Uniqueness and non-existence results for spacelike hypersurfaces with constant mean curvature in $\mathbb{H}^n \times \mathbb{R}_1$

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(joint work with Fernanda E. C. Camargo and Henrique F. de Lima)

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- Recently, Alarcón and Souam have obtained examples of non-trivial entire spacelike graphs with constant mean curvature in $\mathbb{H}^2 \times \mathbb{R}_1$.
- Therefore, in order to obtain existence results for spacelike constant mean curvature surfaces of $\mathbb{H}^2 \times \mathbb{R}_1$, or in general hypersurfaces of $\mathbb{H}^n \times \mathbb{R}_1$, we will need to ask some extra assumptions.

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Omori-Yau maximum principle (Omori, 1967 and Yau, 1975)

Let M^n be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below and consider $f : M^n \to \mathbb{R}$ a smooth function which is bounded from above on M^n. Then there is a sequence of points $\{p_k\}_{k\in\mathbb{N}} \subset M^n$ such that

$$\lim_{k\to\infty} f(p_k) = \sup f \,, \quad \lim_{k\to\infty} |\nabla f(p_k)| = 0 \quad \text{and} \quad \lim_{k\to\infty} \Delta f(p_k) \leq 0.$$

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- Let h denote the **height function** $h = (\pi_{\mathbb{R}})|_{\Sigma}$ of Σ^n. It is not difficult to see that

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- Given $t_0 \in \mathbb{R}$, the spacelike hypersurface $\mathbb{H}^n \times \{t_0\}$ is called a **slice**. Slices are characterized as the spacelike hypersurfaces with $\theta \equiv 0$. Or equivalently, as the spacelike hypersurfaces with constant height function.

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- The r-**mean curvature function** H_r of the spacelike hypersurface Σ^n is defined by

$$\binom{n}{r} H_r(p) = (-1)^r \sum_{i_1 < \ldots < i_r} \lambda_{i_1}(p) \cdots \lambda_{i_r}(p) \ , \quad 1 \leq r \leq n$$

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⋆ In the particular case when $r = 1$, $H_1 = H = -\frac{1}{n}\mathrm{tr}(A)$ is the mean curvature of Σ^n.

⋆ In the case $r = 2$, H_2 defines a geometric quantity related to the scalar curvature of the hypersurface.

Main results

Theorem

Let Σ^n be a complete spacelike hypersurface with constant mean curvature H of $\mathbb{H}^n \times \mathbb{R}_1$. If the height function h of Σ^n satisfies, for some constant $0 < \alpha < 1$,

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- Or equivalently,

Corollary

There do not exist any complete spacelike hypersurface with constant mean curvature $H \neq 0$ in $\mathbb{H}^n \times \mathbb{R}_1$ such that for some constant $0 < \alpha < 1$

$$\|\nabla h\|^2 \leq \frac{n\alpha}{n-1} H^2.$$

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$$\mathrm{Ric}(X,X) = -(n-1)\|X\|^2 - (n-2)\langle X, \nabla h\rangle^2 - \|\nabla h\|^2\|X\|^2 + \left\|AX + \frac{nH}{2}X\right\|^2 - \frac{n^2H^2}{4}\|X\|^2$$

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⇝ The Ricci curvature tensor is bounded from below since $\|\nabla h\|^2$ is bounded from above.

- There exist $\{p_k\}_{k\in\mathbb{N}}$ on Σ^n such that

$$\lim_{k\to\infty} \cosh\theta(p_k) = \sup_{p\in\Sigma^n} \cosh\theta(p) \quad \text{and} \quad \lim_{k\to\infty} \Delta\cosh\theta(p_k) \leq 0.$$

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$\rightsquigarrow H = 0 \rightsquigarrow \|\nabla h\|^2 = 0 \rightsquigarrow \Sigma^n$ is a slice.

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Theorem

Let Σ^n be a complete spacelike hypersurface with constant mean curvature H and H_2 bounded from below of $\mathbb{H}^n \times \mathbb{R}_1$. If the height function h of Σ^n is such that

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- ⋆ The proof of this theorem is analogous to the previous one. However, now $\|A\|^2$ is not necessarily constant, but since it holds $\|A\|^2 = n^2H^2 - n(n-1)H_2$ we can guarantee that $\sup_\Sigma \|A\|^2 < +\infty$. This is enough for our purpose.

Entire spacelike graphs in $\mathbb{H}^n \times \mathbb{R}_1$

- An **entire graph** in $\mathbb{H}^n \times \mathbb{R}_1$ is determined by a smooth function $u \in \mathcal{C}^\infty(\mathbb{H}^n)$ and it is given by

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- For any entire spacelike graph we have the following relation:

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- It is well known that an entire spacelike graph is not necessarily complete. However, it is easy to see that in the case when $|Du|^2 < c < 1$, for some positive constant c, the graph is complete.

A non-parametric version of our results

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Corollary

There do not exist any entire spacelike graph with constant mean curvature $H \neq 0$ in $\mathbb{H}^n \times \mathbb{R}_1$ such that for some constant $0 < \alpha < 1$

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Let $\Sigma^n(u)$ be an entire spacelike graph in $\mathbb{H}^n \times \mathbb{R}_1$ with constant mean curvature H and H_2 bounded from below. If the function u is such that

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Thank you very much for your kind attention!