4-dimensional Kähler–Weyl manifolds

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Weyl geometry

1. (M, g) is a pseudo-Riemannian manifold of signature (p, q) for $m = p + q = 2\bar{m}$ is even. Assume $H^1(M; R) = 0$.

2. (M, g, ∇) is a Weyl manifold if ∇ is a torsion free connection and if $\nabla g = -2\phi \otimes g$.

3. If $\tilde{g} = e^{2f} g$ is a conformally equivalent metric, then $(M, \tilde{g}, \nabla)$ is a Weyl manifold with $\tilde{\phi} = \phi - df$.

4. The Weyl structure is said to be trivial if $\nabla = \nabla^{\tilde{g}}$ is the Levi-Civita connection for a conformally equivalent metric – i.e. $\phi = df$. Let ρ_a be the alternating part of the Ricci tensor of ∇; $d\phi = -\frac{1}{m}\rho_a$. So if $H^1(M;R) = 0$, trivial Weyl structure is equivalent to Ricci symmetric.

(para)-complex geometry

5. One says that (M, g, J_-) is almost pseudo-Hermitian if $J_-^2 = -\mathrm{Id}$ and if $J_-^* g = g$. This implies p and q are even.

6. One says that (M, g, J_+) is almost para-Hermitian if $J_+^2 = \mathrm{Id}$ and if $J_+^* g = -g$. This implies $p = q$.

7. $J_\pm$ is integrable means there exist local coordinates $(x^1, ..., x^{\bar{m}}, y^1, ..., y^{\bar{m}})$ so $J_\pm \partial_{x_i} = \partial_{y_i}$ and $J_\pm \partial_{y_i} = \pm \partial_{x_i}$ i.e. the Njuinhuis tensor $N_\pm = 0$ where

$$N_\pm(x, y) := [x, y] \mp J_\pm[J_\pm x, y]$$
$$\mp J_\pm[x, J_\pm y] \pm [J_\pm x, J_\pm y]$$

Kähler geometry

8. Let $\Omega_\pm(x,y) := g(x,J_\pm y)$ be the Kähler form. The following conditions are equivalent and if any is satisfied, $(M,g,J_\pm)$ is said to be **Kähler**.

(8a) $\nabla^g J_\pm = 0$.

(8b) $d\Omega_\pm = 0$ and $J_\pm$ is integrable.

(8c) Given $P \in M$, there are local para/holomorphic coord. so $dg(P) = 0$.

9. This has topological implications. Let (M, g, J_-) be Kähler where g is Riemannian and M is compact. Then Ω is harmonic and $x = [\Omega] \in H^2(M; R)$ is such that $\{1, x, x^2, ..., x^{\bar{m}}\}$ are non-zero.
Thus, for example, $S^1 \times S^3$ admits no Kähler metric.

Kähler–Weyl structures

10. $(M, g, J_\pm, \nabla)$ is **Kähler–Weyl** if
    a) $J_\pm^* g = \mp g$, $J_\pm^2 = \pm \mathrm{Id}$
       (pseudo/para almost (para)
       Hermitian).
    b) $\nabla g = -2\phi \otimes g$ (Weyl Structure)
    c) $\nabla J_\pm = 0$ (implies $J_\pm$ integrable)

Theorem (Vaisman) If $(M, g, J_\pm, \nabla)$ is Kähler–Weyl and $m \geq 6$, then the Weyl structure is trivial, i.e. there is a conformally equivalent metric so that $(M, \tilde{g}, J_\pm)$ is Kähler.

Theorem (Kokarev-Kotschick) Let $m = 4$. If $(M, g, J_\pm)$ is para or pseudo-Hermitian, then there is a unique torsion free connection ∇ so $(M, g, J_\pm, \nabla)$ is Kähler Weyl; here $\phi = \pm\frac{1}{2}J_\pm^*\delta\Omega_\pm$.

Curvature Decompositions

11. Let (M, g, ∇) be a Weyl manifold.
$R(x,y)z := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]}$
$R(x,y,z,w) = g(R(x,y)z,w)$.
$\rho(x,y) = \text{Tr}(z \to R(z,x)y)$.

One has:

11-a) $R(x,y,z,w) = -R(y,x,z,w)$.

11-b) $R(x,y,z,w) + R(y,z,x,w)$
$+R(z,x,y,w) = 0$.

11-c) $R(x,y,z,w) + R(x,y,w,z)$
$= -\frac{4}{m}\rho_a(x,y)g(z,w)$.

11-d) If $\nabla = \nabla^g$, then
$R(x,y,z,w) + R(x,y,w,z) = 0$.

Let V be a real vector space of dimension m and let $\langle\cdot,\cdot\rangle$ be a non degenerate innerproduct on V of signature (p, q).

12. Let $\mathcal{W} \subset \otimes^4 T^*V$ be the space of Weyl tensors – tensors satisfying 11a)-11c).

13. Let $\mathcal{R} \subset \mathcal{W}$ be the space of Riemann tensors - these satisfy in addition 11d).

Theorem (Higa)$\mathcal{W} = \mathcal{R} \oplus L^2$ as an orthogonal module. The factor of $L^2 \approx \Lambda^2$ is detected by ρ_a.

Theorem

a) Let (M, g, ∇) be a Weyl manifold. The Weyl structure is trivial if and only if $R_P \in \mathcal{R}(T_P M, g_P)$ for all points P of M or, equivalently, if ρ is symmetric.

b) Every element of $\mathcal{R}$ is representable a pseudo Riemannian manifold.

c) Every element of $\mathcal{W}$ is representable by a Weyl structure.

14. If $(M, g, J_\pm, \nabla)$ is Kähler, then $R(x, y, z, w) = \mp R(x, y, J_\pm z, J_\pm w)$.

15. Let $\mathcal{K} \subset \otimes^4 V^*$ be defined by imposing this symmetry. Let $\mathcal{K}_\mathcal{R} := \mathcal{K} \cap \mathcal{R}$ and $\mathcal{K}_\mathcal{W} := \mathcal{K} \cap \mathcal{W}$ be the Kähler–Riemann tensors and the Kähler–Weyl tensors.

Theorem

a) If $m \geq 6$, then $\mathcal{K}_{\mathcal{R}} = \mathcal{K}_{\mathcal{W}}$.

b) If $m = 4$, then $\mathcal{K}_{\mathcal{W}} = \mathcal{K}_{\mathcal{R}} \oplus L_0^2$
for $L_0^2 \approx \{\omega \in \Lambda^2 : \omega \perp \Omega_{\pm}\}$.

c) Every element of $\mathcal{K}_{\mathcal{R}}$ is representable by a Kähler manifold.

d) Every element of $\mathcal{K}_{\mathcal{W}}$ is representable by a Kähler–Weyl manifold.

Gray Symmetrizer

Set $\mathcal{G}_{\pm}(R)(x,y,z,w)$

$$
\begin{aligned}
&:= R(x,y,z,w)\\
&+R(J_{\pm}x,J_{\pm}y,J_{\pm}z,J_{\pm}w)\\
&\pm R(J_{\pm}x,J_{\pm}y,z,w)\pm R(J_{\pm}x,y,J_{\pm}z,w)\\
&\pm R(J_{\pm}x,y,z,J_{\pm}w)\pm R(x,J_{\pm}y,J_{\pm}z,w)\\
&\pm R(x,J_{\pm}y,z,J_{\pm}w)\pm R(x,y,J_{\pm}z,J_{\pm}w).
\end{aligned}
$$

Theorem Let $(M, g, J_\pm)$ be para or pseudo Hermitian. Let $\nabla g = -2\phi \otimes g$. Let $m \geq 4$. Then $\mathcal{G}_\pm(R) = 0$.

Representation Theory

Structure groups act by pullback on $\bigotimes^k V^*$

$$\mathcal{O} = \{T \in GL : T^*\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle\}$$
$$\mathcal{U}_\pm^\star = \{T \in \mathcal{O} : TJ_\pm = J_\pm T \text{ or } TJ_\pm = -J_\pm T\}.$$

Theorem Let $(V, \langle\cdot,\cdot\rangle, J_\pm)$ be an orthogonal, para or pseudo-Hermitian vector space. Let ξ be a $\mathcal{G} = \mathcal{O}$, $\mathcal{U}_-$ or $\mathcal{U}_\pm^\star$ submodule of $\otimes^k V$.

1) $\langle\cdot,\cdot\rangle$ is non-degenerate on ξ.

2) There is an orthogonal direct sum decomposition $\xi = \eta_1 \oplus \ldots \oplus \eta_k$ where the η_i are irreducible $\mathcal{G}$ modules.

3) If ξ_1 and ξ_2 are inequivalent irreducible $\mathcal{U}^\star_\pm$ submodules of ξ, then $\xi_1 \perp \xi_2$.
4) The multiplicity that an irreducible representation appears in ξ is independent of the decomposition in (2).

5) If ξ_1 appears with multiplicity 1 in ξ and if η is any $\mathcal{G}$ submodule of ξ, then either $\xi_1 \subset \eta$ or else $\xi_1 \perp \eta$.
6) If $0 \to \xi_1 \to \xi \to \xi_2 \to 0$ is a short exact sequence of $\mathcal{G}$ modules, then
$\xi \approx \xi_1 \oplus \xi_2$ as a $\mathcal{G}$ module.
Remark: This fails for $\mathcal{U}_+$.

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