Pseudo-umbilical and other umbilical-type surfaces in spacetime

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VI International Meeting on Lorentzian Geometry
Granada, 8 September 2011

Outline

Introduction (with a personal Motivation)

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- The classification was carried out according to the *extrinsic* properties of the surface: it is an algebraic classification based, at each point, on the properties of two independent Weingarten operators.

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  In particular, I used two *null* Weingarten operators A_ℓ and A_k (where ℓ and k are the two independent null vectors fields orthogonal to the surface.)
- Each of the Weingarten operators is a self-adjoint matrix which can be readily classified algebraically according to the signs of their (real) eigenvalues. This produces 8 different types for each matrix, and therefore 64 types of points for generic spacelike surfaces.

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- The parameter can be chosen as the angle between the two ON eigen-bases for A_ℓ and A_k. So, it takes values on a finite closed interval of $\mathbb{R}$.
- Actually, one can prove (see the cited Ref.) that the parameter is simply related to

$$[A_k, A_\ell]$$

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Corollary

In particular, for conformally flat spacetimes (including Lorentz space forms) the neccessary and sufficient condition is that the normal curvature vanishes.

Basic concepts and notation

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- Then, at any $x \in S$ one has the orthogonal decomposition

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- Let $\mathfrak{X}(S)$ (respectively $\mathfrak{X}(S)^{\perp}$) denote the set of smooth vector fields tangent (resp. orthogonal) to S.

- There is a volume element 2-form on each $T_xS^\perp$, denoted by $\epsilon^\perp$. The corresponding Hodge dual operator is written and defined by

$$\star^\perp N \equiv (i_N \epsilon^\perp)^\sharp, \ \ \forall N \in \mathfrak{X}(S)^\perp$$

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- The *Levi-Civita connection* $\bar{\nabla}$ of $\bar{g}$ ($\overline{\nabla}\bar{g} = 0$) can be defined as

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- The basic extrinsic object is $\mathrm{I\!I} : \mathfrak{X}(S) \times \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)^{\perp}$ defined by

$$-\mathrm{I\!I}(X, Y) \equiv (\nabla_X Y)^{\perp} = \nabla_X Y - \overline{\nabla}_X Y$$

  called the *shape tensor* or *second fundamental form tensor* of S in $\mathcal{V}$.

Second fundamental forms

- Given any normal direction $N \in \mathfrak{X}(S)^{\perp}$, the *second fundamental form* of S in $(\mathcal{V}, g)$ relative to N is the 2-covariant symmetric tensor field on S defined by

$$K_N(X,Y) \equiv g\left(N, \mathrm{I\!I}(X,Y)\right), \qquad \forall X, Y \in \mathfrak{X}(S).$$

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- Therefore, at each $x \in S$, $A_N|_x$ is a self-adjoint linear transformation on T_xS. As such, it is always diagonalisable over $\mathbb{R}$.
- The *normal connection* D is then given by

$$D_X N \equiv \left(\nabla_X N\right)^{\perp}$$

Special bases on $\mathfrak{X}(S)^\perp$

- There are two *independent* normal vector fields on S. They can be appropriately chosen to form an ON basis on $\mathfrak{X}(S)^\perp$, in which case we will denote them by $u, n \in \mathfrak{X}(S)^\perp$, with

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- Of course, any two such ON basis are related by a Boost (Lorentz transformation):

$$\begin{pmatrix} u' \\ n' \end{pmatrix} = \begin{pmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{pmatrix} \begin{pmatrix} u \\ n \end{pmatrix}$$

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$$\star^{\perp} u = n, \;\; \star^{\perp} n = u; \qquad \star^{\perp}\ell = \ell, \;\; \star^{\perp} k = -k$$

The mean curvature vector H

- Thus, the shape tensor decomposes as

$$\mathrm{II}(X,Y) = -K_k(X,Y)\;\ell - K_\ell(X,Y)\;k \tag{2}$$

in the null basis, or as

$$\mathrm{II}(X,Y) = -K_u(X,Y)\;u + K_n(X,Y)\;n \tag{2'}$$

in any ON basis, $\forall X, Y \in \mathfrak{X}(S)$.

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  in any ON basis, $\forall X,Y \in \mathfrak{X}(S)$.
- Observe that these formulae are invariant under the boost freedom.
- The *mean curvature vector* $H \in \mathfrak{X}(S)^\perp$ is defined as the trace of the shape tensor with respect to $\bar{g}$, or explicitly

$$H \equiv -(\mathrm{tr}\,A_k)\;\ell - (\mathrm{tr}\,A_\ell)\;k \tag{3}$$

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  in ON bases.

On the causal character of H

- Notice that H and

$$\star^{\perp} H = -(\text{tr}\, A_k)\ \ell + (\text{tr}\, A_\ell)\ k = -(\text{tr}\, A_u)\ n + (\text{tr}\, A_n)\ u$$

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- Actually, the most important surfaces in Gravitation are defined according to the causal orientation of H. For instance, the simple condition

$$H \wedge (\star^{\perp} H) = 0$$

  is equivalent to saying that H is null everywhere on S. Among these surfaces, if H (and hence $\star^{\perp} H$) points along one of the null directions ℓ or k everywhere, then they are called *marginally outer trappped surfaces* (MOTS) (also called *null dual*). If in addition the causal orientation of H does not change on S then this is called a *marginally trapped surface*.

Another extrinsic vector G

- One can also define another normal vector field $G \in \mathfrak{X}(S)^{\perp}$ by using a second invariant of the matrices A_N. Unfortunately, there are no other linear invariants. Anyway, for each $N \in \mathfrak{X}(S)^{\perp}$ we set

$$\sigma_N^2 \equiv (\text{tr} A_N)^2 - 4 \det A_N$$

(called the *shear* along N) and then we define

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- G, as well as

$$\star^{\perp} G = \sigma_k \, \ell - \sigma_\ell \, k$$

are invariant under the boost freedom (1).

The normal connection one-form ω

- For a fixed ON basis on $\mathfrak{X}(S)^{\perp}$, a one-form $\omega \in \Lambda^1(S)$ is defined by

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- For $\sqrt{2}\ell = u + n$ and $\sqrt{2}k = u - n$ one also has

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- Therefore, for all $X \in \mathfrak{X}(S)$

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- $d\omega$ is thus invariant and well-defined (related to the normal curvature).

Curvatures

- The intrinsic curvature for $(S, \bar{g})$ has the usual definition

$$\overline{R}(X,Y)Z \equiv \overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]} Z, \quad \forall X, Y, Z \in \mathfrak{X}(S)$$

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- Similarly, the normal curvature is defined on S by

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- A simple calculation provides

$$R^{\perp}(X,Y)N = d\omega(X,Y) \star^{\perp} N$$

for all $X, Y \in \mathfrak{X}(S)$ and for all $N \in \mathfrak{X}(S)^{\perp}$. Thus, we justify that ω characterizes the normal connection and that $d\omega$ defines its curvature.

Forms of the Gauss equation

For all $X, Y, Z, W \in \mathfrak{X}(S)$

$$R(W,Z,X,Y) = \overline{R}(W,Z,X,Y) + g\left(\mathrm{II}(X,Z), \mathrm{II}(Y,W)\right) - g\left(\mathrm{II}(Y,Z), \mathrm{II}(X,W)\right)$$

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However, as S is 2-dimensional its curvature is uniquely determined by its Gaussian curvature $K(S)$.
Therefore, the previous relation can be written as a single scalar equation.
To that end, we are going to define a new extrinsic object, quadratic in the shape tensor II.

The extrinsic objects $\mathbb{J}$ and B

For any ON basis $\{e_i\}$ in $\mathfrak{X}(S)$, set by definition

$$\mathbb{J}(X,Y) \equiv \sum_{i=1}^{2} g\left(\mathrm{I\!I}(e_i,X), \mathrm{I\!I}(e_i,Y)\right) \qquad \forall X,Y \in \mathfrak{X}(S)\,.$$

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$\mathbb{J}(X,Y)$ is a 2-covariant symmetric tensor field on S.

Then, define $B\ :\mathfrak{X}(S) \to \mathfrak{X}(S)$ by

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$K(S)$ is the Gaussian curvature of S, Ric and $\mathcal{S}$ are the Ricci tensor and the scalar curvature of $(\mathcal{V},g)$.

Forms of the Ricci equation

For all $X, Y \in \mathfrak{X}(S)$, for all $N, M \in \mathfrak{X}(S)^{\perp}$

$$\begin{aligned}(R(X,Y)N)^{\perp} &= \mathrm{II}\,(X, A_N(Y)) - \mathrm{II}\,(Y, A_N(X)) + R^{\perp}(X,Y)N \\ &= \mathrm{II}\,(X, A_N(Y)) - \mathrm{II}\,(Y, A_N(X)) + d\omega(X,Y) \star^{\perp} N\end{aligned}$$

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Umbilical-type surfaces

Definition (Umbilic points on S)

A point $x \in S$ is called umbilic with respect to $N|_x \in T_x^{\perp} S$ (or simply N-umbilic) if the corresponding Weingarten operator is proportional to the Identity

$$A_N|_x = \frac{1}{2} F \, \mathbf{1}$$

Necessarily $F = g(H, N|_x)$.

Equivalently, $K_N|_x = \frac{1}{2} g(H, N|_x) \, \bar{g}|_x$.

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Definition (N-Umbilical surfaces)

Thus, S is said to be *umbilical along a vector field* $N \in \mathfrak{X}(S)^\perp$ if

$$A_N = \frac{1}{2} g(H, N) \mathbf{1}$$

or equivalently, if $K_N = \frac{1}{2} g(H, N)\, \bar{g}$.

N-subgeodesic and 0-isotropic surfaces

Observe: Minimal surfaces ($H = 0$) can only be considered as a limit case of N-umbilical (because necessarily $A_N = 0$).

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A spacelike surface S is called N-subgeodesic if $A_{\star^\perp N} = 0$.

(This means that any geodesic γ within $(S, \bar{g})$ is a sub-geodesic with respect to N on the spacetime $(\mathcal{V}, g)$: $\nabla_{\gamma'}\gamma' = fN$.)

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Observe: Totally geodesic $\iff$ $A_N = 0 \ \forall N \in \mathfrak{X}^{\perp}(S)$.

Pseudo- and Ortho-umbilical surfaces

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S is said to be pseudo-umbilical if it is umbilical with respect to $N = H$, so that $A_H = \frac{1}{2} g(H, H)\, \mathbf{1}$

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Hence, H-subgeodesic $\Longleftrightarrow$ ortho-umbilical.

A surface can be pseudo- and ortho-umbilical at the same time. This requires that H be null necessarily, ergo these are MOTS, and they are necessarily H-subgeodesic (with now $H = \ell$ or k). (I recently learned that these are also called 0-isotropic surfaces).

Totally umbilical surfaces

Definition (totally umbilical surfaces)

S is called totally umbilical if it is umbilical with respect to all possible $N \in \mathfrak{X}(S)^\perp$:

$$\forall N \in \mathfrak{X}(S)^\perp \quad A_N = \frac{1}{2} g(H, N)\,\mathbf{1}.$$

Equivalently,

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Equivalently,

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Result (The meaning of G)

Totally umbilical surfaces can be characterized by

$$G = 0$$

The first main theorem

Theorem

The necessary and sufficient condition for $x \in S$ to be umbilic is that two independent Weingarten operators on T_xS commute (and then, all possible Weingarten operators do).

This is equivalent to the condition that the shape tensor be diagonalizable on S.

The umbilic direction is then uniquely determined and given either by G or $\star^{\perp}G$.

Hence, there exists a (generally unique) ON basis in which all possible Weingarten operators diagonalize.

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- $\Longrightarrow$ Assume that $N \in \mathfrak{X}(S)^{\perp}$ is an umbilical direction, so that in (say) the null basis one has

$$-g(N,k)\,A_\ell - g(N,\ell)\,A_k = \frac{1}{2} g(H,N)\,\mathbf{1}\,.$$

Then, by taking the commutator with A_ℓ or A_k, one immediately derives (for $N \neq 0$):

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Then, by taking the commutator with A_{ℓ} or A_k, one immediately derives (for $N \neq 0$):

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- Now, all possible Weingarten operators are linear combinations of any two of them, that is, for any $M \in \mathfrak{X}(S)^{\perp}$:

$$A_M = aA_k + bA_{\ell}$$

and therefore

$$\left[A_M, A_{\tilde{M}}\right] = 0, \quad \forall M, \tilde{M} \in \mathfrak{X}(S)^{\perp}\,.$$

Proof (sufficiency):

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- Let $\{\lambda_1, \lambda_2\}$ and $\{\nu_1, \nu_2\}$ denote the corresponding eigenvalues for A_k and A_ℓ, respectively.
- Then, the equation for the umbilical direction becomes

$$-g(N,\ell)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} - g(N,k)\begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix} = \frac{1}{2} g(H,N)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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- Introducing here that

$$\begin{aligned} g(H,N) &= -g(N,\ell)\mathrm{tr}A_k - g(N,k)\mathrm{tr}A_\ell \\ &= -g(N,\ell)(\lambda_1+\lambda_2) - g(N,k)(\nu_1+\nu_2) \end{aligned}$$

the system of equations collapses to a <u>single</u> equation

$$g(N,\ell)(\lambda_1-\lambda_2) + g(N,k)(\nu_1-\nu_2) = 0.$$

Proof (sufficiency cont'd):

- Its solution is clearly unique (up to proportionality factors) and explicitly given by

$$\boxed{N_{umb} = (\lambda_1 - \lambda_2)\ell - (\nu_1 - \nu_2)k}$$

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$$\lambda_1 - \lambda_2 = \pm\,\sigma_k, \qquad \nu_1 - \nu_2 = \pm\,\sigma_\ell$$

this unique direction is, at each $x \in S$, either $G|_x$ or $\star^\perp G|_x$.

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  this unique direction is, at each $x \in S$, either $G|_x$ or $\star^{\perp} G|_x$.
- The unique exceptional case is

$$G = 0 \quad \Longleftrightarrow \quad \text{totally umbilical.}$$

Q.E.D.

The causal character of the umbilical direction

From the previous results

$$g\left(N_{umb}, N_{umb}\right) = 2(\lambda_1 - \lambda_2)(\nu_1 - \nu_2) = 4\,\mathrm{tr}(A_k A_\ell) - 2\,\mathrm{tr}A_k\,\mathrm{tr}A_\ell$$

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$$g(H,H) - 2\,\mathrm{tr}B \begin{cases} < 0 & \Rightarrow N_{umb} \quad \text{is timelike} \\ > 0 & \Rightarrow N_{umb} \quad \text{is spacelike} \\ = 0 & \Rightarrow N_{umb} \quad \text{is null} \end{cases}$$

Alternatively, N_{umb} is spacelike (respectively timelike, null) if the ordered eigen-bases for A_ℓ and A_k agree (resp. are opposite, one of them cannot be ordered).

The second main theorem

Theorem

The necessary and sufficient condition for S to be umbilical along a normal direction is

$$R^{\perp}(X,Y)N = (R(X,Y)N)^{\perp}\,, \qquad \forall X,Y\,\mathfrak{X}(S),\ \ \forall N \in \mathfrak{X}(S)^{\perp}$$

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Proof. Taking the following form of the Ricci equation

$$R(M,N,X,Y) = g\left(\left[A_M,A_N\right](Y),X\right) + d\omega(X,Y)\; g(\star^{\perp}N,M)$$

and using the first main theorem one derives the result. Q.E.D.

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Eliminating M, one can also write

$$(R(X,Y)N)^{\perp} = d\omega(X,Y)\star^{\perp}N,\;\; \forall X,Y \in \mathfrak{X}(S),\;\; \forall N \in \mathfrak{X}(S)^{\perp}$$

Some important Corollaries and Consequences

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For locally conformally flat spacetimes (including Lorentz space forms) the neccessary and sufficient condition for S to be umbilical along a normal direction is that its normal curvature vanishes:

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Proof.

It is easily checked that $(R(X,Y)N)^{\perp} = (C(X,Y)N)^{\perp}$ where C denotes de Weyl conformal curvature.
Consequently, if $(\mathcal{V}, g)$ is locally conformally flat, then $(R(X,Y)N)^{\perp} = 0$ for all $X, Y \in \mathfrak{X}(S)$ and for all $N \in \mathfrak{X}(S)^{\perp}$, so that from the previous theorem one gets $d\omega = 0$, or equivalently

$$R^{\perp} = 0$$

Q.E.D.

Corollary: Pseudo-umbilical surfaces

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$$(\lambda_1 - \lambda_2)(\nu_1 + \nu_2) + (\lambda_1 + \lambda_2)(\nu_1 - \nu_2) = 0$$

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Now, this automatically implies that $[A_N, A_M] = 0$ for all $N, M \in \mathfrak{X}(S)^{\perp}$ (unless $H = 0$). Q.E.D.

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Note that the condition can be invariantly characterized by

$$(\mathrm{tr}B)^2 - 4\det B = 0$$

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Proof. The first part follows from the vanishing of $A_{\star^\perp H}$. Then, one needs that $N_{umb} \wedge \star^\perp H = 0$, which readily gives:

$$\begin{aligned}(\lambda_1 - \lambda_2)(\nu_1 + \nu_2) - (\lambda_1 + \lambda_2)(\nu_1 - \nu_2) &= 0 \\ \Longleftrightarrow \quad \lambda_1\nu_2 + \lambda_2\nu_1 &= 0 \\ \Longleftrightarrow \quad g(H,H) - \mathrm{tr}B &= 0\end{aligned}$$

Q.E.D.

Corollary: Ortho-umbilical surfaces (2)

Recall the Gauss equation:

$$2\,K(S) = \mathcal{S} - 4\,\mathrm{Ric}(\ell, k) + 2R(\ell, k, \ell, k) + g(H, H) - \mathrm{tr}B$$

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Corollary

Ortho-umbilical surfaces on Constant-Curvature spacetimes have vanishing normal curvature and

$$K(S) = \mathcal{K}$$

Corollary: Pseudo- and Ortho-umbilical surfaces

IF S is both pseudo-umbilical and ortho-umbilical, then clearly

$$g(H,H) = \text{tr}B \qquad \text{and} \qquad B = \frac{1}{2}\text{tr}B\,\mathbf{1}$$

This actually implies that

$$g(H,H) = 0, \qquad B = 0$$

so that, apart from all the previous consequences for each of them, they have

$$\mathbb{I} = -K_k\,\ell \quad \text{or} \quad -K_\ell\,k$$

and they are H-subgeodesic MOTS and also 0-isotropic.

More Corollaries and Consequences

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ergo

$$A_\xi = -\phi\, \mathbf{1}.$$

Q.E.D.

Final Comments

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  1. Co-dimension two spacelike submanifolds in Lorentzian manifolds of higher dimensions will also have two independent Weingarten operators, and their commutativity can be seen to be a necessary condition for umbilicity. However, it cannot be sufficient in general.
  2. If the co-dimension is greater than two, then there are more than two independent Weingarten operators, and their commutativity is not even a necessary condition.