Lie algebra $\mathfrak{a}$ is *Abelian* if the Lie bracket vanishes, i.e. $[X,Y] = 0$, for all X and Y in $\mathfrak{a}$, and it is *simple* if it is a non-abelian Lie algebra whose only ideals are 0 and $\mathfrak{a}$ itself. Then, a Lie algebra $\mathfrak{a}$ is *semisimple* if it is a direct sum of simple Lie algebras.

Let $\mathfrak{a}, \mathfrak{b}$ be two Lie algebras. Then, $\phi : \mathfrak{a} \longrightarrow \mathfrak{b}$ is a *Lie algebra homomorphism* if it satisfies that $\phi([X,Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{a}$. If it is bijective, then it is called a *Lie algebra isomorphism*, and when $\mathfrak{a} = \mathfrak{b}$ the isomorphism becomes a *Lie algebra automorphism*.

From now on, we will consider only finite-dimensional subalgebras of $\mathfrak{X}(N)$. It is possible to associate to a Lie group G a Lie algebra $\mathfrak{g}$, which indeed is identified with T_eG, as follows. First, define *the left translation of G onto itself by g* as the diffeomorphism $L_g : G \longrightarrow G$ such that $L_g(g') = gg'$. Then, we say that $X \in \mathfrak{X}(G)$ is *left-invariant* if and only if $X = (L_g)_* X$ for all $g \in G$, i.e.,

$$X_{gg'} = (L_g)_{*|g'}(X_{g'}), \quad \forall g, g' \in G, \tag{A.1}$$

and define the set

$$\mathfrak{g} = \{X \in \mathfrak{X}(G) : X \text{ is left-invariant}\} \subset \mathfrak{X}(G).$$

It is easy to prove that given any vector $\vec{x} \in T_eG$ there exists a unique left-invariant vector field X on G such that $X_e = \vec{x}$. Conversely, by formula A.1 it follows that

$$X_g = (L_g)_{*|e}(X_e), \quad \forall g \in G, \forall X \in \mathfrak{g},$$

and any vector field X in $\mathfrak{g}$ is totally defined by just knowing its value on e. Therefore, it is possible to construct an isomorphism $\psi : \mathfrak{g} \longrightarrow T_eG$ between vector spaces, and hence prove that $\mathfrak{g}$ is a vector space that inherits the Lie algebra structure from $\mathfrak{X}(G)$. So $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{X}(G)$ and a Lie algebra by itself, which is called *the Lie algebra of the Lie group G*. To induce in T_eG a Lie algebra structure so that ψ is an isomorphism between Lie algebras, define a bracket operator on T_eG in the following way: for any $\vec{x}, \vec{y} \in T_eG$, let X, Y be the (unique) left-invariant vector fields on $\mathfrak{g}$ such that $X_e = \vec{x}$ and $Y_e = \vec{y}$. Then, define $[\vec{x}, \vec{y}] \equiv [X,Y]_e$.

A *Lie group homomorphism* is a smooth group homomorphism between Lie groups. Then, if $\phi : G \longrightarrow G$ is a *Lie group automorphism*, the differential of ϕ restricted to $\mathfrak{g}$ yields a *Lie algebra automorphism* (which will be denoted also as $d\phi$), i.e., it satisfies that:

(i) $d\phi(X) \in \mathfrak{g}$ for all $X \in \mathfrak{g}$.

(ii) $d\phi([X, Y]) = [d\phi(X), d\phi(Y)]$ for all $X, Y \in \mathfrak{g}$.

(iii) It is bijective.

By the reasoning above, $d\phi : \mathfrak{g} \longrightarrow \mathfrak{g}$ is completely determined by the single map $(d\phi)_e$. Now, define for each $g \in G$ the automorphism $a_g : G \longrightarrow G$ to be the mapping $g' \longrightarrow gg'g^{-1}$ and denote the differential of a_g restricted to $\mathfrak{g}$ as $Ad_g : \mathfrak{g} \longrightarrow \mathfrak{g}$. Let H be a subgroup of G. An object defined on the Lie algebra $\mathfrak{g}$ is *$Ad(H)$-invariant* if it is preserved by $Ad_h : \mathfrak{g} \longrightarrow \mathfrak{g}$ for all $h \in H$.

Let $\mathfrak{a}$ be a Lie algebra and define the map $ad_X : \mathfrak{g} \longrightarrow \mathfrak{g}$ as $Y \longrightarrow [X, Y]$. Then, the bilinear form B defined as $B(X, Y) = \text{trace}(ad_X ad_Y)$ is called *the Killing form of* $\mathfrak{a}$. If $\mathfrak{a}$ is a Lie algebra of a Lie group G, then it is called *the Killing form of G*.

G-invariant metrics

Let φ be a left action of a Lie group G on a semi-Riemannian manifold (N, h). Then, if for each $g \in G$ the map φ_g is an isometry the action is called *isometric* and the semi-Riemannian metric is said to be *G-invariant*. The relation between the G-invariant metrics on a manifold N with a transitive action of G and the Lie algebra $\mathfrak{g}$ of G is then given in the following proposition:

Proposition A.6 ([65], page 311). *Let G be a Lie group acting transitively on the manifold $N = G/H$, with H its isotropy group at a point p, and let $\mathfrak{g}$, $\mathfrak{h}$ be their Lie algebras, respectively. If there exists an $Ad(H)$-invariant subspace $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{p} \oplus \mathfrak{h}$ (direct sum of vector spaces), then:*

(i) $[\mathfrak{h}, \mathfrak{p}] \subset \mathfrak{p}$.

(ii) $\mathfrak{p}$ can be identified with T_pN.

(iii) There exists a one-to-one correspondence between $Ad(H)$-invariant scalar products on $\mathfrak{p}$ and G-invariant metrics on N.

Appendix B

Canonical Form of Self-adjoint Endomorphisms

This appendix is devoted to give a brief summary of the classification of symmetric $(0,2)$-tensors in an Euclidean and a Lorentz vector space, and extend it to tensor fields.

Definition B.1. *Let* $(\mathcal{V}, h)$ *be a semi-Euclidean vector space. Then, an endomorphism* $L : \mathcal{V} \longrightarrow \mathcal{V}$ *is a* self-adjoint endomorphism *if it satisfies that* $h(L(\vec{v}), \vec{w}) = h(\vec{v}, L(\vec{w}))$ *for all* $\vec{v}, \vec{w} \in \mathcal{V}$.

Observe that this definition is equivalent to saying that the metrically equivalent $(0,2)$-tensor field to L is symmetric. In abstract index notation this is written as: $L_{\alpha\beta} = L_{\beta\alpha}$, while the endomorphism L is denoted as $L_\alpha{}^\beta$, and it is self-adjoint if $g_{\alpha\rho} L_\beta{}^\rho = g_{\beta\rho} L_\alpha{}^\rho$.

The classification of self-adjoint endomorphisms and their possible canonical forms on a semi-Euclidean real vector space $(\mathcal{V}, h)$ is well-known for positive-definite scalar products (see for example [54] and Theorem B.8 below) and is based on the fact that all the eigenvalues are real. The Lorentzian case is not so straightforward as the positive-definite case since it admits non-real eigenvalues, and its solution is known as the *Segre classification*. It was developed by Segre in [78] for the 4-dimensional case. An alternative classification based on spinors can be found, which gives a one-to-one correspondence with the Segre classification, in [58]. Plenty of work has been developed around this classification (see, for example, [80, 69, 14]). The classification for arbitrary dimension has also been developed (see, for example, [42]). It turns out that the higher dimensional case does not add more types than those arising in the four dimensional case. The summary of the Lorentzian signature in this appendix is mostly based on [41, 42, 69]. For a full classification in arbitrary signature, see [48].

To give the classification, we must consider a real vector space $\mathcal{V}$ as a part of its complexification $\mathcal{V}^{\mathbb{C}}$.

Complexification of a vector space

The complexification $\mathcal{V}^{\mathbb{C}}$ of a real vector space $\mathcal{V}$ is defined by taking the tensor product of

$\mathcal{V}$ with the complex numbers:

$$V^{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}.$$

To make $\mathcal{V}^{\mathbb{C}}$ into a complex vector space define complex multiplication as follows:

$$z(\vec{v} \otimes \zeta) = \vec{v} \otimes (z\zeta) \qquad \text{for all } \vec{v} \in \mathcal{V} \text{ and } z, \zeta \in \mathbb{C}.$$

The natural embedding of $\mathcal{V}$ into $\mathcal{V}^{\mathbb{C}}$ is given by

$$\vec{v} \mapsto \vec{v} \otimes 1$$

and the vector space $\mathcal{V}$ may then be regarded as a real subspace of $\mathcal{V}^{\mathbb{C}}$. It is a common practice to drop the tensor product symbol and just write $\zeta\vec{v}$ instead of $\vec{v} \otimes \zeta$. Then, every *complex vector* $\vec{v} \in \mathcal{V}^{\mathbb{C}}$ can be written uniquely in the form

$$\vec{v} = \vec{v}_1 + i\vec{v}_2,$$

where $\vec{v}_1$ and $\vec{v}_2$ are (real) vectors in $\mathcal{V}$. The conjugate of $\vec{v} \in \mathcal{V}^{\mathbb{C}}$ is given by

$$\bar{\vec{v}} = \vec{v}_1 - i\vec{v}_2.$$

Multiplication by the complex number $z_1 + iz_2$ is then given by the usual rule

$$(z_1 + iz_2)(\vec{v}_1 + i\vec{v}_2) = (z_1\vec{v}_1 - z_2\vec{v}_2) + i(z_2\vec{v}_1 + z_1\vec{v}_2).$$

With this rule for multiplication by complex numbers, we can then regard $\mathcal{V}^{\mathbb{C}}$ as the direct sum of two copies of $\mathcal{V}$:

$$\mathcal{V}^{\mathbb{C}} \cong \mathcal{V} \oplus i\mathcal{V}$$

Analogously, given a real linear transformation $L : \mathcal{V} \to \mathcal{W}$ between two real vector spaces there is a natural complex linear transformation $L^{\mathbb{C}} : V^{\mathbb{C}} \to W^{\mathbb{C}}$, called *the complexification of* L, given by $L^{\mathbb{C}}(\vec{v} \otimes z) = L(\vec{v}) \otimes z$. Without loss of generality, we will denote $L^{\mathbb{C}}$ with the same symbols as the real transformation L. Then,

$$L(\vec{v}_1 + i\vec{v}_2) = L(\vec{v}_1) + iL(\vec{v}_2),$$

where on the lefthand side L is the complex linear transformation and on the right side the real one. Therefore, it follows that

$$\overline{L(\vec{v})} = L(\bar{\vec{v}}). \tag{B.1}$$

Nevertheless, there exist two ways to extend a scalar product h on a real vector space $\mathcal{V}$ into $\mathcal{V}^{\mathbb{C}}$. As a sesquilinear form to obtain an Hermitic form (so that in the first slot h acts as a $\mathbb{C}$-linear map, but in the second the conjugate of the complex number comes out), or as a bilinear form (in both slots h acts as a $\mathbb{C}$-linear map). We have chosen the last one for this appendix, so that:

$$h(\vec{v}_0 + i\vec{v}_1, \vec{w}_0 + i\vec{w}_1) = h(\vec{v}_0, \vec{w}_0) - h(\vec{v}_1, \vec{w}_1) + i(h(\vec{v}_0, \vec{w}_1) + h(\vec{v}_1, \vec{w}_0)). \tag{B.2}$$

An analogous development with the Hermitian form would also be possible.

The Eigenvalue Problem on a semi-Euclidean Vector Space

The classification of a self-adjoint endomorphism L (denoted in abstract index form as $L_\alpha{}^\beta$) and its possible canonical forms on a semi-Euclidean real vector space $(\mathcal{V}, h)$ is based on the eigenvalue problem, that is, the solution to the problem

$$L(\vec{v}) = \lambda\vec{v}, \qquad \text{for some } \vec{v} \in \mathcal{V} \text{ and } \lambda \in \mathbb{R}.$$

Observe that for the metrically equivalent $(0,2)$-tensor L (denoted in abstract index form as $L_{\alpha\beta}$) this is written as:

$$L(\vec{v}, \cdot) = \lambda h(\vec{v}, \cdot), \qquad \text{for some } \vec{v} \in \mathcal{V} \text{ and } \lambda \in \mathbb{R}.$$

It is possible to talk about complex eigenvalues of L, since this is equivalent to consider the eigenvalues of its (real) matrix in any basis considering it as a complex one. Besides, if L is self-adjoint, so it is its complexification with respect to the bilinear extension of the metric h. Then, as a first result on a semi-Euclidean vector space, one has:

Lemma B.2. *A self-adjoint endomorphism L in a semi-Euclidean vector space $(\mathcal{V}, h)$ satisfies:*

(i) Eigenvectors associated to different (complex) eigenvalues are mutually orthogonal.

(ii) In a Lorentzian scalar product $h = g$, two linearly independent lightlike eigenvectors are associated to the same eigenvalue.

Proof. Let λ, μ be two eigenvalues with associated eigenvectors $\vec{v}$, $\vec{w}$ respectively. Then, $h(L(\vec{v}), \vec{w}) = h(\vec{v}, L(\vec{w}))$ yields

$$(\lambda - \mu)h(\vec{v}, \vec{w}) = 0.$$

So, if $\lambda \neq \mu$ necessarily $h(\vec{v}, \vec{w}) = 0$, and if $\vec{v}$, $\vec{w}$ are two linearly independent lightlike vectors in a Lorentz vector space, then, since $g(\vec{v}, \vec{w}) \neq 0$, it follows that $\lambda = \mu$. ■

Observe that if an endomorphism has a (non-real) complex eigenvalue $z = z_0 + iz_1 \in \mathbb{C} - \mathbb{R}$ with associated complex eigenvector $\vec{v} = \vec{v}_0 + i\vec{v}_1$, taking conjugates in $L(\vec{v}) = z\vec{v}$ and using (B.1) one obtains that the complex conjugate $\overline{z}$ is also an eigenvalue of L with associated eigenvector $\overline{\vec{v}}$. Clearly $\vec{v}$ and $\overline{\vec{v}}$ are $\mathbb{C}$-linearly independent (otherwise, if $\vec{v} = \zeta\,\overline{\vec{v}}$ from $L(\vec{v}) = z\vec{v}$ one would have $\zeta\,\overline{z}\,\overline{\vec{v}} = \zeta z\,\overline{\vec{v}}$, i.e., $\overline{z} = z$). Therefore, $\vec{v}_0, \vec{v}_1 \neq 0$.

Besides, expanding $L(\vec{v}) = z\vec{v}$ and making equal the real and the imaginary parts of each side of the equality, one obtains:

$$L(\vec{v}_0) = z_0\vec{v}_0 - z_1\vec{v}_1 \tag{B.3a}$$

$$L(\vec{v}_1) = z_1\vec{v}_0 + z_0\vec{v}_1 \tag{B.3b}$$

Then:

Proposition B.3. *Let V be a vector space and L an endomorphism on it with a complex eigenvalue $z = z_0 + iz_1 \in \mathbb{C} - \mathbb{R}$ and associated complex eigenvector $\vec{v} = \vec{v}_0 + i\vec{v}_1$. Then, $\vec{v}_0, \vec{v}_1$ are $\mathbb{R}$-linearly independent, so $\pi = span_{\mathbb{R}}\{\vec{v}_0, \vec{v}_1\}$ is an L-invariant plane and the endomorphism $L_{|\pi}$ is well-defined.*

Even more, if $(\mathcal{V}, h)$ is a a semi-Euclidean vector space and L is self-adjoint, then

$$h(\vec{v}_0, \vec{v}_0) + h(\vec{v}_1, \vec{v}_1) = 0. \tag{B.4}$$

Proof. Since $\vec{v}$ and its conjugate $\vec{\overline{v}}$ are $\mathbb{C}$-linearly independent, and

$$\vec{v}_0 = \frac{1}{2}(\vec{v} + \vec{\overline{v}}), \qquad \vec{v}_1 = \frac{1}{2i}(\vec{v} - \vec{\overline{v}}), \tag{B.5}$$

then $\vec{v}_0, \vec{v}_1$ are $\mathbb{C}$-linearly independent, and, thus, $\mathbb{R}$-linearly independent.

If L is self-adjoint, since $z_1 \neq 0$, by (B.3) the condition $h(L(\vec{v}_0), \vec{v}_1) = h(\vec{v}_0, L(\vec{v}_1))$ implies $h(\vec{v}_0, \vec{v}_0) = -h(\vec{v}_1, \vec{v}_1)$. ■

From the above proposition, one deduce the following corollaries for a self-adjoint endomorphism in different signature type vector spaces:

Corollary B.4. *All the eigenvalues of a self-adjoint endomorphism L in a Euclidean vector space $(\mathcal{V}, \overline{g})$ are real.*

Proof. If $\vec{v} = \vec{v}_0 + i\vec{v}_1$ is a complex eigenvector associated to a non-real complex eigenvalue, from Proposition B.3 one has that $\overline{g}(\vec{v}_0, \vec{v}_0) = -\overline{g}(\vec{v}_1, \vec{v}_1) \geq 0$, so $\vec{v}_0 = \vec{v}_1 = 0$, which is not possible. ■

Corollary B.5. *Let $(\mathcal{V}, h)$ be a Lorentz vector space. If a self-adjoint endomorphism L admits a non-real complex eigenvalue $z = z_0 + iz_1$, then:*

(1) If $\vec{v} = \vec{v}_0 + i\vec{v}_1$ is a non-real complex eigenvector, then $span_{\mathbb{R}}\{\vec{v}_0, \vec{v}_1\}$ is an L-invariant timelike plane π and it always exist two linearly independent lightlike vectors $\vec{l}, \vec{k}$ that span the plane $\pi = span\{\vec{v}_0, \vec{v}_1\}$ and such that

$$L(\vec{l}) = \lambda_0\vec{l} - \lambda_1\vec{k}, \qquad L(\vec{k}) = \lambda_1\vec{l} + \lambda_0\vec{k},$$

where $\lambda_0 = z_0 + z_1, \lambda_1 = z_0 - z_1$.

(2) The only non-real eigenvalues are z and $\overline{z}$.

Proof. (1) By proposition B.3 the vectors $\vec{v}_0, \vec{v}_1$ are linearly independent and $g(\vec{v}_0, \vec{v}_0) + g(\vec{v}_1, \vec{v}_1) = 0$, so either $g(\vec{v}_0, \vec{v}_0) = g(\vec{v}_1, \vec{v}_1) = 0$ and $\vec{v}_0 = \vec{l}, \vec{v}_1 = \vec{k}$ span a timelike plane with $\lambda_0 = z_0, \lambda_1 = z_1$ or $g(\vec{v}_0, \vec{v}_0) = -g(\vec{v}_1, \vec{v}_1) \neq 0$ and thus $\vec{v}_0$ is spacelike and $\vec{v}_1$ is timelike (or vice versa). To find the required lightlike vectors in this case, make $\vec{v}_0, \vec{v}_1$ orthogonal as follows. Take $\eta \in \mathbb{C}$. Then, $\vec{v}\,' = \eta\vec{v}$ is also an eigenvector associated to z. Therefore, since

$$g(\vec{v}\,', \vec{v}\,') = \eta^2 g(\vec{v}, \vec{v}) \in \mathbb{C},$$

taking η as one of the possible quadratic roots of $\pm\frac{1}{g(\vec{v},\vec{v})} \in \mathbb{C}$ it is possible to re-scale $\vec{v}$ so that

$$g(\vec{v},\vec{v}) = \pm 1.$$

Thus, taking $g(\vec{v},\vec{v}) = 1$, since L is self-adjoint, one has that $g(\vec{v}_0,\vec{v}_0) = 1/2$, $g(\vec{v}_1,\vec{v}_1) = -1/2$ and $g(\vec{v}_0,\vec{v}_1) = 0$. Define $\vec{l} = \vec{v}_0 + \vec{v}_1$ and $\vec{k} = \vec{v}_0 + \vec{v}_1$, so that $\lambda_0 = z_0 + z_1$ and $\lambda_1 = z_0 - z_1$.

(2) Assume that there exist two different complex eigenvalues $z, \zeta \in \mathbb{C} - \mathbb{R}$ where $z \neq \overline{\zeta}$ and with associated eigenvectors $\vec{v} = \vec{v}_0 + i\vec{v}_1$ and $\vec{w} = \vec{w}_0 + i\vec{w}_1$ respectively so that both $\text{span}_{\mathbb{R}}\{\vec{v}_0,\vec{v}_1\}$ and $\text{span}_{\mathbb{R}}\{\vec{w}_0,\vec{w}_1\}$ are timelike L-invariant planes. Using (i) in Lemma B.2 it follows that $\vec{v},\vec{w}$ are mutually orthogonal, which implies (via equation (B.5)) that the timelike planes $\text{span}_{\mathbb{R}}\{\vec{v}_0,\vec{v}_1\}$ and $\text{span}_{\mathbb{R}}\{\vec{w}_0,\vec{w}_1\}$ are two orthogonal timelike planes, and this is not possible in a Lorentz vector space. ∎

We also need some previous results concerning eigenvalues, eigenvectors and L-invariant subspaces.

Lemma B.6. *An endomorphism L in a semi-Euclidean vector space $(\mathcal{V}, h)$ satisfies that:*

(i) There always exists an L-invariant $\mathbb{R}$-plane.

(ii) If L is self-adjoint, the orthogonal of an L-invariant subspace is also L-invariant.

Proof. (i) If L has a complex eigenvalue by Proposition B.3 there exists an L-invariant plane. If not, by a basic result in algebra L is triangularizable, i.e., there exists a basis of (real) vectors $\{\vec{v}_0,\ldots,\vec{v}_{n-1}\}$ such that the matrix associated to L in this basis is written as:

$$(L) = \begin{pmatrix} \lambda_{00} & \lambda_{01} & \ldots & \lambda_{0(n-1)} \\ 0 & \lambda_{11} & \ldots & \lambda_{1(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_{(n-1)(n-1)} \end{pmatrix},$$

so $\text{span}_{\mathbb{R}}\{\vec{v}_0,\vec{v}_1\}$ is an L-invariant plane.

(ii) Let $\mathcal{W}$ be an L-invariant subspace. Then, as $L(\vec{w}) \in \mathcal{W}$ for all $\vec{w} \in \mathcal{W}$,

$$h(L(\vec{w}),\vec{w}\,') = 0, \qquad \forall \vec{w} \in \mathcal{W}, \vec{w}\,' \in \mathcal{W}^{\perp}.$$

But since L is self-adjoint,

$$0 = h(L(\vec{w}),\vec{w}\,') = h(\vec{w},L(\vec{w}\,')), \qquad \forall \vec{w} \in \mathcal{W}, \vec{w}\,' \in \mathcal{W}^{\perp}.$$

Hence, $L(\vec{w}\,') \in \mathcal{W}^{\perp}$ and $\mathcal{W}^{\perp}$ is L-invariant. ∎

Therefore,

Lemma B.7. *If a self-adjoint endomorphism L in a Lorentz vector space $(\mathcal{V}, g)$ has a degenerate L-invariant subspace, then there is a lightlike eigenvector contained in it.*

Proof. If $\mathcal{W}$ is a degenerate subspace in a Lorentz vector space, it contains a unique lightlike line, which is the intersection of $\mathcal{W}$ with its orthogonal $\mathcal{W}^{\perp}$. Therefore, if $\mathcal{W}$ is L-invariant, by lemma B.6 so it is $\mathcal{W}^{\perp}$ and necessarily the lightlike vector is an eigenvector. ∎

In particular, for positive definite scalar products the elementary canonical form of a self-adjoint endomorphism is also deduced.

Theorem B.8 ([54], page 80)**.** *Let $(\mathcal{V}, \overline{g})$ be an Euclidean vector space. Then, any self-adjoint endomorphism $L : \mathcal{V} \longrightarrow \mathcal{V}$ is diagonalizable. Even more, there exists an orthonormal vector basis $\{\vec{v}_0, \ldots, \vec{v}_{n-1}\}$ such that $L(\vec{v}_i) = \mu_i \vec{v}_i$ for some constants μ_i and $i = 0, \ldots, n-1$.*

Proof. Let λ be an eigenvalue of L, which must be real by Corollary B.4. Then, the eigenvalue problem $(L - \lambda I)(v) = 0$ has at least one solution, $\vec{v}_0$. If one chooses the L-invariant subspace $\mathcal{W} = \text{span}\{\vec{v}_0\}^{\perp}$ (Lemma B.6) with $V = \text{span}\{\vec{v}_0\} \oplus \mathcal{W}$, then $L_{|\mathcal{W}}$ is a well-defined self-adjoint endomorphism in the $(n-1)$-dimensional vector space $\mathcal{W}$, and the result follows by induction. ∎

Classification in a Lorentz Vector Space

We need the following general result in order to prove the classification:

Proposition B.9. *Let L be a self-adjoint endomorphism in a Lorentz vector space $(\mathcal{V}, g)$ of dimension greater than two. Then,*

$$\textit{there exists an } L\textit{-invariant} \begin{cases} \bullet \ \textit{spacelike} \\ \bullet \ \textit{timelike} \\ \bullet \ \textit{lightlike} \end{cases} \mathbb{R}\textit{-plane iff } L \textit{ has} \begin{cases} \bullet \ \textit{two independent and orthogonal spacelike eigenvectors} \\ \bullet \ n-2 \textit{ independent and orthogonal spacelike eigenvectors} \\ \bullet \ \textit{a (maybe non-unique) lightlike eigenvector} \end{cases}$$

Proof. If the L-invariant plane π is spacelike, observe that $L_{|\pi} : \pi \longrightarrow \pi$ is a self-adjoint endomorphism in $\mathbb{R}^2 \approx \pi$, with a positive-definite scalar product. Therefore, by Theorem B.8 $L_{|\pi}$ is diagonalizable and there exist two linearly independent and orthogonal spacelike eigenvectors. The converse is trivial.

If the L-invariant plane π is timelike, then by (ii) in Lemma B.6 the spacelike subspace $\pi^{\perp}$ is L-invariant, and $L_{|\pi^{\perp}}$ is a self-adjoint endomorphism in $\mathbb{R}^{n-2} \approx \pi^{\perp}$, with a positive-definite scalar product. So by Theorem B.8 $L_{|\pi^{\perp}}$ is diagonalizable and there exists $(n-2)$ linearly independent spacelike eigenvectors, as required. Conversely, the orthogonal complement to the L-invariant subspace spanned by the $(n-2)$ spacelike eigenvectors is the required L-invariant timelike plane, by (ii) in Lemma B.6.

If the L-invariant plane π is lightlike, the result follows from Lemma B.7. For the converse, observe that from (i) in Lemma B.6 we can always find an L-invariant plane. If it is not lightlike, we have already proved above that (for dimensions greater than two) there exists a spacelike eigenvector, which together with the given lightlike eigenvector span the required plane. ■

As an example of the non-uniqueness in the lightlike vector, take $L = \lambda(\vec{v}_0 \otimes \vec{v}_1{}^\flat - \vec{v}_0 \otimes \vec{v}_0{}^\flat) + \mu_i \vec{v}_i \otimes \vec{v}_i{}^\flat$, with $\{\vec{v}_\alpha\}$ an orthonormal basis. The lightlike vectors $\frac{1}{\sqrt{2}}(\vec{v}_0 \pm \vec{v}_1)$ are eigenvectors of L and the plane $\pi = \text{span}_{\mathbb{R}}\{\frac{1}{\sqrt{2}}(\vec{v}_0 + \vec{v}_1), \vec{v}_2\}$ is an L-invariant lightlike plane.

The classification in the Lorentzian case is given in the following theorem:

Theorem B.10. *Let $(\mathcal{V}, g)$ be a Lorentz vector space of dimension n and $L : \mathcal{V} \longrightarrow \mathcal{V}$ a self-adjoint endomorphism. Then, there exists a null basis $\{\vec{l}, \vec{k}, \vec{v}_2, \dots, \vec{v}_{n-1}\}$ ($\vec{l}, \vec{k}$ lightlike vectors with orthogonal spanned by the orthonormal vectors $\vec{v}_i$'s, with $g(\vec{l}, \vec{k}) = -1$) such that:*

- **Segre type** $[z\bar{z}\dots 1]$**:**

$$L = -\lambda_0(\vec{k} \otimes \vec{l}^\flat + \vec{l} \otimes \vec{k}^\flat) + \lambda_1(\vec{k} \otimes \vec{k}^\flat - \vec{l} \otimes \vec{l}^\flat) + \sum_{i=2}^{n-1} \lambda_i \vec{v}_i \otimes \vec{v}_i{}^\flat$$

with $\lambda_1 \neq 0$ and the associated matrix in the null frame is

$$(L) = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & \dots & 0 \\ -\lambda_1 & \lambda_0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{n-1} \end{pmatrix}$$

- **Segre type** $[1, 1 \dots 1]$**:**

$$L = -\lambda_0(\vec{l} \otimes \vec{k}^\flat + \vec{k} \otimes \vec{l}^\flat) - \lambda_1(\vec{l} \otimes \vec{l}^\flat + \vec{k} \otimes \vec{k}^\flat) + \sum_{i=2}^{n-1} \lambda_i \vec{v}_i \otimes \vec{v}_i{}^\flat$$

and the associated matrix in the null frame is

$$(L) = \begin{pmatrix} \lambda_0 & \lambda_1 & 0 & \dots & 0 \\ \lambda_1 & \lambda_0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{n-1} \end{pmatrix}$$

In this case, there exists also an orthonormal basis of eigenvectors.

- **Segre type** $[211\dots1]$**:**

$$L = -\lambda_0(\vec{l}\otimes\vec{k}^\flat + \vec{k}\otimes\vec{l}^\flat) \pm \vec{l}\otimes\vec{l}^\flat + \sum_{i=2}^{n-1}\lambda_i\vec{v}_i\otimes\vec{v}_i^{\,\flat}$$

and the associated matrix in the null frame is

$$(L) = \begin{pmatrix} \lambda_0 & \pm1 & 0 & \dots & 0 \\ 0 & \lambda_0 & 0 & \dots & 0 \\ 0 & 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \lambda_{n-1} \end{pmatrix}$$

- **Segre type** $[31\dots1]$**:**

$$L = \lambda_0(-\vec{l}\otimes\vec{k}^\flat - \vec{k}\otimes\vec{l}^\flat + \vec{v}_2\otimes\vec{v}_2^{\,\flat}) + \vec{l}\otimes\vec{v}_2^{\,\flat} - \vec{v}_2\otimes\vec{l}^\flat + \sum_{i=3}^{n-1}\lambda_i\vec{v}_i\otimes\vec{v}_i^{\,\flat}$$

and the associated matrix in the null frame is

$$(L) = \begin{pmatrix} \lambda_0 & 0 & 1 & 0 & \dots & 0 \\ 0 & \lambda_0 & 0 & 0 & \dots & 0 \\ 0 & 1 & \lambda_0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \lambda_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & \lambda_{n-1} \end{pmatrix}$$

Proof. If L admits complex eigenvalues, by Corollary B.5 there are only two complex conjugate eigenvalues $z = z_0 + iz_1, \bar{z}$, and there exist two linearly independent lightlike vectors $\vec{l}$, $\vec{k}$ with the required properties that span the L-invariant timelike plane $\pi = \mathrm{span}_{\mathbb{R}}\{\vec{v}_0, \vec{v}_1\}$. Diagonalize the self-adjoint endomorphism $L_{\pi^\perp}$ defined in a $(n-2)$-dimensional Euclidean vector space using Theorem B.8 to prove the result for Segre type $[z\bar{z}1\dots1]$.

If there is no complex eigenvalues, assume first that L is diagonalizable. Then, V is a direct sum of eigenspaces and by Lemma B.2 they are mutually orthogonal. Thus, there is no degenerate eigenspaces. So there exists an orthonormal basis $\{\vec{v}_0, \vec{v}_1, \dots, \vec{v}_{n-1}\}$ with $\vec{v}_0$ timelike, the remaining spacelike, and $L(\vec{v}_\alpha) = \mu_\alpha\vec{v}_a$ for some $\mu_\alpha \in \mathbb{R}$ and for all $\alpha = 0, \dots, n-1$. Take $\vec{l} = \vec{v}_0 + \vec{v}_1$ and $\vec{k} = \vec{v}_0 - \vec{v}_1$ to obtain the required basis, so that $\lambda_0 = \mu_0 + \mu_1$ and $\lambda_1 = \mu_0 - \mu_1$, and obtain the Segre type $[1, 1\dots1]$.

Finally, assume that L is not diagonalizable. If λ is an eigenvalue with algebraic multiplicity m_λ and geometric one m'_λ, so that $m_\lambda - m'_\lambda = s$, then

$$(L - \lambda\mathrm{Id})^s \neq 0 \qquad \text{but} \qquad (L - \lambda\mathrm{Id})^{s+1} = 0$$

and if $\vec{v}_0 \notin \mathrm{Ker}((L-\lambda\mathrm{Id})^s)$ so that $(L-\lambda\mathrm{Id})^s(\vec{v}_0) \neq 0$, then

$$\{\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_s\} = \{\vec{v}_0, (L-\lambda\mathrm{Id})(\vec{v}_0), \ldots, (L-\lambda\mathrm{Id})^s(\vec{v}_0)\}$$

is a set of linearly independent vectors. Since L is self-adjoint:

$$\begin{aligned}
g(\vec{v}_{s-1}, \vec{v}_{s-1}) &= g((L-\lambda\mathrm{Id})^{s-1}(v_0), (L-\lambda\mathrm{Id})^{s-1}(v_0)) \\
&= g((L-\lambda\mathrm{Id})^{s+1}(v_0), (L-\lambda\mathrm{Id})^{s-3}(v_0)) = 0, \\
g(\vec{v}_s, \vec{v}_{s-1}) &= g((L-\lambda\mathrm{Id})^{s}(v_0), (L-\lambda\mathrm{Id})^{s-1}(v_0)) \\
&= g((L-\lambda\mathrm{Id})^{s+1}(v_0), (L-\lambda\mathrm{Id})^{s-2}(v_0)) = 0, \\
(\vec{v}_s, \vec{v}_s) &= g((L-\lambda\mathrm{Id})^{s}(v_0), (L-\lambda\mathrm{Id})^{s}(v_0)) \\
&= g((L-\lambda\mathrm{Id})^{s+1}(v_0), (L-\lambda\mathrm{Id})^{s-1}(v_0)) = 0,
\end{aligned}$$

so $\vec{v}_{s-1}, \vec{v}_s$ are two linearly independent and mutually orthogonal lightlike vectors, which is impossible in a Lorentz vector space unless $s \leq 2$. We will reason only the case $s = 2$, $s = 1$ being simpler. Then,

$$(L-\lambda\mathrm{Id})^2 \neq 0 \qquad \text{but} \qquad (L-\lambda\mathrm{Id})^3 = 0$$

and

$$\{\vec{v}_0, \vec{v}_1, \vec{l}\} = \{\vec{v}_0, (L-\lambda\mathrm{Id})(\vec{v}_0), (L-\lambda\mathrm{Id})^2(\vec{v}_0)\}$$

is a set of linearly independent vectors, with $\vec{l}$ lightlike. Besides, for a different eigenvalue $\mu \neq \lambda$ $m_\mu - m'_\mu = 0$ must hold, as otherwise there would exist another lightlike eigenvector associated to μ, and this is not possible by Lemma B.2. Furthermore, if there were another causal (lightlike or timelike) eigenvector associated to λ, then in any case there would exist a timelike eigenvector $\vec{w}$. Take the L-invariant subspace $\mathcal{W} = \mathrm{span}\{\vec{w}\}^\perp$. Then, $L_{|\mathcal{W}}$ would be a well-defined self-adjoint endomorphism in the $(n-1)$-dimensional Euclidean vector space $\mathcal{W}$, thus, diagonalizable, and since $V = \mathrm{span}\{\vec{w}\} \oplus \mathcal{W}$ the endomorphism L would also be diagonalizable.

In conclusion, if L is not diagonalizable there is a unique lightlike eigenvector $\vec{l}$ up to a constant, and the eigenvectors associated to eigenvalues different from λ span an L-invariant $(n-3)$-dimensional Euclidean vector subspace $\mathcal{W}$. Besides, it is possible to construct an orthonormal basis of eigenvectors in $\mathcal{W}$. Therefore, from now on, we will restrict ourselves to $\mathcal{W}^\perp$ and the problem is then reduced to three dimensions.

Fixed $\vec{l}$, from Proposition B.9 there exists an L-invariant lightlike plane in $\mathcal{W}^\perp$ spanned by $\vec{l}$ and a spacelike unit vector $\vec{v}_2$ orthogonal to $\vec{l}$. Extend $\{\vec{l}, \vec{v}_2\}$ to a null frame $\{\vec{l}, \vec{k}, \vec{v}_2\}$ in $\mathcal{W}^\perp$ with $g(\vec{l}, \vec{k}) = -1$. Therefore, as L is self-adjoint:

$$\begin{aligned}
L(\vec{l}) &= \lambda \vec{l} \\
L(\vec{v}_2) &= \beta_2 \vec{l} + \lambda_2 \vec{v}_2, \\
L(\vec{k}) &= \mu \vec{l} + \lambda \vec{k} + \beta_2 \vec{v}_2.
\end{aligned}$$

The changes between null frames that maintain the vector $\vec{l}$ fixed (and preserve the orientation of the plane span$\{\vec{l}, \vec{v}_2\}$ are given by:

$$\vec{l}' = \vec{l} \tag{B.6a}$$
$$\vec{v}_2{}' = \vec{v}_2 + \gamma_2 \vec{l}, \tag{B.6b}$$
$$\vec{k}' = \vec{k} + \gamma_2 \vec{v}_2 + \frac{1}{2}(\gamma_2)^2 \vec{l} \tag{B.6c}$$

Then, if $\lambda_2 \neq \lambda$, we can set $\beta_2 = 0$ (take $\gamma_2 = \frac{\beta_2}{\lambda_2 - \lambda}$ in (B.6)). So rescaling $\vec{l}$ and $\vec{k}$ as

$$\vec{l}\,' = \sqrt{|\mu|}\vec{l} \qquad \text{and} \qquad \vec{k}\,' = \frac{1}{\sqrt{|\mu|}}\vec{k}$$

one obtains the canonical form for Segre type $[211\dots1]$. If instead $\lambda_2 = \lambda$, to obtain the required canonical form for $[31\dots1]$, take $\gamma_2 = \frac{\mu}{2\beta_2}$ in (B.6)) so that $\mu' = 0$, and rescale $\vec{v}_2{}'' = \frac{\vec{v}_2{}'}{\sqrt{|\beta_2'|}}$ so that $\beta_2'' = 1$. ■

The coma in Segre notation used above separates the eigenvalue associated to the timelike eigenvector in case this exists, and the notation $z\bar{z}$ is used to indicate a pair of complex conjugate eigenvalues. The numbers indicate the Jordan blocks dimensions, that is, the difference between algebraic and geometric multiplicity plus one. Sometimes, round brackets are used to reinforce degeneracies (i.e., multiplicities of the eigenvalues), so for example $[(1, 1\dots1)]$ is the Segre type of the (Lorentzian) scalar product g.

One can easily derive from the theorem the following:

Corollary B.11. *Let $(\mathcal{V}, g)$ be a Lorentz vector space of dimension n and $L : \mathcal{V} \longrightarrow \mathcal{V}$ a self-adjoint endomorphism. Then:*

(1) If L has Segre type $[z\bar{z}\dots1]$, the eigenvalues are $\lambda_0 \pm i\lambda_1$ with $\lambda_0, \lambda_1 \neq 0, \lambda_2, \dots, \lambda_{n-1}$ and the associated eigenvectors are $\vec{l} \pm i\vec{k}, \vec{v}_2, \dots, \vec{v}_{n-1}$, which form a complex basis for $\mathcal{V}^{\mathbb{C}}$. Therefore, there is neither a timelike nor a lightlike eigenvector, and there are two conjugate eigenvectors with associated complex conjugate eigenvalues. Their real and imaginary part span a timelike plane.

(2) If L has Segre type $[1, 1\dots1]$, the eigenvalues are $\lambda_0 \pm \lambda_1, \lambda_2, \dots, \lambda_{n-1}$ and the associated eigenvectors are, respectively, $\vec{l} \pm \vec{k}, \vec{v}_2, \dots, \vec{v}_{n-1}$, which form an orthonormal basis for $\mathcal{V}$. Therefore, there exists a timelike eigenvector, and there exist two lightlike eigenvectors if and only if the associated eigenvalue to the timelike eigenvector is multiple (that is, it coincides with some $\lambda_i, i > 1$ or $\lambda_1 = 0$).

(3) If L has Segre type $[211\dots1]$ or $[31\dots1]$, the eigenvalues are $\lambda_0, \lambda_2, \dots, \lambda_{n-1}$ and $\lambda_0, \lambda_3, \dots, \lambda_{n-1}$ respectively, and the associated eigenvectors are, $\vec{l}, \vec{v}_2, \dots, \vec{v}_{n-1}$ and $\vec{l}, \vec{v}_3, \dots, \vec{v}_{n-1}$ respectively; there does not exist a timelike eigenvector, and there exists a unique lightlike eigenvector (up to a constant), which is associated to the eigenvalue λ_0, of (algebraic) multiplicity 2 for Segre type $[211\dots1]$ and multiplicity 3 for Segre type $[31\dots1]$.

Besides, for dimensions greater than three, L always has a spacelike eigenvector.

More breafly, the Segre types can also be explained in terms of the existence of a lightlike eigenvector as follows:

- If there is no lightlike eigenvector: the Segre type is $[1, 1 \ldots 1]$, including its degeneracies except from $[(1, 1) \ldots 1]$,
- if there is more than one independent lightlike eigenvector: the Segre type is $[(1, 1) \ldots 1]$, including its degeneracies,
- if there is only one lightlike eigenvector (up to a constant) the Segre type is $[211 \ldots 1]$ or $[31 \ldots 1]$.

Segree type for tensor fields

To end this Appendix, we add a useful result on manifolds:

Lemma B.12. *Let (M, g) be a connected Lorentzian manifold and L a $(0, 2)$-tensor field. If L is parallel its Segre type is the same everywhere, and the eigenvalues are constant.*

Proof. Let $p \in M$. Then, there exists a vector basis in T_pM such that $L_{|p}$ can be written in one of the forms given in Theorem B.10. Since L is parallel, it remains constant under parallelly transported vectors along curves. So extend this basis to a frame on a starshaped neighborhood of p by geodesical extension (see definition 2.3.4) to obtain the first assertion.

For the second, if λ_p is an eigenvalue of L_p and $\vec{v}$ an associated eigenvector, then

$$L_p(\cdot, \vec{v}) - \lambda_p g_p(\cdot, \vec{v}) = 0.$$

Take $q \in M$ different from p. If γ is the geodesic joining p to q and V the geodesical extension of $\vec{v}$ along such geodesic, differentiate covariantly the above equation along the geodesic to obtain that $\nabla_\gamma \lambda = 0$. Therefore, the eigenvalue remains constant along γ, and consequently all over M. ■

More information about Segre type in a $(0, 2)$-tensor field can be found in [80].

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Index