On the Geometric Structure of Ori Spacetimes

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joint work with J. Dietz and A. Dirmeier

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CTCs

In Lorentzian manifolds, closed timelike curves – CTCs – are the worldlines of material particles that are closed. So from the mathematicians point of view they are simple entities.

Chronology

Spacetimes without CTCs are denoted as chronological; a lot of well known spacetimes have this property, like Minkowski, Friedmann and Robertson-Walker spacetimes (globally hyperbolic).

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The main problem (from the physicists point of view) is that most of the models are not very physical (energy conditions, matter ...).

Short list

Of interest in the last decades were the following models allowing CTCs (the list is not complete):

- Lanczos-cylinder (1924)
- Van Stockum-dust (1936)
- Gödel spacetime (1949)
- Kerr-vacuum (1963)
- Tipler-cylinder (1974)
- Generalized Gödel-type models (2010)

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- Tipler-cylinder (1974)
- Generalized Gödel-type models (2010)
- Ori spacetimes (1993,...,2011)

Ori (1993)

Perturbation of the Minkowski spacetime (cylindrical coordinates):

$$ds^2 = dt^2 - dr^2 - r^2 d\phi^2 - dz^2 - d\sigma^2$$

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with

$$\begin{aligned} d\sigma^2 &= 2rh(\rho)\left(at \cdot dt - b\left((r - r_0)dr + zdz\right)\right) d\phi + \\ & \quad r^2 h^2(\rho)\left(b^2\rho^2 - a^2 t^2\right) d\phi^2. \end{aligned}$$

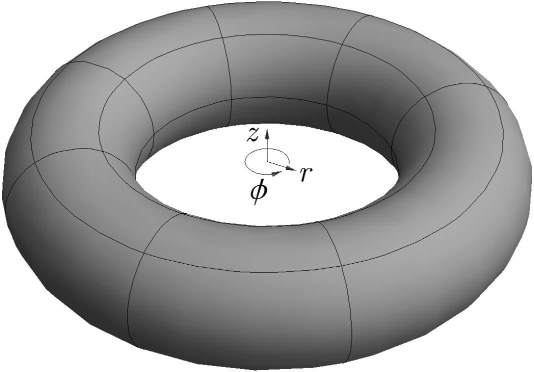
Ori (1993)

Here

$$h(\rho) = \begin{cases} \left(1 - \left(\frac{\rho}{d}\right)^4\right)^3, & \rho < d \\ 0, & otherwise, \end{cases}$$

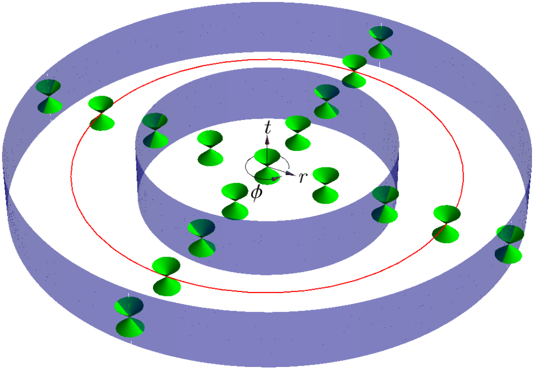
where $a, b, r_0 > 0$, $0 < d < r_0$ are parameters and $\rho^2 = (r - r_0)^2 + z^2$. The factor $h(\rho)$ restricts the perturbation to the interior of the described torus.

Ori (1993)



Inside the torus

The light cones tip over with increasing t.



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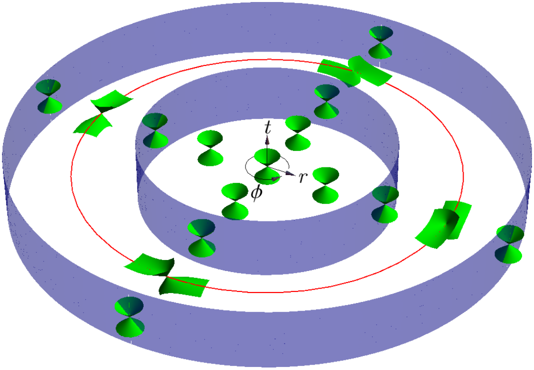

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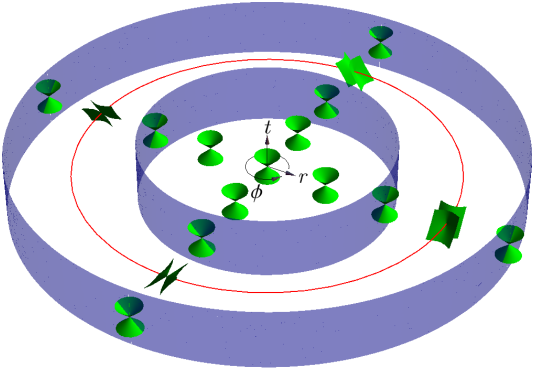

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The line element reads

$$\begin{aligned} ds^2 &= 2dzdt - dx^2 - dy^2 - (e\rho^2 - t)dz^2 - \\ &\quad 2\left((2e - a)xdx + (2e - a)ydy\right) dz \end{aligned}$$

with $\rho^2 = x^2 + y^2$, $e, a > 0$. Here the z-coordinate is periodic; $z \in [0, L]$, $L > 0$; $z = 0$ and $z = L$ are identified.

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- The hypersurfaces $t = const$ are mixed at $t \geq 0$:
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- Since $g_{zz} = e\rho^2$, the curves ∂_z are timelike at $t > e\rho^2$

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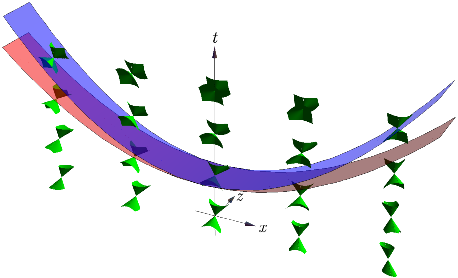

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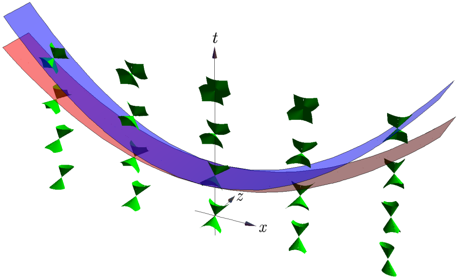

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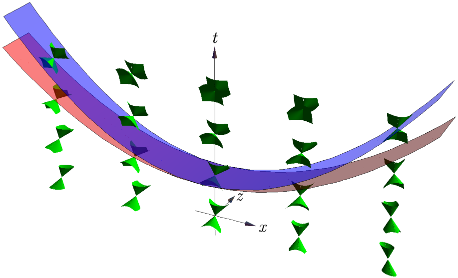

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Causality border for the directions ∂_z

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- Base: $Z = \mathbb{S}^1 \times \mathbb{R}^+$ with $g_1 = \left(1 - \frac{2m}{r}\right) d\nu^2 + 2 d\nu dr$

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- Fiber: H^2 with $g_2 = d\theta^2 + \sinh^2(\theta) d\phi^2$ (metric of the hyperbolic plane with radius 1)
  - $(0, \infty) \times \mathbb{S}^1 \to \mathbb{R}^2, \quad (\theta, \phi) \mapsto (\sinh\theta\cos\phi, \sinh\theta\sin\phi)$

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- Result: $\mathcal{C}^2=\dot{r}^2+V(r)$ with

$$V(r)=\left(1-\frac{2m}{r}\right)\left(k-\frac{\mathcal{H}^2}{r^2}\right).$$

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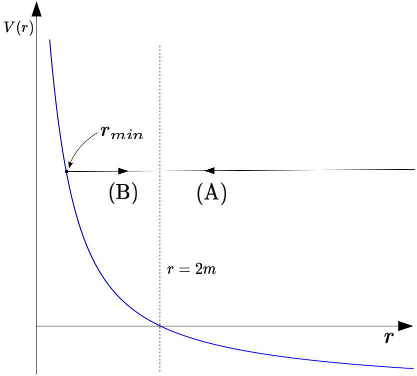
Causal geodesics:



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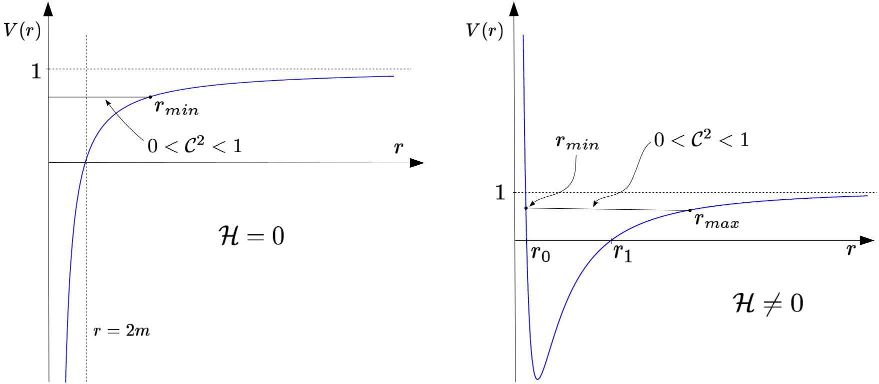
Spacelike geodesics:



Generalized spacetime (2011, J. Dietz, A. Dirmeier, M. S.)

Consider the manifold $M = \mathbb{S}^1 \times \mathbb{R}^3$ with the metric

$$g = g_{\nu\nu} d\nu^2 - 2d\nu dr + 2g_{\nu\phi} d\nu d\phi - 2a \sinh^2(\theta) dr d\phi + \rho^2 d\theta^2 + g_{\phi\phi} d\phi^2$$

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with

$$g_{\nu\nu} = 1 - \frac{2mr}{\rho^2},$$
$$g_{\nu\phi} = -\frac{2mar}{\rho^2} \sinh^2(\theta),$$
$$g_{\phi\phi} = \left(r^2 + a^2 - \frac{2ma^2 r}{\rho^2} \sinh^2(\theta) \right) \sinh^2(\theta),$$

where $\rho^2 = r^2 + a^2 \cosh^2(\theta)$.

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The so called Kerr function

$$\Delta(r) = r^2 - 2mr + a^2$$

has two zeros $0 < r_0 < r_1 < 2m$ given by

$$r_i = m + (-1)^i \sqrt{m^2 - a^2}.$$

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This separates M into the regions $\mathrm{R_I}$, $\mathrm{R_{II}}$, $\mathrm{R_{III}}$ for which $r_1 < r$, $r_0 < r < r_1$ and $r < r_0$, respectively.

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Furthermore, we have two horizons H_i characterized by $r = r_i$ with their union H.

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Fix some $r_s > r_1$ that determines the spacelike hypersurface

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and consider the subset

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The constant $\theta_0 > 0$ will be subject to constraints, such that the future domain of dependence $D^{+}(\mathscr{C})$ contains points on the horizon H_1 in its closure.

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