VI International Meeting on Lorentzian Geometry
Granada (Spain), September 6-9, 2011
GELOGRA 2011
Plenary Talk: Calabi-Bernstein results and parabolicity of maximal surfaces in Lorentzian product spaces.

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A maximal surface in a 3-dimensional Lorentzian manifold is a spacelike surface with zero mean curvature. Here by *spacelike* we mean that the induced metric from the ambient Lorentzian metric is a Riemannian metric on the surface. The terminology *maximal* comes from the fact that these surfaces locally maximize area among all nearby surfaces having the same boundary. Besides their mathematical interest, maximal surfaces and, more generally, spacelike surfaces with constant mean curvature are also important in General Relativity.

One of the most important global results about maximal surfaces is the Calabi-Bernstein theorem for maximal surfaces in the 3-dimensional Lorentz-Minkowski space $\mathbb{R}^3_1$, which, in parametric version, states that the only complete maximal surfaces in $\mathbb{R}^3_1$ are the spacelike planes. This result can be seen also in a non-parametric form, establishing that the only entire maximal graphs in $\mathbb{R}^3_1$ are the spacelike planes; that is, the only entire solutions to the maximal surface equation

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=0, \quad |Du|^2<1$$

on the Euclidean plane $\mathbb{R}^2$ are affine functions.

In this lecture, we present new Calabi-Bernstein results for maximal surfaces immersed into a Lorentzian product space of the form $M^2\times\mathbb{R}_1$, where M^2 is a connected Riemannian surface and $M^2\times\mathbb{R}_1$ is endowed with the Lorentzian metric $\langle,\rangle=\langle,\rangle_M-dt2$. In particular, when M is a (necessarily complete) Riemannian surface with non-negative Gaussian curvature K_M, we prove that any complete maximal surface in $M^2\times\mathbb{R}_1$ must be totally geodesic. Besides, if M is non-flat we conclude that it must be a slice $M\times\{t_0\}, t_0\in\mathbb{R}$ (here by *complete* it is meant, as usual, that the induced Riemannian metric on the maximal surface from the ambient Lorentzian metric is complete). We prove that the same happens if the maximal surface is complete with respect to the metric induced from the

Riemannian product $M^2 \times \mathbb{R}$. This allows us to give also a non-parametric version of the Calabi-Bernstein theorem for entire maximal graphs in $M^2 \times \mathbb{R}_1$, under the same assumptions on K_M. Moreover, we also construct counterexamples which show that our Calabi-Bernstein results are no longer true without the hypothesis $K_M \geq 0$.

On the other hand, we introduce a local approach to our Calabi-Bernstein results, which is based on some parabolicity criteria for maximal surfaces with non-empty smooth boundary in $M^2 \times \mathbb{R}_1$. In particular, we derive that every maximal graph over a starlike domain $\Omega \subseteq M$ is parabolic. This allows us to give an alternative proof of the non-parametric version of the Calabi-Bernstein result for entire maximal graphs in $M^2 \times \mathbb{R}_1$. Finally, we also introduce a second local approach to our results, which is given by means of a local integral inequality for the squared norm of the second fundamental form of the surface.

The results in this lecture are part of our recent research work developed jointly with Alma L. Albujer, and it can be found in the following references:

- A. L. Albujer, *New examples of entire maximal graphs in* $\mathbb{H}^2 \times \mathbb{R}_1$, Differential Geometry and its Applications 26 (2008), 456–462.
- A. L. Albujer and L. J. Alías, *A local estimate for maximal surfaces in Lorentzian product spaces*, Matemática Contemporanea 34 (2008), 1–10.
- A. L. Albujer and L. J. Alías, *Calabi-Bernstein results for maximal surfaces in Lorentzian product spaces*, Journal of Geometry and Physics 59 (2009), 620–631.
- A. L. Albujer and L. J. Alías, *Parabolicity of maximal surfaces in Lorentzian product spaces*, Mathematische Zeitschrift 267 (2011), 453–464.