Totally geodesic hypersurfaces in Robertson-Walker spaces with flat fibers

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Introduction

Totally geodesic submanifolds keep recieving a lot of attention, because the knowledge of totally geodesic submanifolds means better understaning of the geometry of the manifold.

In many cases, totally geodesic submanifolds are just copies of the ambient manifold with lower dimension. This is the case for very well geometries, like real, complex and quaternionic space forms.

In a Riemannian symmetric space, a complete, connected submanifold is totally geodesic if and only if it is also a symmetric subspace.

A few related results:

- totally geodesic submanifolds of Riemannian symmetric spaces (S. Klein, Q. Jin),
- Finslerian totally geodesic submanifolds in the complex and quaternionic projective space (G. Berck),
- totally geodesic co-dimension 1 foliations of some Lorentzian manifolds (R. Roșca, K. Yokumoto),
- spacelike hypersurfaces in Lorentzian products (G. Li, I. Salavessa).

Robertson-Walker spacetimes play an important rule in Relativity, and their geometry (surfaces, submanifolds,...) is studied by mathematicians (A. Albujer, L.J. Alías, F. Camargo, B.-Y. Chena, H. de Lima, S. Montiel, A. Romero, M. Sánchez, J. Van der Veken,...).

These results and ideas led us to study totally geodesic submanifolds of Robertson-Walker spacetimes.
For the simplicity, we focus on spacetimes $M^{n+1} = -I \times_f \mathbb{R}^n$.

It is well known that the fibers of a Robertson-Walker spacetimes are not totally geodesic unless they coincide with critical points of the warping function.

Hypersurface in M^{n+1}

Let us consider the warped product $M = -I \times_f \mathbb{R}^n$, with coordinates $(t, \tilde{x}, x_n) = (t, x^1, \dots, x^{n-1}, x^n)$ and a hypersurface $N \subset M$ defined locally as the graph of the smooth mapping $\underline{\psi\colon L^n \subset -I \times_f \mathbb{R}^{n-1} \to M}$ given by

$$\psi(t, \tilde{x}) = \psi(t, x^1, \dots, x^{n-1}) = (t, x^1, \dots, x^{n-1}, z(t, \tilde{x})),$$

or the mapping $\underline{\varphi\colon U \subset \mathbb{R}^n \to M}$ given by

$$\varphi(x^1, \dots, x^n) = (t(x^1, \dots, x^n), x^1, \dots, x^n),$$

respectively. We will use both these mappings conveniently.

Lemma

Let N be a submanifold of a smooth manifold M with a torsion free affine connection. N is totally geodesic submanifold of M if and only if for any vector fields X and Y tangent to N, the covariant derivative $\nabla_X Y$ is also tangent to N at every point of N.

Using the mapping φ, we obtain the tangent vector fields

$$\varphi_*(\partial_i) \;=\; \partial_i + \frac{\partial t}{\partial x^i}\partial_t = \partial_i + t_i\partial_t, \qquad i = 1, \dots, n.$$

The nonzero Christoffel symbols for the pseudo-Riemannian connection in M are

$$\Gamma^0_{ii} = ff', \quad \Gamma^i_{0i} = f'/f, \qquad i = 1, \dots, n.$$

We obtain for the covariant derivatives the formulas

$$\begin{aligned}
\nabla_{\varphi_*(\partial_i)}\varphi_*(\partial_i) &= \nabla_{\partial_i + t_i\partial_t}(\partial_i + t_i\partial_t) = \\
&= (ff' + t_{ii})\partial_t + 2t_i\frac{f'}{f}\partial_i, \\
\nabla_{\varphi_*(\partial_i)}\varphi_*(\partial_j) &= \nabla_{\partial_i + t_i\partial_t}(\partial_j + t_j\partial_t) = \\
&= t_{ij}\partial_t + t_j\frac{f'}{f}\partial_i + t_i\frac{f'}{f}\partial_j.
\end{aligned}$$

The covariant derivatives of these tangent vector fields must be tangent to N. We obtain the system of PDEs

$$\begin{aligned}
t_{ii} &= 2{t_i}^2\frac{f'}{f} - ff', \\
t_{ij} &= 2t_it_j\frac{f'}{f}, \qquad i,j = 1,\dots,n,\ i \neq j.
\end{aligned} \tag{1}$$

Because f is the function of t, this system of second order PDEs cannot be transformed into a system of first order.

Using the mapping ψ, we have the tangent vector fields

$$
\begin{aligned}
\psi_*(\partial_t) &= \partial_t + \frac{\partial z}{\partial t}\partial_{x^n} = \partial_t + z_t\partial_n, \qquad z_t = \frac{\partial z}{\partial t},\\
\psi_*(\partial_i) &= \partial_{x^i} + \frac{\partial z}{\partial x^i}\partial_{x^n} = \partial_i + z_i\partial_n, \quad z_i = \frac{\partial z}{\partial x^i}, \quad i = 1, \ldots, n-1.
\end{aligned}
$$

We introduce the new functions

$$
g(t,\tilde{x}) = z_t(t,\tilde{x}), \quad h^i(t,\tilde{x}) = z_i(t,\tilde{x}), \quad i = 1, \ldots, n-1. \quad (2)
$$

Now, we obtain

$$
\begin{aligned}
\nabla_{\psi_*(\partial_t)}\psi_*(\partial_t) &= g^2 ff'\partial_t + (g_t + 2g\frac{f'}{f})\partial_n,\\
\nabla_{\psi_*(\partial_t)}\psi_*(\partial_i) &= gh^i ff'\partial_t + \frac{f'}{f}\partial_i + ((h^i)_t + h^i\frac{f'}{f})\partial_n,\\
\nabla_{\psi_*(\partial_i)}\psi_*(\partial_i) &= (1 + (h^i)^2)ff'\partial_t + (h^i)_i\partial_n,\\
\nabla_{\psi_*(\partial_i)}\psi_*(\partial_j) &= h^i h^j ff'\partial_t + (h^j)_i\partial_n.
\end{aligned}
$$

We obtain the following system of partial differential equations:

$$\begin{array}{rcl} g_t & = & g^3 ff' - 2gf'/f, \\ g_i = (h^i)_t & = & g^2 h^i ff', \\ (h^i)_i & = & g(1 + (h^i)^2) ff', \\ (h^i)_j = (h^j)_i & = & g h^i h^j ff', \qquad i, j = 1, \ldots, n-1,\ i \neq j. \end{array} \quad (3)$$

This is originally the system of second order partial differential equations for the function $z(t, \tilde{x})$.

The crucial simplification is done by the substitution $g = z_t, h^i = z_i$, which transforms this system into the system of first order partial differential equations for the functions $g(t, \tilde{x})$ and $h^i(t, \tilde{x})$.

Solving the system of PDEs

Case 1, $f' = 0$:
We have $f = \mathrm{const}$, $g_t = g_i = (h^i)_t = (h^i)_j = 0$ for all i, j, hence

$$f(t) = f_0, \quad z(t, \tilde{x}) = at + \sum_{i=1}^{n-1} b^i x^i + c, \quad f_0, a, b^i, c \in \mathbb{R}.$$

In this case, the entire solution holds for $L^n = -I \times \mathbb{R}^{n-1}$.

Case 2, $g = 0$:
We easily obtain $(h^i)_j = 0$ for all i, j, which implies for arbitrary f,

$$z(t, \tilde{x}) = \sum_{i=1}^{n-1} b^i x^i + c, \quad c, b^i \in \mathbb{R}.$$

In this case, the entire solution holds for $L^n = -I \times \mathbb{R}^{n-1}$.

Case 3: $f' \neq 0 \neq g$:

The first equation in (3) can be integrated:

$$\begin{aligned} g_t &= g^3 ff' - 2gf'/f, \\ g(t,\tilde{x}) &= \frac{\varepsilon}{f(t)\sqrt{1+f^2(t)F(\tilde{x})}}, \qquad \varepsilon = \pm 1. \end{aligned} \tag{4}$$

By taking differentiation on this formula, we obtain

$$g_i = \frac{\partial g}{\partial x^i} = -\frac{\varepsilon f(t) F_i(\tilde{x})}{2[1+f(t)^2 F(\tilde{x})]^{3/2}}, \qquad i = 1, \dots, n-1.$$

If we use expressions for g and g_i in (3_2), we obtain

$$h^i = \frac{-\varepsilon f(t)^2 F_i(\tilde{x})}{2f'(t)\sqrt{1+f(t)^2 F(\tilde{x})}}, \qquad i = 1, \dots, n-1.$$

By considering integrations with respect to x^i, we obtain

$$z(t,\tilde{x}) = -\frac{\varepsilon\sqrt{1+f^2(t)F(\tilde{x})}}{f'(t)} + z_0(t). \tag{5}$$

Lemma

For the solution (5) *of the system* (3), *it holds*
1) $z_0(t) = z_0$ *is a constant,*
2) $f(t) = c_2 \exp^{c_1 t}, \qquad c_1, c_2 \in \mathbb{R}, \quad c_1 \neq 0,\ c_2 > 0.$

Proof. By using $z(t, \tilde{x})$, we compare $\frac{\partial z(t,\tilde{x})}{\partial t}$ and $g(t, \tilde{x})$.
We obtain

$$-\varepsilon \frac{f(f')^2 F - f''(1 + f^2 F)}{(f')^2 \sqrt{1 + f^2 F}} + z_0(t)' = \frac{\varepsilon}{f\sqrt{1 + f^2 F}}.$$

After the simplification, we obtain

$$\varepsilon z_0(t)' \quad = \quad \frac{\left((f')^2 - ff''\right)\sqrt{1 + f^2 F}}{f(f')^2}.$$

If we assume that $z_0(t)$ is nonconstant on an open interval, we obtain $(f')^2 - ff'' \neq 0$, and also

$$F(\tilde{x}) = \frac{1}{f^2(t)}\left(\left[\frac{\varepsilon f(t)(f'(t))^2 z_0'(t)}{(f'(t))^2 - f(t)f''(t)}\right]^2 - 1\right),$$

where the right-hand side is the function of t only.
It leads to $F_i = 0$ for all $i = 1, \ldots, n-1$.

By the formula for h^i, we get $h^i = 0$ for all i.
Next, by the original system of equations (3),
we obtain $0 = gff'$, which is a contradiction
to our assumptions $f' \neq 0$ and $g \neq 0$.

As a result, we have to assume that $z_0(t) = z_0$ is a constant and $(f')^2 - ff'' = 0$, which implies

$$f(t) = c_2 \exp^{c_1 t}, \quad c_1, c_2 \in \mathbb{R}, \quad c_1 \neq 0,\ c_2 > 0.$$

□

Lemma

Let the formula (5) *be the solution of the system* (3).
The function $F(\tilde{x})$ *must be in the form*

$$F(\tilde{x}) = -{c_1}^2 \sum_{i=1}^{n-1} (x^i)^2 + \sum_{i=1}^{n-1} d_i x^i + d_0, \qquad d_0, d_i \in \mathbb{R}.$$

Proof. From the formula for $z(t, \tilde{x})$, we obtain

$$\begin{aligned} h^i = z_i &= -\frac{\varepsilon f^2 F_i}{2f'\sqrt{1+f^2F}}, \\ (h^i)_i = z_{ii} &= -\frac{\varepsilon f^2}{4f'} \cdot \frac{2F_{ii}(1+f^2F) - f^2(F_i)^2}{(1+f^2F)^{\frac{3}{2}}}, \\ (h^i)_j = z_{ij} &= -\frac{\varepsilon f^2}{4f'} \cdot \frac{2F_{ij}(1+f^2F) - f^2 F_i F_j}{(1+f^2F)^{\frac{3}{2}}}. \end{aligned}$$

We substitute these expressions and the formula for $g(t, \tilde{x})$ into the original equations (3_3) and (3_4).
After the simplification, we obtain the conditions

$$F_{ij} = 0, \qquad f^2 F_{ii} + 2(f')^2 = 0, \qquad i, j = 1, \ldots, n.$$

Here we use the formula for f. We obtain

$$F_{ii} = -2(c_1)^2, \qquad F_{ij} = 0, \qquad i, j = 1, \ldots, n$$

and the statement follows. □

Geometrical realization of the hypersurface N

We will look at the hypersurface N via the map φ. We have

$$z(t,\tilde{x}) = -\frac{\varepsilon\sqrt{1+f^2(t)F(\tilde{x})}}{f'(t)} + z_0(t).$$

We substitute conveniently $z = x_n$, $t = t(x_1,\ldots,x_n)$
and formulas for f and F. We obtain

$$t(x^1,\ldots,x^n) \;\; = \;\; -\frac{1}{2c_1}\log({c_1}^2\,{c_2}^2\,(x^n - x_0^n)^2 - {c_2}^2 F(\tilde{x})).$$

Obviously, we can introduce new constants $k, x_0^1, x_0^2, \ldots x_0^{n-1}$
instead of integration constants $d_0,\ldots,d_{n-1}$
and rewrite the last formula in the form

$$t(x^1,\ldots,x^n) \;\; = \;\; -\frac{1}{2c_1}\log({c_1}^2\,{c_2}^2\,\sum_{i=1}^{n}(x^i - x_0^i)^2 + k). \quad (6)$$

Theorem

Let $M = -I \times_f \mathbb{R}^n$ be a Robertson-Walker spacetime with nonconstant function $f(t)$.
Nontrivial totally geodesic hypersurfaces in M exist only if $\underline{f(t) = c_2 \exp^{c_1 t}, c_1, c_2 \in \mathbb{R}, c_1 \neq 0,\ c_2 > 0}$.
Each nontrivial totally geodesic hypersurface N in M is defined globally as the graph of the function $\varphi\colon U \subset \mathbb{R}^n \to M$, given by the formula (6).

For $\underline{k > 0}$, the definition domain is $\mathbb{R}^n$ and the hypersurface N is spacelike.
For $\underline{k < 0}$, the definition domain is $\mathbb{R}^n \setminus C$, where C is the circle around $(x_0^1, \dots, x_0^n)$ with the diameter k. The hypersurface N is timelike in this case.
For $k = 0$, the definition domain is $\mathbb{R}^n \setminus (x_0^1, \dots, x_0^n)$ and the pullback $\varphi^*(g)$ of the metric g to N is degenerate.

Proof.

It remains to prove the statements about <u>the causal character</u> of N. We use the coordinate tangent vector fields

$$(\partial_i)_* = \partial_i + t_i \partial_t = \partial_i - \frac{c_1 {c_2}^2 (x^i - x_0^i)}{{c_1}^2 {c_2}^2 \sum_{i=1}^n (x^i - x_0^i)^2 + k} \partial_t.$$

Now we consider tangent vector fields W_i defined as

$$W_i = (x^{i+1} - x_0^{i+1})(\partial_i)_* - (x^i - x_0^i)(\partial_{i+1})_*, \qquad i = 1, \dots, n-1.$$

All these vector fields are spacelike and mutually orthogonal.
The vector field W_n will be defined as

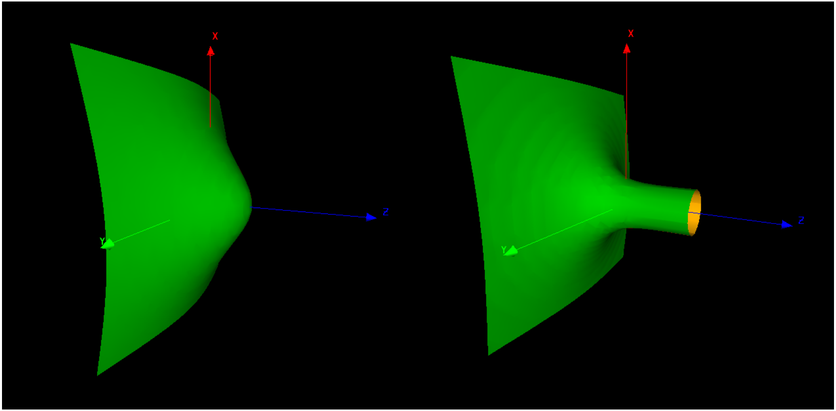
$$\begin{aligned} W_n &= \sum_{i=1}^n (x^i - x_0^i)(\partial_i)_* = \\ &= \sum_{i=1}^n (x^i - x_0^i)\partial_i - \frac{c_1 {c_2}^2 \sum_{i=1}^n (x^i - x_0^i)^2}{{c_1}^2 {c_2}^2 \sum_{i=1}^n (x^i - x_0^i)^2 + k} \partial_t. \end{aligned}$$

The vector field W_n is orthogonal to all vector fields W_i.
The norm of W_n is

$$\begin{aligned}\|W_n\| &= f(t)^2\sum_{i=1}^{n}(x^i - x_0^i)^2 - \Big[\frac{c_1{c_2}^2\sum_{i=1}^{n}(x^i - x_0^i)^2}{{c_1}^2{c_2}^2\sum_{i=1}^{n}(x^i - x_0^i)^2 + k}\Big]^2 = \\ &= \frac{{c_2}^2\, k\, \sum_{i=1}^{n}(x^i - x_0^i)^2}{[{c_1}^2{c_2}^2\sum_{i=1}^{n}(x^i - x_0^i)^2 + k]^2}.\end{aligned}$$

It follows easily that $\|W_n\| > 0$ (or $= 0$, or < 0, respectively) for $k > 0$ (or $k = 0$, or $k < 0$, respectively).

We see that $\{W_i\}_{i=1}^n$ is the pseudo-orthogonal basis of the tangent space $T_{\varphi(p)}N$ with the appropriate signature at any point $\varphi(p)$ of N except the point $\varphi[(x_0^1, \ldots, x_0^n)]$ for $k > 0$. In this case and at this point, $\{(\partial_i)_*\}_{i=1}^n$ is the orthogonal basis. $\square$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