On the space of geodesics of Riemannian and Lorentzian space forms

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The space of geodesics of a given type (timelike or spacelike) $L^{\pm}(\mathbb{S}_p^{n+1})$ of a non-flat pseudo-Riemannian space form $\mathbb{S}_p^{n+1}$ of signature $(p, n+1-p)$ enjoys a natural invariant structure: in the case of periodic geodesics, it is (pseudo)-Kähler, while in the case of unbounded geodesics, it enjoys a para-Kähler structure. We shall prove that the corresponding pseudo-Riemannian metric is Einstein.

We also study the normal congruence of a hypersurface S of $\mathbb{S}_p^{n+1}$, which happens to be a Lagrangian submanifold $\bar{S}$ of $L^{\pm}(\mathbb{S}_p^{n+1})$, and relate the geometries of S and $\bar{S}$. In particular $\bar{S}$ is totally geodesic if and only if S has parallel second fundamental form. Moreover we may express the minimality of $\bar{S}$ by a functional relation between the principal curvatures of S.

The case of geodesics in three-dimensional space forms is special: the four-dimensional spaces $L^{\pm}(\mathbb{S}_p^3)$ enjoy an additional pseudo- or para-Kähler structure, whose corresponding metric is not anymore Einstein, but is scalar-flat. Again the metric properties of a normal congruence $\bar{S}$ are related to that of the underlying surface; in particular $\bar{S}$ is flat if and only if S is Weingarten.