LORENTZIAN BOBILLIER FORMULA

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I.Introduction

A brief introduction to the history of Bobillier formula

- Bobillier's well known theorem on the centers of curvature was the first theorem concerning second order properties of general motion.

- In 1988, M. Fayet presented a new formula relative to the curvatures in an one parameter planar Euclidean motion. This formula analytically solves the problem that Bobillier's construction solved graphically as given by R. Garnier in 1956 and E. A. Dijskman in 1976.

- M. Fayet is proved that the Bobillier formula may also be obtained without using the Euler Savary formula which is derived by Euler in 1765 and Savary 1845 and this relation (Euler Savary formula) is well documented by N. Rosenauer and, A. H. Willis in 1953 and H. R. Müller in 1963.

- The Bobillier formula is proved and also illustrated by elementary tasks in 2004 by A. Muminagic.

- By taking Lorentzian plane instead of Euclidean plane, A.A. Ergin introduced one parameter planar motion and gave the relations between the velocities, accelerations and pole curves of this Lorentzian motion.

- In the Lorentzian plane Euler Savary formula is studied by M. Ergüt in 1988, I. Aytun in 2002 and T. Ikawa in 2003.

What we are trying to do in our study

- **Bobillier formula for the Lorentzian planar motion** is not studied yet. Thus, the study is proposed to serve such a need.

II. Lorentzian Planar Motions

and

Lorentzian Euler Savary Formula

Lorentzian Planar Motions

- Let P_0 and P_1 be fixed and moving planes in Lorentzian space, respectively.
- The perpendicular coordinate system of the planes P_0 and P_1 are $\{O_0; \vec{p}_{01}, \vec{p}_{02}\}$ and $\{O_1; \vec{p}_{11}, \vec{p}_{12}\}$, respectively.
- Suppose that M and M' are nonnull (timelike or spacelike) points linked to moving plane P_1.
- Then the conjugate points γ and γ' of these nonnull points are the curvature centers of the trajectory drawn by M and M' in the fixed plane P_0.
- The normals of this trajectory pass from an instantaneous center of rotation denoted by I and called as pole point.

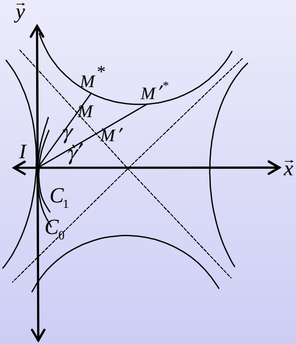

Figure 2.1 Timelike $\overrightarrow{IM}, \overrightarrow{IM'}$ vectors

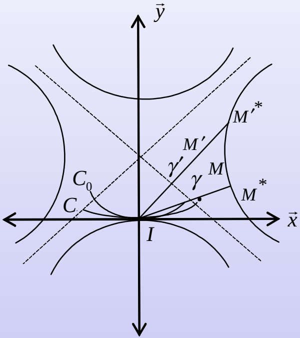

Figure 2.2 Spacelike $\overrightarrow{IM}, \overrightarrow{IM'}$ vectors

Lorentzian Planar Motions (Expression of Pole Curves)

- At each t moment there is a rotation pole and the geometric locus of the pole points is called fixed pole curve C_0 in the plane P_0 and moving pole curve C_1 in the plane P_1 during the one-parameter Lorentzian motion $P_1 \backslash P_0$.
- If θ is the rotation angle of the motion $P_1 \backslash P_0$. then each nonnull point of P_1 makes a rotation motion with $\dot{\theta}$ angular velocity at the center I.
- C_0 and C_1 roll upon each other without sliding during the one parameter Lorentzian planar motion, namely, C_0 and C_1 pole curves are always tangent to each other and have the same velocity at each t moment.
- The causal character of C_0 and C_1 are determined with respect to the causal character of the tangent of this curve in Lorentzian plane.

Lorentzian Planar Motions (Geometric Point of View)

- From the Figure 2.1; γ and γ' are timelike curvature centers of a trajectory drawn by the timelike points M and M' linked to the P_1 in P_0. Also, $\vec{x}$ and $\vec{y}$ are, respectively, common normal and common tangent of timelike pole curves C_0 and C_1.
- In the Figure 2.2, it is indicated that γ and γ' are spacelike curvature centers of a trajectory in the fixed plane drawn by spacelike points M and M' linked to moving plane P_1. The common tangent and the common normal of the spacelike pole curves C_0 and C_1 are $\vec{x}$ and $\vec{y}$, respectively.

Lorentzian Planar Motions (Investigation of Distances)

- Let $\overrightarrow{X}$, $\overrightarrow{X'}$ and $\overrightarrow{X''}$ be timelike (spacelike) unit vectors, then these unit vectors can be given as follows;

$$\overrightarrow{X} = \frac{\overrightarrow{IM}}{\left\|\overrightarrow{IM}\right\|}, \ \overrightarrow{X'} = \frac{\overrightarrow{IM'}}{\left\|\overrightarrow{IM'}\right\|}, \ \overrightarrow{X''} = \frac{\overrightarrow{IM''}}{\left\|\overrightarrow{IM''}\right\|}, \qquad (1)$$

see Figure 2.3 (see Figure 2.4)

Figures

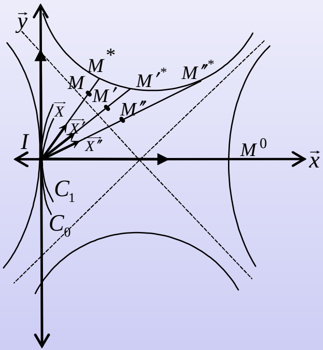

Figure 2.3 Timelike $\vec{X}, \vec{X}'$ ve $\vec{X}''$ vectors

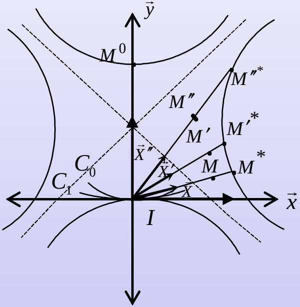

Figure 2.4 Spacelike $\vec{X}, \vec{X}'$ ve $\vec{X}''$ vectors

- If the abscissa of M, M^* and γ timelike (spacelike) points on the axis (I,X) are ρ_1, ρ and ρ_0, respectively, then there are the relationships

$$\left\langle \overrightarrow{IM}, \overrightarrow{X} \right\rangle = \varepsilon\rho_1, \left\langle \overrightarrow{IM^*}, \overrightarrow{X} \right\rangle = \varepsilon\rho, \ \left\langle \overrightarrow{I\gamma}, \overrightarrow{X} \right\rangle = \varepsilon\rho_0 \qquad (2)$$

where $\varepsilon = -1$ if the pole curves are timelike or $\varepsilon = +1$ if the pole curves are spacelike.

- Similarly, we can give

$$\left\langle \overrightarrow{IM'^*}, \overrightarrow{X'} \right\rangle = \varepsilon\rho', \left\langle \overrightarrow{IM''^*}, \overrightarrow{X''} \right\rangle = \varepsilon\rho''.$$

Lorentzian Euler Savary Formula

- Let the diameter of the hyperbolic (Lorentzian) inflection circle be h. This means that h is a distance from a timelike (spacelike) point M^0 on the hyperbolic (Lorentzian) inflection circle at the direction of the common normal to the instantaneous rotation center I, see Figure 2.3 (Figure 2.4).

- Then there is a relationship between h and ρ as follows

$$h \sinh \theta = \rho$$

where θ is a hyperbolic angle of the motion $P_1 \backslash P_0$.

Lorentzian Euler Savary Formula

- If the canonical relative systems of a plane with respect to other planes are taken into consideration then the Lorentzian Euler Savary formula can be constructed for timelike and spacelike pole curves, separately.
- It is proved that this formula remains unchanged whether the pole curves are spacelike or timelike in 2002 by I. Aytun.

- The Lorentzian Euler Savary formula, which gives the relation between the curvatures of the trajectory curves drawn by the points of the moving plane in fixed plane, is

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right)\sinh\theta = \frac{1}{R_1} - \frac{1}{R_0} \tag{3}$$

  where R_0 and R_1 are the abscissae on $(O, \vec{x})$ (on $(O, \vec{y})$) of the curvature centers of the timelike (spacelike) pole curves C_0 and C_1, respectively.
- ρ_0 and ρ_1 are the distance from the timelike (spacelike) points γ and M to the center I, respectively, see Figure 2.5 (see Figure 2.6)

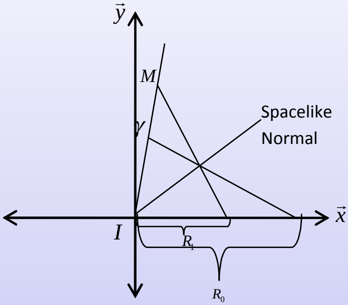

Figure 2.5 R_0 and R_1 lengths

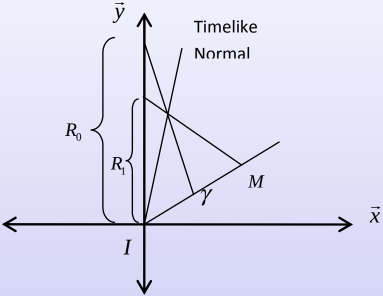

Figure 2.6 R_0 and R_1 lengths

Lorentzian Euler Savary Formula (First and Second Form)

- Since there is the relation $\frac{1}{\rho} = \frac{1}{\rho_1} - \frac{1}{\rho_0}$ the formula given by the equation (3) is

$$\left(\frac{1}{\rho_1} - \frac{1}{\rho_0}\right) \sinh\theta = \left(\frac{1}{R_1} - \frac{1}{R_0}\right) = \frac{1}{h}$$

in which $\frac{1}{h} = \frac{1}{R_1} - \frac{1}{R_0}$ (first form) or $\frac{1}{h} = \pm\frac{\omega}{V}$ (second form) where ω is the angular velocity of the motion of the plane P_1 with respect to P_0 and V is the common velocity of I on the pole curves C_0 and C_1.

III. Lorentzian Bobillier Formula

Deduced from

Lorentzian Euler Savary Formula

Investigation of the Timelike (Spacelike) Points Q for the Lorentzian Bobillier Formula

- By the aid of Lorentzian Euler Savary formula we will investigate the timelike (spacelike) points Q defined by $\overrightarrow{IQ} = \varepsilon\frac{1}{\rho}\overrightarrow{X}$ where $\varepsilon = -1$ if C_0 and C_1 are timelike and $\varepsilon = +1$ if C_0 and C_1 are spacelike pole curves.

- Let timelike (spacelike) points Q, Q and Q'' be the images of the timelike (spacelike) points M^*, M'^* and M''^* of hyperbolic (Lorentzian) inflection circle which respectively belong to (I, X), (I, X') and (I, X'') by an inversion at the rotation center, then it is easily seen that there are the relationships, see Figure 3.1 (see Figure 3.2)

$$\left\langle \overrightarrow{IQ}, \overrightarrow{X} \right\rangle = \varepsilon\frac{1}{\rho}, \left\langle \overrightarrow{IQ'}, \overrightarrow{X'} \right\rangle = \varepsilon\frac{1}{\rho'}, \left\langle \overrightarrow{IQ''}, \overrightarrow{X''} \right\rangle = \varepsilon\frac{1}{\rho''}, \left\langle \overrightarrow{IQ^0}, \overrightarrow{x} \right\rangle = \varepsilon\frac{1}{h}. \quad (4)$$

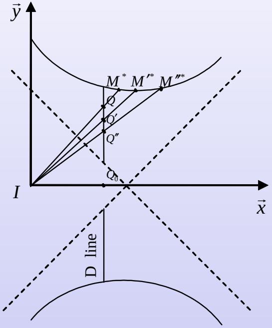

Figure 3.1 Timelike Q points

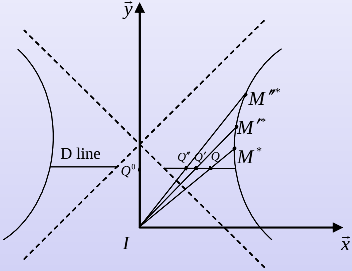

Figure 3.2 Spacelike Q points

- From the equations (3) and (4) the relationship

$$\overrightarrow{IQ}\sinh\theta = \frac{1}{\rho}\overrightarrow{X}\sinh\theta = \frac{1}{h}\overrightarrow{X}$$

is obtained. Similarly,

$$\overrightarrow{IQ'}\sinh\theta' = \frac{1}{\rho'}\overrightarrow{X'}\sinh\theta' = \frac{1}{h}\overrightarrow{X'}$$

and

$$\overrightarrow{IQ''}\sinh\theta'' = \frac{1}{\rho''}\overrightarrow{X''}\sinh\theta'' = \frac{1}{h}\overrightarrow{X''}$$

is given.

- If the last three equations are taken into consideration it is easily seen that

$$\left\langle \overrightarrow{IQ}, \overrightarrow{X} \right\rangle \sinh\theta = \left\langle \overrightarrow{IQ'}, \overrightarrow{X'} \right\rangle \sinh\theta' = \left\langle \overrightarrow{IQ''}, \overrightarrow{X''} \right\rangle \sinh\theta'' = \frac{1}{h}.$$

This means that the set of the timelike (spacelike) points Q is a straight line D parallel to axis $\vec{y}$ (axis $\vec{x}$). Thus the line D is an image of the hyperbolic (Lorentzian) inflection circle by this inversion at the rotation center I, see Figure 3.1 (Figure 3.2).

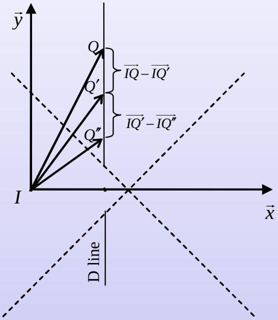

Figure 3.3 $\overrightarrow{IQ}-\overrightarrow{IQ'}$ and $\overrightarrow{IQ'}-\overrightarrow{IQ''}$ vectors

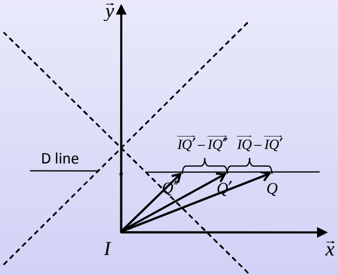

Figure 3.4 $\overrightarrow{IQ}-\overrightarrow{IQ'}$ and $\overrightarrow{IQ'}-\overrightarrow{IQ''}$ vectors

- Since the timelike (spacelike) vectors $\left(\overrightarrow{IQ} - \overrightarrow{IQ'}\right)$ and $\left(\overrightarrow{IQ'} - \overrightarrow{IQ''}\right)$ are linearly dependent, the Lorentzian cross product of these vectors is

$$(\overrightarrow{IQ} \times \overrightarrow{IQ'}) - (\overrightarrow{IQ'} \times \overrightarrow{IQ'}) - (\overrightarrow{IQ} \times \overrightarrow{IQ''}) + (\overrightarrow{IQ'} \times \overrightarrow{IQ''}) = \vec{0}$$

see Figure 3.3 (Figure 3.4).

Lorentzian Bobillier Formula

- By considering the equations $\overrightarrow{IQ} = \varepsilon \frac{1}{\rho} \overrightarrow{X}$, $\overrightarrow{IQ'} = \varepsilon \frac{1}{\rho'} \overrightarrow{X'}$, $\overrightarrow{IQ''} = \varepsilon \frac{1}{\rho''} \overrightarrow{X''}$

$$\left(\varepsilon^2 \frac{1}{\rho'} \overrightarrow{X} \times \frac{1}{\rho} \overrightarrow{X'}\right) + \left(\varepsilon^2 \frac{1}{\rho} \overrightarrow{X''} \times \frac{1}{\rho''} \overrightarrow{X}\right) + \left(\varepsilon^2 \frac{1}{\rho''} \overrightarrow{X'} \times \frac{1}{\rho'} \overrightarrow{X''}\right) = \vec{0}$$

  is obtained.

- Since $\rho\rho'\rho'' \neq 0$ and $\varepsilon^2 = 1$

$$\rho'' \left(\overrightarrow{X} \times \overrightarrow{X'}\right) + \rho' \left(\overrightarrow{X} \times \overrightarrow{X''}\right) + \rho \left(\overrightarrow{X'} \times \overrightarrow{X''}\right) = \vec{0}$$

  can be written.

Lorentzian Bobillier Formula

- If the definition of Lorentzian cross product is taken into consideration, then the last equation becomes

$$\rho \sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right) + \rho' \sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right) + \rho'' \sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right) = 0$$

where $\frac{1}{\rho'} = \frac{1}{\rho'_1} - \frac{1}{\rho'_0}$ and $\frac{1}{\rho''} = \frac{1}{\rho''_1} - \frac{1}{\rho''_0}$.

- This is called **Lorentzian Bobillier formula** which is totally based on Lorentzian Euler Savary formula.

IV. Direct way toward Lorentzian Bobillier Formula

- The following approach will allow us to obtain the Lorentzian Bobillier formula, directly. Then the Lorentzian Euler Savary formula will appear as a particular case.

- Let $\vec{V}^0\,(M)$, $\vec{J}^0\,(M)$ be absolute velocity vector and absolute acceleration vector of the timelike (spacelike) point M , respectively.

- If ω is the angular velocity of the motion $P_1 \backslash P_0$ then $\omega = \frac{\Delta\theta}{\Delta t}$ where θ is the rotation angle.

- By taking a vector $\vec{z}$ which is orthogonal to the planes P_0 and P_1 the angular velocity vector can be defined by $\vec{\omega} = \omega\vec{z}$.

- The sliding velocity vector of the point M is

$$\vec{V}_1(M) = \vec{\omega} \times \overrightarrow{IM}.$$

- In the one parameter Lorentzian planar motion the relationship

$$\vec{V}^0(M) = \vec{V}_1^0(I) + \vec{V}_1(M) \tag{5}$$

holds where $\vec{V}^0(M), \vec{V}_1^0(I)$ and $\vec{V}_1(M)$ denote the absolute, relative and sliding velocity vectors of the motion, $P_1 \backslash P_0$ respectively.

- Substituting the equality of the sliding velocity vector into the equation (5) and differentiating it with respect to t, we obtain

$$\vec{J}^0(M) = \vec{J}_1^0(I) + \left(\dot{\omega}\vec{z} \times \overrightarrow{IM}\right) + \left(\omega\vec{z} \times \left(\omega\vec{z} \times \overrightarrow{IM}\right)\right) \tag{6}$$

where $\vec{J}_1^0(I)$ is the acceleration vector of the point M on P_1 that coincides instantaneously with I.

- If the Lagrange identity in the sense of Lorentz is taken into consideration, then the equation (6) becomes

$$\vec{J}^0(M) = \vec{J}_1^0(I) + \left(\dot{\omega}\vec{z} \times \overrightarrow{IM}\right) + \langle \omega\vec{z}, \omega\vec{z} \rangle \overrightarrow{IM} - \left\langle \overrightarrow{IM}, \omega\vec{z} \right\rangle \omega\vec{z}.$$

- Since $\overrightarrow{IM}$ is orthogonal to the angular velocity vector, $\left\langle \overrightarrow{IM}, \omega\vec{z} \right\rangle = 0$. On the other hand $\langle \omega\vec{z}, \omega\vec{z} \rangle = -\varepsilon\omega^2$ where $\varepsilon = -1$ if $\vec{z}$ is spacelike or $\varepsilon = 1$ if $\vec{z}$ is timelike.

- Therefore the last equation is found as follows;

$$\vec{J}^0(M) = \vec{J}_1^0(I) + \left(\dot{\omega}\vec{z} \times \overrightarrow{IM}\right) - \varepsilon\omega^2\overrightarrow{IM}.$$

- If we decomposed the absolute acceleration vector of the point M to tangent and normal components

$$\vec{J}^{0}(M) = \vec{J}_t(M) + \vec{J}_n(M)$$

where

$$\vec{J}_t(M) = \vec{J}_1^{0}(I) + \left(\dot{\omega}\vec{z} \times \overrightarrow{IM}\right)$$

and

$$\vec{J}_n(M) = -\varepsilon\omega^2\overrightarrow{IM}.$$

- By the fact that the absolute velocity and acceleration vectors of the timelike (spacelike) point M^* on the hyperbolic (Lorentzian) circle are linearly dependent, there is

$$\vec{V}^0(M^*) \times \vec{J}^0(M^*) = \overrightarrow{0}.$$

- If we substitute the equations (5) and (6) into the last equation, we find the following equation;

$$\left(\vec{V}_1^0(I) + \left(\omega\vec{z} \times \overrightarrow{IM^*}\right)\right) \times \left(\vec{J}_1^0(I) + \left(\dot{\omega}\vec{z} \times \overrightarrow{IM^*}\right) - \varepsilon\omega^2\overrightarrow{IM^*}\right) = \overrightarrow{0}.$$

- By applying the Lorentzian cross product and considering $\overrightarrow{V_1^0}(I) = \overrightarrow{0}$, we obtain

$$-\omega\left\langle\vec{z},\vec{J}_1^0(I)\right\rangle\overrightarrow{IM^*} + \omega\left\langle\overrightarrow{IM^*},\vec{J}_1^0(I)\right\rangle\vec{z} + \omega\dot{\omega}\left(\left(\vec{z}\times\overrightarrow{IM^*}\right)\times\left(\vec{z}\times\overrightarrow{IM^*}\right)\right)$$
$$+\omega^3\varepsilon\left\langle\vec{z},\overrightarrow{IM^*}\right\rangle\overrightarrow{IM^*} - \varepsilon\omega^3\left\langle\overrightarrow{IM^*},\overrightarrow{IM^*}\right\rangle\vec{z} = \vec{0}.$$

- It is known that the relationships

$$\left\langle \vec{z}, \vec{J}_1^0(I) \right\rangle = 0, \left\langle \vec{z}, \overrightarrow{IM^*} \right\rangle = 0, \left\| \overrightarrow{IM^*} \right\|^2 = \varepsilon \left\langle \overrightarrow{IM^*}, \overrightarrow{IM^*} \right\rangle$$

and

$$\left(\vec{z} \times \overrightarrow{IM^*} \right) \times \left(\vec{z} \times \overrightarrow{IM^*} \right) = \vec{0}$$

are hold. Then we find that

$$\left\langle \overrightarrow{IM^*}, \vec{J}_1^0(I) \right\rangle \vec{z} - \varepsilon^2 \omega^2 \left\| \overrightarrow{IM^*} \right\|^2 \vec{z} = \vec{0}.$$

- There is always a constant angle between the vector $\vec{J}_1^0(I)$ and the normal vector $\overrightarrow{IM^*}$ if the pole trajectory. Let us call this angle α . The angle between the timelike (spacelike) vectors $\vec{J}_1^0(I)$ and $\overrightarrow{IM^*}$ is a hyperbolic angle and the last equation becomes

$$\varepsilon \left\| \overrightarrow{IM^*} \right\| J_1^0(I) \cosh \alpha - \varepsilon^2 \omega^2 \left\| \overrightarrow{IM^*} \right\|^2 = 0.$$

- From the equation (2) and some some rearrangements it becomes

$$\rho = \varepsilon \frac{J_1^0(I) \cosh \alpha}{\omega^2}.$$

- Since the hyperbolic angle α is also an angle between the timelike (spacelike) vectors $\vec{J}_1^0(I)$ and $\overrightarrow{X}$, it is found that

$$\rho = \frac{\left\langle \vec{J}_1^0(I), \overrightarrow{X} \right\rangle}{\omega^2}. \tag{7}$$

The analogous equations can be written for points M' and M'' as follows;

$$\rho' = \frac{\left\langle \vec{J}_1^0\left(I\right), \overrightarrow{X'} \right\rangle}{\omega^2}, \tag{8}$$

and

$$\rho'' = \frac{\left\langle \vec{J}_1^0\left(I\right), \overrightarrow{X''} \right\rangle}{\omega^2}. \tag{9}$$

- So, from the equation (7), (8), (9), ρ, ρ' and ρ'' may be seen as the Lorentzian orthogonal projections of the same timelike (spacelike) vector $\frac{\vec{J}_1^0(I)}{\omega^2}$ on the timelike (spacelike) unit vectors X, $X'$and X'' which are linearly dependent. The dependence between X, X' and X'' may be written as follows;

$$\lambda \overrightarrow{X} + \mu \overrightarrow{X'} + \vartheta \overrightarrow{X''} = \vec{0}. \tag{10}$$

By successive Lorentzian cross products with X and X', the quantities λ, μ and ϑ are obtained as follows

$$\lambda = \sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right), \mu = \sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right), \vartheta = \sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right). \tag{11}$$

- Substituting the equation (11) into the (10) the linear combination becomes

$$\sinh\left(\overrightarrow{X'},\overrightarrow{X''}\right)\overrightarrow{X}+\sinh\left(\overrightarrow{X''},\overrightarrow{X}\right)\overrightarrow{X'}+\sinh\left(\overrightarrow{X},\overrightarrow{X'}\right)\overrightarrow{X''}=0.$$

- The Lorentzian scalar product of the previous equation with the vector $\frac{\vec{J}_1^0(I)}{\omega^2}$ is

$$\sinh\left(\overrightarrow{X'},\overrightarrow{X''}\right)\frac{\left\langle\overrightarrow{X},\vec{J}_1^0(X)\right\rangle}{\omega^2}+\sinh\left(\overrightarrow{X''},\overrightarrow{X}\right)\frac{\left\langle\overrightarrow{X'},\vec{J}_1^0(X)\right\rangle}{\omega^2}+\sinh\left(\overrightarrow{X},\overrightarrow{X'}\right)\frac{\left\langle\overrightarrow{X''},\vec{J}_1^0(X)\right\rangle}{\omega^2}=0. \qquad (12)$$

- Finally, taking into account (7), (8) and (9) **Lorentzian Bobillier Formula** is obtained again, but using a direct way without the use of Lorentzian Euler Savary formula,

$$\rho \sinh\left(\overrightarrow{X'}, \overrightarrow{X''}\right) + \rho' \sinh\left(\overrightarrow{X''}, \overrightarrow{X}\right) + \rho'' \sinh\left(\overrightarrow{X}, \overrightarrow{X'}\right) = 0. \qquad (13)$$

Therefore, the following theorem can be given.

Theorem

In one parameter Lorentzian planar motion of moving plane P_1 with respect to fixed plane P_0, the relationship between the centers of curvatures concerning second order instantaneous properties is given by the **Lorentzian Bobillier formula** *given in the equation (13).*

Let us investigate a particular case of Theorem 1.

Corollary

Let a timelike (spacelike) point M_1 linked to moving plane P_1 be coincident with instantaneous pole center I. Then $\vec{V}^0(M_1) = \vec{0}$ and similarly $\vec{J}^0(M_1) = \vec{0}$. Under this condition the length ρ' vanishes and for timelike pole curves and spacelike pole curves Lorentzian Bobillier formula becomes

$$\rho \sinh(\vec{y}, \vec{x}) + \rho'' \sinh\left(\overrightarrow{X}, \vec{y}\right) = 0,$$

and

$$\rho \sinh\left(\overrightarrow{x}, \overrightarrow{y}\right) + \rho'' \sinh\left(\overrightarrow{X}, \overrightarrow{x}\right) = 0,$$

respectively.
Let θ be a hyperbolic angle between X and y (x) for timelike (spacelike) pole curves. On the other hand x and y are orthogonal in the sense of Lorentzian. Thus, the last two formulas can be given by

$$\rho = \rho'' \sinh \theta$$

which was the Lorentzian Euler Savary formula.

V. References

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Thank You for Your Attention