Space of geodesics of pseudo-Riemannian space forms and normal congruences of hypersurfaces

Henri Anciaux

Instituto de Matemática e Estatística, Universidade de São Paulo

henri.anciaux@gmail.com

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$\Rightarrow$ To be posted on arXiv very soon!

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$L^\pm(\mathbb{S}_p^{n+1})$ are $2n$-dimensional symplectic manifolds.

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- *The metric $\mathbb{G}$ is Einstein, with scalar curvature $S = \epsilon n$;*

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- *The metric* $\mathbb{G}$ *is Einstein, with scalar curvature* $S = \epsilon n$;
- $\omega = \epsilon \mathbb{G}(\mathbb{J}., .)$ *is the canonical symplectic form of* $L^\pm(\mathbb{S}^{n+1}_p)$.

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- *The metrics* $\mathbb{G}$ *and* $\mathbb{G}'$ *have the same Levi-Civita connection.*

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$\Rightarrow$ Not difficult to calculate its curvature using Gauss equation.

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$\Rightarrow$ We can check that $\mathbb{J}'$ is parallel w.r.t. $\mathbb{G}$, but so far we haven't proved that it is integrable.

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$\Rightarrow$ Therefore $\mathbb{J}'$ is parallel w.r.t. $\mathbb{G}'$.

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- The (para)-complex structure induces a (anti)-isometry of $T\mathcal{L}$ onto $N\mathcal{L}$;
- The extrinsic curvature of $\mathcal{L}$ is described by the tri-symmetric tensor

$$h(X, Y, Z) := \mathbb{G}(\nabla_X Y, \mathbb{J}Z) = \epsilon\omega(\nabla_X Y, Z).$$

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Normal congruences as Lagrangian submanifolds

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Let $\mathcal{S}$ *a non-degenerate hypersurface of* $\mathbb{S}^{n+1}_p$. *Then its* normal congruence $\bar{\mathcal{S}}$, *i.e. the set of geodesics which are normal to* $\mathcal{S}$,

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Conversely, a simply connected, n-dimensional submanifold of $L^{\pm}(\mathbb{S}^{n+1}_p)$ is the normal congruence of some hypersurface of $\mathbb{S}^{n+1}_p$ if and only if it is Lagrangian.

Geometry of $\bar{\mathcal{S}}$ with respect to the Einstein metric $\mathbb{G}$

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where A is the shape operator of $\mathcal{S}$.

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Finally, if A of $\mathcal{S}$ is complex-diagonalizable with principal curvatures $\kappa_1, ..., \kappa_n$, the mean curvature vector of $\bar{\mathcal{S}}$ is

$$\vec{H} = \frac{1}{n}\mathbb{J}\nabla\left(\sum_{i=1}^{n} \mathrm{arctan}\epsilon(\kappa_i)\right),$$

where $\mathrm{arctan}\epsilon = \int \frac{dz}{z^2-\epsilon}$.

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If $n=2$, $\bar{\mathcal{S}}$ is $\mathbb{G}$-minimal if and only if $\mathcal{S}$ is linear Weingarten, i.e. it satisfies the equation

$$aH + b(K - \epsilon) = 0.$$

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$$\bar{g}' = \varpi(A., .) + \varpi(., A.),$$

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where A and ϖ are the shape operator and the area element of $\mathcal{S}$. In particular, $\bar{g}'$ is degenerate at the umbilic points of $\mathcal{S}$ and Lorentzian elsewhere, with null lines the principal directions of $\mathcal{S}$. Finally, if (e_1, e_2) is an orthonormal, principal frame along $\mathcal{S}$, (i.e. $Ae_1 = \kappa_1 e_1$ and $Ae_2 = \kappa_2 e_2$), the mean curvature vector of $\bar{\mathcal{S}}$ is

$$\vec{H}' = \mathbb{J}' \frac{e_1(\kappa_2)e_1 + e_2(\kappa_1)e_2}{2(\kappa_1 - \kappa_2)^2}.$$

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Corollary

Suppose A is real-diagonalizable. Then $\bar{\mathcal{S}}$ is $\mathbb{G}'$-flat (or $\mathbb{G}$-flat) if and only if the surface $\mathcal{S}$ is Weingarten, i.e. there exists a functional relation $f(\kappa_1, \kappa_2) = 0$ satisfied by the two principal curvatures of $\mathcal{S}$.

Marginally trapped surfaces

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If $\mathcal{S}$ is a flat surface, a tube (i.e. a surface with one constant principal curvature) or a surface of revolution, then $\bar{\mathcal{S}}$ is $\mathbb{G}'$-marginally trapped (i.e. $\mathbb{G}'(\vec{H}', \vec{H}') = 0$).

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- *No invariant metric on* $L^{\pm}(\mathbb{R}^{n+1}, \langle ., . \rangle_p)$, $n \geq 3$;

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- *No invariant metric on* $L^{\pm}(\mathbb{R}^{n+1}, \langle ., . \rangle_p)$, *n* ≥ 3*;*
- *There does exist an invariant structure* $(\mathbb{G}', \mathbb{J}')$ *on* $L^{\pm}(\mathbb{R}^3, \langle ., . \rangle_p)$, *with exactly the same properties than that of* $L^{\pm}(\mathbb{S}^3_p)$.

Concluding remarks

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$\Rightarrow$ What about the spaces $L(\mathbb{CP}^n), L(\mathbb{CH}^n)$...?

¡Gracias por su atención!