Lorentzian quasi-Einstein manifolds

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(M, g) is a **quasi-Einstein manifold** if there exist a smooth function $f : M \to \mathbb{R}$ and two constants μ, $\lambda \in \mathbb{R}$ such that

$$\rho + \mathrm{Hes}_{\mathbf{f}} - \mu \mathbf{df} \otimes \mathbf{df} = \lambda \mathbf{g}.$$

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G. Catino, C. Mantegazza, L. Mazzieri and M. Rimoldi

Let (M,g) be a complete locally conformally flat quasi-Einstein Riemannian manifold. Then

- if $\mu = -\frac{1}{n}$, it is globally conformally equivalent to a space form.
- if $\mu \neq -\frac{1}{n}$, around any regular point of f, it is locally a warped product with $(n+1)$-dimensional fibers of constant sectional curvature.

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$$\rho + \mathrm{Hes}_{\mathbf{f}} - \mu \, \mathbf{df} \otimes \mathbf{df} = \lambda \mathbf{g}.$$

Aim

To classify locally conformally flat quasi-Einstein manifolds in the Lorentzian setting by focusing on their local structure.

Particular cases

- They are natural generalizations of Einstein manifolds (if f is constant)

$$\rho = \lambda g$$

- If $\mu = 0$, quasi-Einstein manifolds correspond to gradient Ricci solitons
- If $\mu = -\frac{1}{n}$ and (M, g) is quasi-Einstein, the metric $\widetilde{g} = e^{-\frac{2}{n}f} g$ is Einstein
- If $\mu = \frac{1}{m}$ the first necessary condition for a warped product $M \times_f F$, where $m = \dim F$, to be Einstein $\rho^M - \frac{\dim F}{f} \mathrm{Hes}_f = \lambda g$, corresponds with the condition for M to be quasi-Einstein for the function $\phi = -m \ln f$ and $\mu = \frac{1}{\dim F}$.

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$$\begin{aligned}\rho_{\widetilde{g}} &= \rho_g + \mathrm{Hes}_f + \frac{1}{n} df \otimes df + \frac{1}{n}(\Delta f - |\nabla f|^2) g \\ &= \frac{1}{n}(\Delta f - |\nabla f|^2 + n\lambda) e^{\frac{2}{n}f} \widetilde{g}.\end{aligned}$$

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Quasi-Einstein manifolds are:

- *isotropic* if $\|\nabla f\| = 0$,
- *non isotropic* if $\|\nabla f\| \neq 0$.

Lemma

A Lorentzian quasi-Einstein manifold satisfies

- $\tau + \Delta f - \mu\|\nabla f\|^2 = (n+2)\lambda$
- $\nabla\tau = 2\rho(\nabla f) - 2(n+2)\lambda\mu\nabla f - 2\mu^2\|\nabla f\|^2\nabla f + 2\mu\tau\nabla f + \mu\nabla\|\nabla f\|^2$.

Lemma

A Lorentzian isotropic quasi-Einstein manifold satisfies

- $\nabla\tau = 2(\lambda - \mu((n+2)\lambda - \tau))\nabla f$,
- $\rho(\nabla f) = \lambda\nabla f$.

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Lemma

Let (M, g) be a locally conformally flat quasi-Einstein manifold. If $\mu \neq -\frac{1}{n}$, ∇f is an eigenvector of the Ricci operator.

Main theorem

Let (M, g) be a locally conformally flat quasi-Einstein manifold.

1. If $\mu = -\frac{1}{n}$ (M, g) is globally equivalent to a space form.
2. If $\mu \neq -\frac{1}{n}$, then
   - If $\|\nabla f\| \neq 0$, $M = I \times_\varphi N$ with metric $\varepsilon dt^2 + \varphi^2 g_N$
   - If $\|\nabla f\| = 0$, (M, g) is locally isometric to a plane wave.

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Non isotropic quasi-Einstein manifolds

$\|\nabla f\| \neq 0$

- ∇f is an eigenvector of the Ricci operator.
- The level sets of f are totally umbilical hypersurfaces and hence (M, g) decomposes locally as a twisted product[a].
- $\rho(\nabla f, X) = 0$, $\forall X \perp \nabla f$, and therefore the twisted product reduces to a warped product[b], $(I \times N, \varepsilon dt^2 + \psi(t)^2 g_N)$ where (N, g_N) is a Riemannian or a Lorentzian manifold of constant sectional curvature.

[a] R. Ponge and H. Reckziegel; Twisted Products in Pseudo-Riemannian Geometry, *Geom. Dedicata*, **48** (1993), 15–25.

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Isotropic quasi-Einstein manifolds

$\|\nabla f\| = 0$

- ∇f is an eigenvector of the Ricci operator, which is 2-step nilpotent.
- $\lambda = 0$ and ∇f is a geodesic ($\nabla_{\nabla f} \nabla f = 0$) and recurrent ($\nabla_X \nabla f = \text{hes}_f(X) = \sigma(X)\nabla f$) vector field.
- ∇f spans a parallel null line field $\mathscr{D}$. Hence the curvature satisfies $R(\mathscr{D}, \mathscr{D}^\perp, \cdot, \cdot) = R(\mathscr{D}, \mathscr{D}, \cdot, \cdot) = R(\mathscr{D}^\perp, \mathscr{D}^\perp, \mathscr{D}, \cdot) = 0.$
- $R(\mathscr{D}^\perp, \mathscr{D}^\perp, \cdot, \cdot) = 0$ and therefore (M, g) is a *pr*-wave[a].

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Isotropic quasi-Einstein manifolds

$\|\nabla f\| = 0$ (continuation)

⋆ The Ricci tensor is isotropic ($\rho(Y,\cdot) = 0, \forall Y \in \nabla f^{\perp}$) and thus (M,g) is a *pp*-wave[a].

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Definition

$(\mathbb{R}^{n+2}, g_{pp})$ is a *pp*-wave if

$$g_{pp} = 2dudv + H(u, x_1, \ldots, x_n)du^2 + \sum_{i=1}^{n} dx_i^2. \tag{1}$$

and it is locally conformally flat if

$$H(u, x_1, \ldots, x_n) = a(u)\sum_i x_i^2 + \sum_i b_i(u)x_i + c(u).$$

If $(\mathbb{R}^{n+2}, g_{pp})$ is locally conformally flat quasi-Einstein, then

$$f(u, x_1, \ldots, x_n) = -\frac{\log(f_0(u) + \sum_i \kappa_i x_i)}{\mu}.$$

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with: $$2\kappa_i(1 + n\mu)\rho_{uu} = 0, \forall i.$$

One of the following holds:

1. $(\mathbb{R}^{n+2}, g_{pp})$ is flat.
2. If $\mu = -\frac{1}{n}$, the manifold is locally conformal to an Einstein manifold.
3. If $\mu \neq -\frac{1}{n}$, all the κ_i vanish and f is only a function of u that should satisfy: $f''(u) - \mu(f'(u))^2 = na(u)$.

Remark

In the particular case that $\mu = 0$ we get exactly what it happened for gradient Ricci solitons, $f''(u) = na(u)$.

$$f(u, x_1, \ldots, x_n) = -\frac{\log(f_0(u) + \sum_i \kappa_i x_i)}{\mu},$$

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