A Finslerian notion of causal structure

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An outline

- (local) Causal structures: definition, motivation, and history
- The equivalence method: Riemannian, Finsler, conformal, causal
- Structure equations and local invariants
- Causal vs. Finsler
- Null Jacobi fields and tidal force
- Half-flat indefinite causal structures in dimension 4
- Discussing Petrov type and almost Einstein condition for causal str

Definitions

The relation of causal str to conformal pseudo-Riem str is intended to be an analogue of what Finsler str are to Riem str:
Pseudo-Riemannian metric on M^{n+1} is uniquely determined by

$$TM \supset \Sigma^{2n+1} = \{v \in TM \mid g(v,v) = 1\}$$

Roughly speaking, if Σ_x not quadratic one has a (local) Finsler metric

$$TM \supset \Sigma^{2n+1} = \{v \in TM \mid F(v) = 1\}$$

assuming *radial transversality* and *non-deg of the 2nd fund form* of $\Sigma_x \subset T_xM,\ \forall x \in M$.

Definitions

A more general notion of Finsler structures due to Bryant:

Definition (**Bryant**) A *generalized pseudo-Finsler structure* on M^{n+1} is denoted by (M,Σ) together with an immersion $\iota:\Sigma\to TM$ where Σ is a connected, smooth manifold of dimension $2n+1$ and ι is a radially transverse immersion satisfying

- The map $\pi\circ\iota:\Sigma\to M$ is a submersion with connected fibers.
- In the fibration $\pi\circ\iota:\Sigma\to M$, the fibers $\Sigma^n_x := (\pi\circ\iota)^{-1}(x)$ are mapped to immersed connected hypersurfaces via $\iota_x:\Sigma_x\to T_xM$ with non-deg 2nd fund form everywhere.

$(M,\Sigma) \overset{locally}{\cong} (\tilde{M},\tilde{\Sigma})$

at $x\in M, \tilde{x}\in\tilde{M}$

if $\exists$ diffeo $\phi: U\to\tilde{U}$

where $x\in U\subset M, \tilde{x}\in\tilde{U}\subset\tilde{M}$

$$\begin{array}{ccc} \Sigma|_U & \xrightarrow{\phi_*} & \tilde{\Sigma}|_{\tilde{U}} \\ \mu\downarrow & & \downarrow\tilde{\mu} \\ U & \xrightarrow{\phi} & \tilde{U} \end{array}$$

$\tilde{x}=\phi(x)$

$\phi_*(\Sigma_y)=\tilde{\Sigma}_{\phi(y)}$

$\forall y\in U$

Definitions: remarks

- Not every generalized Finsler structure is realizable as a Finsler metric.
- Note that Σ can be open and immersed as an open submanifold.
- There is no requirement for Σ_x to be compact or ι be an embedding.
- All classical constructions for Finsler structures, e.g., canonical connections, structure bundle and curvature will go through.
- For the local aspects of Finsler geometry ι can be assumed to be an embedding in a sufficiently small neighborhood of Σ.
- When imposing certain DEs on Finsler structures it is more natural to work in the generalized setting an then restrict to the classical setting afterwards.

Definitions

The conformal class of a Pseudo-Riem metric (pseudo-conformal structures) on M^{n+1} is uniquely determined by its null cones

$$\mathbb{P}TM \supset \mathcal{C}^{2n} = \{v \in TM \mid g(v,v) = 0\}$$

Assigning a null cone at each tangent space is the main ingredient for understanding causal properties of M.
Roughly speaking, if $\mathcal{C}_x$ not quadratic, $\mathcal{C}$ is a field of cones, locally described by

$$\mathbb{P}TM \supset \mathcal{C}^{2n} = \{v \in TM \mid G(v) = 0\}.$$

If the projective 2nd fund form of $\mathcal{C}_x \subset \mathbb{P}T_xM$, $\forall x \in M$ is non-degenerate one obtains a *causal structure.*

Definitions

More precise definition:

Definition A *causal structure* on M^{n+1} is denoted by $(M, \mathcal{C})$ together with an immersion $\iota : \mathcal{C} \to \mathbb{P}TM$ where $\mathcal{C}$ is a connected, smooth manifold of dimension $2n$ and ι is an immersions satisfying

- The map $\pi \circ \iota : \mathcal{C} \to M$ is a submersion with connected fibers.
- In the fibration $\pi \circ \iota : \mathcal{C} \to M$, the fibers $\mathcal{C}_x^{n-1} := (\pi \circ \iota)^{-1}(x)$ are mapped to immersed connected *tangentially non-degenerate* projective hypersurfaces via $\iota_x : \mathcal{C}_x \to \mathbb{P}T_xM$, i.e., they have non-deg projective 2nd fund form everywhere.

$(M, \mathcal{C}) \overset{locally}{\cong} (\tilde{M}, \tilde{\mathcal{C}})$

at $x \in M, \tilde{x} \in \tilde{M}$

if $\exists$ diffeo $\phi : U \to \tilde{U}$

where $x \in U \subset M, \tilde{x} \in \tilde{U} \subset \tilde{M}$

$$\begin{array}{ccc} \mathcal{C}|_U & \xrightarrow{\phi_*} & \tilde{\mathcal{C}}|_{\tilde{U}} \\ \mu \downarrow & & \downarrow \tilde{\mu} \\ U & \xrightarrow{\phi} & \tilde{U} \end{array}$$

$\tilde{x} = \phi(x)$

$\phi_*(\mathcal{C}_y) = \tilde{\mathcal{C}}_{\phi(y)}$

$\forall y \in U$

Remarks

- Locally the projective 2nd fundamental form of a hypersuface in $\mathbb{P}^n$ is proportional to the 2nd fundamental form of the affine hypersuface obtained by taking an affine chart for $\mathbb{P}^n$.
- $\mathcal{C}^{2n}$ is called the*(projective) null cone bundle* of the causal structure. We do not assume that its fibers are convex or closed in $\mathbb{P}T_xM$.
- Note that $\mathcal{C}$ can be open and be immersed as an *open hypersurface* in $\mathbb{P}TM$.
- For the local aspects of causal geometry ι can be assumed to be an embedding in a sufficiently small neighborhood of $\mathcal{C}$.

Definitions and examples

Locally a causal structure can be expressed as

$$\mathcal{C} \supset U = \{(x,[y]) \in \mathbb{P}TM | L(x;y) = 0\}.$$

$L : TM \to \mathbb{R}$ or $\mathbb{C}$ satisfies

$$\begin{cases} L(x;\lambda y) = \lambda^r L(x;y) \text{ for some } r \\ \left[\dfrac{\partial^2 L}{\partial y^i \partial y^j}\right] \text{ has max rank over } L = 0. \end{cases}$$

$L(x;y)$, $S(x;y)L(x;y) \longrightarrow$ same causal str (S nowhere vanishing.)

Example : $L(x;y) = (y^1)^2 + (y^2)^2 + (y^3)^2 - (y^4)^2$: *flat* 4D causal structure.

Example : $L(x;y) = \frac{1}{3}(y^2)^3 + y^0 y^3 y^3 - y^1 y^2 y^3$:
Null cones are projectively equivalent to Cayley's cubic surface.

Definitions and examples

Definition : $(M, \mathcal{C})$ is called *locally V-isotrivial* if $\mathcal{C}_x \cong V \subset \mathbb{P}^n, \ \forall x \in M$

$(M, \mathcal{C})$ is called *locally V-isotrivially flat* if $(M, \mathcal{C}) \overset{loc}{\cong} (U, U \times V)$ where $V \subset \mathbb{P}^n$ is a projective hypersurface.
i.e., locally it can be expressed as

$$\{(x; [y]) \in \mathbb{P}TM \mid L(y) = 0\}$$

with $L(y)$ not depending on x.

Being locally V-isotrivial is the causal analogue of being locally Minkowskian in Finsler geometry.

Motivation 1: Geometrization of DEs

This program, pioneered by Cartan and Chern, was aimed to characterize geometric structures arising from certain classes of differential equations.

$$\left\{\begin{matrix}\text{Contact equivalence class of} \\ y''' = f(x,y,y',y'')\end{matrix}\right\} \overset{locally}{\longleftrightarrow} \text{Certain foliations of } J^2(\mathbb{R},\mathbb{R})$$

$J^2(\mathbb{R},\mathbb{R})$: 2nd jet space of functions with coordinates (x, y, p, q), where $p = y', q = y''$.

A *contact equivalence class* is an equivalence relation defined by *contact transformation*: diffeomorphisms of $J^1(\mathbb{R},\mathbb{R})$

$$x \mapsto \bar{x} = \chi(x,y,y'), \quad y \mapsto \bar{y} = \phi(x,y,y'), \quad y' \mapsto \bar{y}' = \psi(x,y,y'),$$

satisfying

$$\psi = \frac{\mathrm{D}\phi}{\mathrm{D}\chi}, \quad \mathrm{D} = \partial_x + y'\partial_y + y''\partial_{y'} + y'''\partial_{y''}$$

Motivation 1: Geometrization of differential equations

Theorem (Holland-Sparling following the works of Cartan, Chern, Sato-Yashikawa, Newman-Kozameh, Nurowski-Godlinski,...)

$$\left\{\begin{matrix}\text{contact equivalent classes} \\ \text{of 3rd order ODEs}\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{causal structures} \\ (M^3, \mathcal{C}^4)\end{matrix}\right\}$$

3-dimensional causal structures $(M^3, \mathcal{C}^4)$

$$J^1(\mathbb{R},\mathbb{R}) \cong \mathcal{K}^3 \xleftarrow{\quad\rho\quad} J^2(\mathbb{R},\mathbb{R}) \cong \mathcal{C}^4 \xrightarrow{\mu} M^3 \cong \text{Space of solutions}$$

$\mathcal{K}^3$: Locally defined space of *"null geodesics"* which has a contact structure.
M^3: Locally defined space of solutions of a 3rd order ODE.
An extension of this result, with relation to Newman's *null surface formulation of General Relativity* will be described later.

Motivation 2: Hwang-Mok program and VMRTs

Their program is to give a differential geometric characterization of uniruled projective manifolds using their variety of minimal rational tangents (VMRT).

On a uniruled manifold M consider the scheme of rational curves with minimal degree wrt to $-K_M$.

At a general point $x \in M$,
$\mathcal{C}_x =$VMRT at $x =${tangent directions to such curves at x}.

Assume for generic x, $\mathcal{C}_x \begin{Bmatrix} \text{smooth} \\ \text{of codimension one} \\ \text{of degree } \geq 2 \end{Bmatrix} : M$ has causal str.

Theorem (Hwang, 2013) Causal structures arising from smooth VMRTs are locally isotrivially flat.
This gives a generalization of results on holomorphic conformal str.

History

- Apart from the works presented so far, the notion of causal structure closely related to what I defined can be found in Irvine Segal's book *Mathematical cosmology and extragalactic astronomy*, 1976.
- In Hwang-Mok program a *cone structure* is defined as a field of cones which can have any codimension or degeneracy condition.
- I borrowed the term causal from the work of Holland-Sparling on scalar 3rd order ODEs. They defined causal structure in 3D via an axiomatic approach.

The equivalence problem: Riemannian geometry

Cartan invented a powerful machinery to solve the equivalence problem of geometric structures by constructing a principal bundle and obtaining invariants.

Given a Riemannian metric $g \in \Gamma(S^2(T^*M))$, define $(\omega^0, \cdots, \omega^n)$ s.t.

$$g = (\omega^0)^2 + \cdots + (\omega^n)^2.$$

The choice of ω^i's are ambiguous up to an $\mathrm{O}(n+1)$-action.

Now, consider the principal $\mathrm{O}(n+1)$-bundle $\pi : \mathcal{F} \to M$.

There is a taugological lift of the 1-forms ω^i's to $\mathcal{F}$, defining the so-called tautological $\mathbb{R}^{n+1}$-valued 1-form on $\mathcal{F}$ (aka the soldering form).

The equivalence problem: Riemannian geometry

The *structure group* $\mathrm{O}(n+1)$ acts transitively on the fibers of $\mathcal{F}$, i.e., the right action

$$R_g(f_x) = g^{-1} f_x$$

where $f_x : T_x M \to \mathbb{R}^{n+1}$ is a coframe on M and $f = (x; f_x) = (x; \omega^0_x, \cdots, \omega^n_x) \in \mathcal{F}$, with $\omega^i \subset T^* M$.

Define the tautological $\mathbb{R}^{n+1}$-valued 1-form ω on $\mathrm{pr} : \mathcal{F} \to M$ by setting at $f = (x, f_x) \in \mathcal{F}$

$$\omega_f(v) := f_x(\mathrm{pr}_*(v)) \in \mathbb{R}^{n+1}, \quad v \in T_f \mathcal{F}$$

The 1-forms $\mathrm{pr}^*(\omega^i)$ furnish a basis of semi-basic forms on $\mathcal{F}$.

In practice, $\mathcal{F}|_U \cong U \times \mathrm{O}(n+1)$. Take a coframe $(\omega^0, \cdots, \omega^n)^T$.

$$\underline{\omega}(x; g) = (\underline{\omega}^0, \cdots, \underline{\omega}^n)(x; g) := g^{-1}(\mathrm{pr}^*\omega^0, \cdots, \mathrm{pr}^*\omega^n)^T = g^{-1}\omega_e.$$

$$\mathrm{d}\underline{\omega} = \mathrm{d}g^{-1} \wedge \omega + g^{-1}\mathrm{d}\omega = -g^{-1}\mathrm{d}g \wedge g^{-1}\omega + g^{-1}\mathrm{d}\omega = -\alpha \wedge \underline{\omega} + T(\underline{\omega} \wedge \underline{\omega})$$

The equivalence problem: Riemannian geometry

The 1-forms α are the Maurer-Cartan forms of $\mathrm{O}(n+1)$.
dropping the underline, we obtain

$$\mathrm{d}\omega^i = -\omega^i_j \wedge \omega^j + \frac{1}{2} T^i_{jk} \omega^j \wedge \omega^k,$$

where $\omega^i_j = -\omega^j_i$, $T^i_{jk} = -T^i_{kj}$, (T is called *torsion*.)
The 1-forms ω^i_j are ambiguous up to a linear combination of ω^i's.
Pin down ω^i_j (Levi-Civita conn) so that $T^i_{jk} = 0$, i.e.,

$$\omega^i_j \to \omega^i_j + \frac{1}{2}(T^i_{jk} - T^j_{ik})\omega^k + \delta_{kl} T^k_{ij} \omega^l.$$

After one more differentiation it follows that

$$\mathrm{d}\omega^i = -\omega^i_j \wedge \omega^j$$
$$\mathrm{d}\omega^i_j = -\omega^i_k \wedge \omega^k_j + \frac{1}{2} R^i_{jkl} \omega^k \wedge \omega^l$$

where $R^i_{jkl} = -R^j_{ikl}$, $R^i_{jkl} = -R^i_{jlk}$, $R^i_{jkl} + R^i_{klj} + R^i_{ljk} = 0$.

The equivalence problem: Conformal geometry

Define $(\omega^0, \cdots, \omega^n)$ so that

$$g = (\omega^0)^2 + \cdots + (\omega^n)^2.$$

The choice of ω^i's are ambiguous up to a $\mathrm{CO}(n+1)$-action.
Consider the principal $\mathrm{CO}(n+1)$-bundle $\pi : \mathcal{F} \to M$.
Using the same absorption, it follows

$$\mathrm{d}\omega^i = -(\omega^i_j + \lambda\delta^i_j) \wedge \omega^j$$

However, ω^i_j, λ still have ambiguity, i.e., one needs to *prolong.*
After prolongation, the structure equations is obtained

$$\begin{aligned}
\mathrm{d}\omega^i &= -(\omega^i_j + \lambda\delta^i_j) \wedge \omega^j \\
\mathrm{d}\omega^i_j &= -\omega^i_k \wedge \omega^k_j + \pi_j \wedge \omega^i - \pi_i \wedge \omega^j + \frac{1}{2} W^i_{jkl}\omega^k \wedge \omega^l, \\
\mathrm{d}\lambda &= \pi_i \wedge \omega^i, \\
\mathrm{d}\pi_i &= (\omega^i_j + \lambda\delta^i_j) \wedge \pi^j + \frac{1}{2} W_{ijk}\omega^i \wedge \omega^j
\end{aligned}$$

Curvature decomposition

The quantities W^i_{jkl}, W_{ijk} are called the Weyl tensor and the Cotton-York tensor of the conformal structure satisfying

$$W^i_{jkl} = -W^j_{ikl},\ W^i_{jkl} = -W^i_{jlk},\ W^i_{jkl} + W^i_{klj} + W^i_{ljk} = 0,\ W^i_{jil} = 0$$

$$W_{ijk} = -W_{ilk},\ W_{ijk} + W_{jki} + W_{kij} = 0.$$

The Cotton-York tensor satisfies $W_{jkl} = \frac{1}{n-3}\frac{\partial}{\partial \omega^i} W^i_{jkl}$.

The symmetries of W^i_{jkl} is like R^i_{jkl} and is trace-free: $W^i_{jil} = 0$.

In fact, the Riemann tensor as an $\mathrm{O}(n+1)$-module decomposes as

$$R^i_{jkl} = W^i_{jkl} - \delta^i_l u_{jk} + \delta^j_l u_{ik} + \delta^i_k u_{jl} - \delta^j_k u_{il},$$

where u_{ij} is the Rho tensor

$$u_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{R}{2(n-1)}\delta_{ij}\right)$$

The equivalence problem: Finsler geometry

As originally done by Cartan and Chern, *Hilbert form* η^0 is the essential notion in Finsler geometry:

If the indicatrix bundle is given by $F(v) = 1$ then $\eta^0 = \partial_{x_i} F \mathrm{d}y^i$.

η^0 can be defined more abstractly: at $u = (x; v) \in \Sigma$

$$\eta^0_u = \mathrm{Ann}(T_v \Sigma_x), \quad \eta^0_u(v) = 1.$$

The non-deg of the indicatrices implies η^0 is a contact 1-form:

$$\eta^0 \wedge (\mathrm{d}\eta^0)^n \neq 0.$$

The geodesic flow is the *Reeb vector field* of η^0.
Moreover,

$$\mathrm{d}\eta^0 = -\zeta_i \wedge \eta^i,$$

The 1-forms $(\eta^1, \cdots, \eta^n)$ are semi-basic wrt $\pi : \Sigma \to M$.

$(\eta^0, \cdots, \zeta_n)$ give a coframe on Σ ambiguous up to certain matrix group.

The equivalence problem: Finsler geometry

After a few steps, one can obtain a canonical coframe $(\eta^i, \zeta_a, \psi^a_b)$ on a principal $\mathrm{O}(n)$-bundle satisfying structure equations

$$
\begin{aligned}
\mathrm{d}\eta^0 &= -\zeta_i \wedge \eta^i, \\
\mathrm{d}\eta^i &= \delta^{ij}\zeta_j \wedge \eta^0 - \psi^i{}_j \wedge \eta^j - I^{ij}{}_k \zeta_j \wedge \eta^k, \\
\mathrm{d}\zeta_i &= \quad \psi^j{}_i \wedge \zeta_j + R_{0i0j}\eta^0 \wedge \eta^j + \tfrac{1}{2} R_{0ijk}\eta^j \wedge \eta^k + J_i{}^j{}_k \zeta_j \wedge \eta^k \\
\mathrm{d}\psi^i{}_j &= -\psi^i{}_k \wedge \psi^k_j + R^i{}_{j0k}\eta^0 \wedge \eta^k + \frac{1}{2} R^i{}_{jkl}\eta^k \wedge \eta^l + P^i{}_j{}^k{}_l \zeta_k \wedge \eta^l
\end{aligned}
$$

The *flag curvature*, i.e. the symm tensor $\mathcal{R} = R_{0i0j}\omega^j \circ \omega^j$, is a fund inv.

The cubic form $\mathcal{I} = I^{ijk}\theta_i \circ \theta_j \circ \theta_k$ is the other fund inv and when restricted to an indicatrix it coincides with its centroaffine cubic form.

If $I = 0$, then $J = 0$ and the structure equations are the same as a Riemannian metric.

The equivalence problem: causal geometry

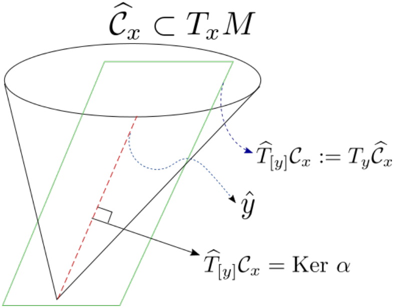
At $(x;[y]) \in \mathcal{C}$, with $\mu : \mathcal{C}^{2n} \to M^{n+1}$, $\mu^{-1}(x) = \mathcal{C}_x^{n-1}$ define

$$\mathcal{V}^{n-1} := \mu_*^{-1}(0) \subset \mathcal{J}^n := \mu_*^{-1}(\hat{y}) \subset \mathcal{P}^{2n-1} := \mu_*^{-1}(\widehat{T}_y \mathcal{C}_x) \subset T_{(x;[y])}\mathcal{C}$$

Let

$$\omega^0 = \mathrm{Ann}(\mathcal{P}) := \mathrm{Ann}(T_{[y]}\mathcal{C}_x)$$
$$\{\omega^0, \cdots, \omega^{n-1}\} = \mathrm{Ann}(\mathcal{J})$$
$$\{\omega^0, \cdots, \omega^n\} = \mathrm{Ann}(\mathcal{V}),$$
with $\{\omega^0, \cdots, \omega^n, \theta_1, \cdots, \theta_{n-1}\}$

being a coframe on $\mathcal{C}$.



ω^0 is called *projective Hilbert form.*

The equivalence problem: causal geometry

If fibers $\mathcal{C}_x$ are *tangentially non-degenerate*, then for suitable θ_a's

$$\mathrm{d}\omega^0 = -2\phi_0 \wedge \omega^0 - \theta_a \wedge \omega^a$$

Let $0 \leq i, j, k \leq n, \quad 1 \leq a, b, c \leq n-1.$

The 1-forms (ω^i, θ_a) give a *1-adapted* coframe on $\mathcal{C}$.

$$\omega^0 \wedge (\mathrm{d}\omega^0)^{n-1} \neq 0 \Rightarrow \mathrm{rank}(\omega^0) = 2n - 1.$$

Hence, ω^0 induces a *quasi-contact* structure on $\mathcal{C}^{2n}$.

$$\exists \text{ line bundle } \ell \subset T\mathcal{C} : \omega^0(\ell) = 0, \quad \ell \lrcorner \mathrm{d}\omega^0 = 0.$$

The curves tangent to ℓ are called the *characteristic curves* of $(M, \mathcal{C})$.

Locally, $\frac{\partial}{\partial \omega^n}$ is a section of ℓ.

Their projection to M give an analogue of *null geodesics*.

The equivalence problem: causal geometry

After several coframe adaptation and a prolongation, a canonical coframing on certain principal bundle is obtained.

Fundamental invariants are:

- A totally symmetric trace-free cubic form $\mathcal{F} := F^{abc}\theta_a \circ \theta_b \circ \theta_c$ which pulls back to the Fubini cubic form of the fibers.
- A symmetric trace-free bilinear form $\mathcal{W} := W_{nanb}\omega^a \circ \omega^b$. In conformal geometry, at $(x;[y]) \in \mathcal{C}$

$$W_{anbn} \propto W_{aibj}y^i y^j.$$

- The vanishing of the fundamental invariants implied that the causal structure is the flat conformal structure.

Causal vs. Finsler

First structure equation for Causal structures:

$$\begin{aligned}
\mathrm{d}\omega^0 =&\ -2\phi_0\wedge\omega^0-\theta_a\wedge\omega^a,\\
\mathrm{d}\omega^a =&\ -\gamma^a\wedge\omega^0-\phi^a{}_b\wedge\omega^b-(\phi_0+\phi_n)\wedge\omega^a-\varepsilon^{ab}\theta_b\wedge\omega^n\\
&\ -K^a{}_b\,\theta^b\wedge\omega^0-F^a{}_{bc}\,\theta^b\wedge\omega^c,\\
\mathrm{d}\omega^n =&\ -\gamma_a\wedge\omega^a-2\phi_n\wedge\omega^n-L^a\,\theta_a\wedge\omega^0\\
\mathrm{d}\theta_a =&\ -\pi_a\wedge\omega^0-\pi_n\wedge\omega_a+\phi^b{}_a\wedge\theta_b-(\phi_0-\phi_n)\wedge\theta_a\\
&\ +W_{anbn}\,\omega^b\wedge\omega^n+\tfrac{1}{2}W_{anbc}\,\omega^b\wedge\omega^c-f_a{}^b{}_c\,\theta_b\wedge\omega^c.
\end{aligned}$$

First structure equations for Finsler structures:

$$\begin{aligned}
\mathrm{d}\eta^0 &= -\zeta_i\wedge\eta^i,\\
\mathrm{d}\eta^i &= \delta^{ij}\zeta_j\wedge\eta^0-\psi^i{}_j\wedge\eta^j-I^{ij}{}_k\zeta_j\wedge\eta^k,\\
\mathrm{d}\zeta_i &= \quad\psi^j{}_i\wedge\zeta_j+R_{0i0j}\eta^0\wedge\eta^j+\tfrac{1}{2}R_{0ijk}\eta^j\wedge\eta^k+J_i{}^j{}_k\zeta_j\wedge\eta^k.
\end{aligned}$$

Causal vs. Finsler

Finsler	**Causal**
Indicatix bdle $\Sigma^{2n+1} \to M^{n+1}$ Loc. expressed as $F = 1$	(Proj.) null cone bdle $\mathcal{C}^{2n} \to M^{n+1}$ Loc. expressed as $L = 0$
Hilbert form $\eta^0 = \frac{\partial F}{\partial y^i} \mathrm{d}x^i$	Pojective Hilbert form $\omega^0 = \frac{\partial L}{\partial y^i} \mathrm{d}x^i$
η^0 : contact form on Σ^{2n+1}	ω^0 : quasi-contact form on $\mathcal{C}^{2n}$
$\mathrm{d}\eta^0 = -\zeta_1 \wedge \eta^1 - \cdots - \zeta_n \wedge \eta^n,$ $\eta^0 \wedge (\mathrm{d}\eta^0)^n \neq 0$	$\mathrm{d}\omega^0 = -\theta_1 \wedge \omega^1 - \cdots - \theta_{n-1} \wedge \omega^{n-1}$ $-2\phi_0 \wedge \omega^0, \quad \omega^0 \wedge (\mathrm{d}\omega^0)^{n-1} \neq 0$
Geodesics: integral curves of the Reeb vector field $\eta^0(\mathbf{u}) = 1, \mathrm{d}\eta^0(\mathbf{u}, .) = 0$	Null geodesics: integral curves of the characteristic line field $\omega^0(\mathbf{v}) = 0, \mathrm{d}\omega^0(\mathbf{v}, .) = 0$
$\Sigma_x \subset T_x M$ is Legendrian $\Sigma_x^n = \mathrm{Ker}\{\eta^i\}$	$\mathcal{C}_x \subset \mathbb{P}T_x M$ are quasi-Legendrian $\mathcal{C}_x^{n-1} = \mathrm{Ker}\{\omega^i\}$
$g = (\eta^0)^2 + \delta_{ij}\eta^i\eta^j$ is well-def on Σ (osc. quadric)	$[g] = [2\omega^0\omega^n + \varepsilon_{ab}\omega^a\omega^b]$ is well-def on $\mathcal{C}$ (osc. quadric)

Causal vs. Finsler

Finsler	Causal
Cartan's conn on Σ	reg. norm. Cartan conn on $\mathcal{C}$ Parabolic geometry of type $(B_{n-1}, P_{12}), (D_n, P_{12}), n \geq 4$ $(D_3, P_{123}), (B_2, P_{12})$
Essential invariants I_{ijk} : centro-affine invariant of Σ_x R_{i0j0} : Flag curvature	Essential invariants (Harmonic) F_{abc} : Fubini cubic form of $\mathcal{C}_x$ W_{anbn} : Weyl shadow flag curvature
$I_{ijk} = 0 \Rightarrow$ Riem. geom. on M	$F_{abc} = 0 \Rightarrow$ Conformal pseudo-Riem. geom. on M
$R_{i0j0} = 0 \Rightarrow \beta$-int Segre str on $\mathcal{K}$ (space of geod)	$W_{anbn} = 0 \Rightarrow \beta$-int Lie contact str on $\mathcal{K}$ (space of null geod)

Some remarks

- Locally isotrivially flat $\Longleftrightarrow W_{anbn} = 0, \quad \frac{\partial}{\partial \omega^i} F_{abc} = 0.$
- One could have started with the dual cones $\mathcal{C}^* \subset \mathbb{P}T^*M$, via a Legendre transformation in which case ω^0 is pulled-back to the natural contant 1-form on $\mathbb{P}T^*M$.
- The submaximal 4D causal structure that is not conformal is unique and isotrivially flat. Its null cones are Cayley's cubic surface and its Lie algebra of infinitesimal symmetries is 8D.

Some remarks: causal structure from a system of PDEs

Following K. Yamaguchi, one can show for $n+1 \geq 4$:

Theorem :

$$\left\{\begin{array}{l}\text{causal str.}\\ \text{on } (M^{n+1}, \mathcal{C}^{2n})\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\text{contact classes of certain}\\ \text{systems of } \frac{n(n-1)}{2} - 1\\ \text{2nd order PDEs}\\ \text{of finite type}\end{array}\right\}$$

$$\begin{array}{ccccc} J^2(\mathbb{R}^{n-1}, \mathbb{R}) \cong LG(n-1, T\mathcal{K}) & \hookleftarrow & \mathcal{C}^{2n} & & \\ \downarrow & \swarrow \rho & & \searrow \mu & \\ J^1(\mathbb{R}^{n-1}, \mathbb{R}) \cong \quad \mathcal{K}^{2n-1} & & & & M^{n+1} \end{array}$$

$\mathcal{K}^{2n-1}$: Locally defined space of *"null geodesics"* which has a contact structure.

$LG(n-1, T\mathcal{K})$: bundle of Lagrangian-Grassmannians of $T\mathcal{K}$.

M^{n+1}: Locally defined space of solutions of the system of PDEs.

Null Jacobi fields and tidal force

Characteristic curves for causal structures are defined topologically and do not arise from a variational problem.
The variational problem of char curves is given by the triple $(\mathcal{I}, \omega^n; \omega^0)$, where

$$\mathcal{I} = \{\omega^0, \cdots, \omega^{n-1}, \theta_1, \cdots, \theta_{n-1}\}$$

Given a char curve: $\gamma : I = (a, b) \to M$, we have

$$\gamma^*(\mathcal{I}) = 0, \quad \gamma^*(\omega^n) \neq 0$$

and we consider the functional $\Phi : \mathcal{U}(I) \to \mathbb{R}$

$$\Phi([\gamma]) := \int_I \gamma^* \omega^0.$$

where $\mathcal{U}(I) := \{\gamma : I \to \mathbb{R} | \gamma^* \mathcal{I} = 0\}$/parametrization.

Null Jacobi fields and tidal force

Consider a variation $\Gamma : (-\epsilon, \epsilon) \to \mathcal{U}(I)$.
Parametrize the curves of $\Gamma(s)$

$$\widehat{\Gamma} : (-\epsilon, \epsilon) \to M \quad \text{where} \quad \widehat{\Gamma}(s,t) = (\Gamma(s))(t)$$

As a result, $\Gamma^*(s_0, t)\mathcal{I} = 0$ for any fixed s_0.
Note that

$$T_\gamma \mathcal{U}(I) := \left\{ \frac{\partial \widehat{\Gamma}}{\partial s}(0,t) : \Gamma : (-\epsilon, \epsilon) \to \mathcal{U}(I) \text{ is a compact variation of } \gamma \right\}.$$

The variational equations of a characteristic curve can be thought of as the first order approximation of $T_\gamma \mathcal{U}(I)$.
If $\eta \in \mathcal{I}$ then

$$\mathcal{L}_{\frac{\partial}{\partial s}} \widehat{\Gamma}^*(s,t)\eta \equiv 0 \quad \text{modulo} \quad \{\mathrm{d}s\}.$$

Setting $s = 0$, the expression above reads as

$$\gamma^*(\hat{J} \lrcorner \, \mathrm{d}\eta + \mathrm{d}(\hat{J} \lrcorner \, \eta)) = 0,$$

where $\hat{J}(t) = \widehat{\Gamma}_*(0,t)\frac{\partial}{\partial s}$

Null Jacobi fields and tidal force

The tangential component of the vector field $\hat{J}(t)$ along $\gamma(t)$ has no effect on the variational equations.
Hence $\frac{\partial}{\partial\omega^n}$-component of $\hat{J}$ will be dropped.
$\hat{J}$ along the curve $\gamma(t)$ is

$$\hat{J}(t) = V^a(t)\frac{\partial}{\partial\omega^a} + V^{\underline{a}}(t)\frac{\partial}{\partial\theta^a}.$$

The vector field $J = \pi_*\hat{J}$ is called a *null Jacobi field* along the null geodesics $\pi(\gamma(t)) \subset M$. The variational equations for $\hat{J}(t)$ can be written as

$$\begin{aligned}
&\hat{J} \lrcorner \, \mathrm{d}\omega^0 + \mathrm{d}(\hat{J} \lrcorner \, \omega^0) = 0, \\
&\hat{J} \lrcorner \, \mathrm{d}\omega^a + \mathrm{d}(\hat{J} \lrcorner \, \omega^a) = 0, \qquad \mod \quad \mathcal{I}. \\
&\hat{J} \lrcorner \, \mathrm{d}\theta_a + \mathrm{d}(\hat{J} \lrcorner \, \theta_a) = 0,
\end{aligned}$$

Null Jacobi curves and tidal force

The first equation has no new information. The last two equations give

$$\begin{aligned} &\mathrm{d}\, V^a + V^b \phi^a{}_b - V^{\underline{a}} \omega^n \equiv 0, \\ &\mathrm{d}\, V^{\underline{a}} + V^{\underline{b}} \phi^a{}_b + W^a{}_{nbn} V^b \omega^n \equiv 0, \quad \mathrm{mod} \quad \mathcal{I}. \end{aligned}$$

Define the covariant derivative of $\hat{J}$ along characteristic curves as

$$D_{\mathbf{v}} \hat{J} \equiv \mathbf{v} \lrcorner \left(\mathrm{d}\, V^a \frac{\partial}{\partial \omega^a} + V^b \phi^a{}_b \frac{\partial}{\partial \omega^a} \right), \quad \mathrm{mod} \quad \frac{\partial}{\partial \theta^1}, \cdots, \frac{\partial}{\partial \theta^{n-1}},$$

where $\mathbf{v}$ is the characteristic vector field satisfying $\omega^n(\mathbf{v}) = 1$.

Null Jacobi fields and tidal force

it follows that

$$D_{\mathbf{v}} D_{\mathbf{v}} \hat{J} \equiv -W^a{}_{nbn} V^b \frac{\partial}{\partial \omega^a}, \quad \mathrm{mod} \quad \frac{\partial}{\partial \theta^1}, \cdots, \frac{\partial}{\partial \theta^{n-1}}$$

$$\equiv -\overset{\mathrm{sf}}{W} \hat{J}, \quad \mathrm{mod} \quad \frac{\partial}{\partial \theta^1}, \cdots, \frac{\partial}{\partial \theta^{n-1}}$$

Note that $W^a{}_{nbn} = \omega^a(W(\frac{\partial}{\partial \omega^b}, \mathbf{v})\mathbf{v}$. Hence, defining

$$J' = \pi_*(D_{\mathbf{v}} \hat{J}),$$

the null Jacobi equation can be expressed as

$$J'' + W(J, \mathbf{v})\mathbf{v} = 0,$$

where $J = \pi_* \hat{J}$ and $W(J, \mathbf{v})\mathbf{v}$ denotes $\pi_* \left(W(\hat{J}, \mathbf{v})\mathbf{v} \right)$.
Following this line of argument, and defining expansion, vorticity and shear in terms of *null Jacobi tensor*, one can obtain the Raychaudhuri equation.

Analogue of self-duality in 4D

Penrose's twistor theory: In conformal structure of signature (2,2), null cones are doubly ruled by 1-parameter family of α-planes and β-planes. β-surface: a surface whose tangent space is a β-plane everywhere.

$$\text{selfduality} \iff \beta\text{−integrability}.$$

i.e., $\exists$ a 3-parameter family of surfaces (β-surfaces) such that at each point and through each β-plane at that point, there passes a unique surface tangent to that β-plane.

self-duality ($W^{asd} = 0$) is usually defined by the hodge star operator

$$W = W^{sd} \oplus W^{asd}.$$

For a 4D causal structure of indefinite signature $[g]$ (null cones are indefinite proj surf) the essential invariants are W^+, W^- and F^+, F^-.
$F^+ = F^- = 0 \Rightarrow W^+$ generates W^{sd} and W^- generates W^{asd}.
$F^- F^+ = 0 \Rightarrow$ the null cones are ruled.
$F^-, W^- = 0 \Rightarrow \exists$ a 3-parameter family of surfaces s.t. along each ruling-plane at every point there passes a unique surface.

Double fibrations

For 4D indefinite self-dual causal structure:

$$T^3 \xleftarrow[F^-,\, W^- = 0]{} \mathcal{C}^6 \xrightarrow[F^-,\, F^+ = 0]{} M^4$$

$$\text{path geom} \longleftarrow \text{causal} \longrightarrow \text{conformal}$$

T^3 is the space of β-surfaces.

If $F^+ = 0$, then $(\mathcal{C}, M)$ gives a self-dual conformal strucure and T is equipped with a *torsion-free* path geometry

The torsion of the path geometry is given by F^+ and its derivatives.

If $F^-, W^-, W^+ = 0$ then T has a projective str.

4D causal

Theorem

$$\text{indefinite selfdual causal on } M^4 \Longleftrightarrow \text{path geom. on } T^3$$

Remarks : A path geometry in 3D can be expressed locally in terms of a pair of ODEs under point equivalence relation

$$z_1'' = F_1(t, z_1, z_2, z_1', z_2'), \quad z_2'' = F_2(t, z_1, z_2, z_1', z_2')$$

Theorem : The submaximal indefinite 4D causal structure that does not descend to a pseudo-conformal structure has 8-dimensional Lie algebra of infinitesimal symmetries and is locally isotrivially flat whose null cone is Cayley's cubic scroll.

It corresponds to the following pair of ODEs:

$$z_1'' = z_2, \quad z_2'' = 0.$$

In terms of affine coordinates for $\mathbb{P}TM$:

$$y_3 = y_1 y_2 + \frac{1}{3}(y_1)^3.$$

Future directions

- Find a generalization of the almost Einstein condition arising from BGG operators in parabolic geometry.
- Find an analogue of principal null directions (planes) for 4D causal structures.
- Is there an analogue of Goldberg-Sachs theorem in the causal setting?
- When is a 4D causal structure locally equivalent to the twisted product of two Finsler metrics?

Thank you for your attention!