Spacelike hypersurfaces of constant higher order mean curvature in generalized Robertson-Walker spacetimes

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The results that will be presented in the talk can be found in the paper

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written in collaboration with L.J. Alías and M. Rigoli to appear on Math.Proc. Cambridge Philos. Soc.

In what follows we consider a n-dimensional Riemannian manifold $\mathbb{P}^n$ and let I be an open interval of the real line. We let $M^{n+1} := -I \times_\rho \mathbb{P}^n$ to denote the Lorentzian warped product endowed with the Lorentzian metric

$$\langle , \rangle = -dt^2 + \rho^2(t) \langle , \rangle_{\mathbb{P}} .$$

Following the terminology introduced by Alías, Romero and Sánchez we will refer to $-I \times_\rho \mathbb{P}^n$ as a generalized Robertson- Walker spacetime.

GRW spacetimes

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Following the terminology introduced by Alías, Romero and Sánchez we will refer to $-I \times_\rho \mathbb{P}^n$ as a generalized Robertson- Walker spacetime. In any GRW spacetime there is a distinguished family of spacelike hypersurfaces, the so-called *slices*. Each of them is defined as a leaf of the foliation $t \to \mathbb{P}_t := \{t\} \times \mathbb{P}$ of M determined by the vector field $T := \frac{\partial}{\partial t}$ and is a totally umbilical spacelike hypersurface with constant mean curvature.

Problem

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Recall that a spacetime obeys the timelike convergence condition (TCC) if its Ricci curvature is nonnegative on timelike directions. It is not difficult to see that a generalized Robertson-Walker spacetime $-I \times_\rho \mathbb{P}^n$ obeys TCC if and only if

$$\mathrm{Ric}_{\mathbb{P}} \geq (n-1) \sup_I ((\log \rho)'' \rho^2) \langle , \rangle_{\mathbb{P}} , \tag{1}$$

and

$$\rho'' \leq 0. \tag{2}$$

Any of the two conditions above implies separately that spacelike slices are the only compact CMC spacelike hypersurfaces in $-I \times_\rho \mathbb{P}^n$. In particular, the weaker condition $(\log \rho)'' \leq 0$ instead of (2) is sufficient to obtain the uniqueness.

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- L. J. Alías and A. G. Colares, Uniqueness of spacelike hypersurfaces with constant higher order mean curvature in generalized Robertson-Walker spacetimes, *Math. Proc. Cambridge Philos. Soc.* **143** (2007), no. 3, 703–729.
- L. J. Alías, A. Romero and M. Sánchez, Spacelike hypersurfaces of constant mean curvature and Calabi-Bernstein type problems. *Tohoku Math. J.* **49** (1997), 337–345.
- S. Montiel, Uniqueness of spacelike hypersurfaces of constant mean curvature in foliated space- times. *Math. Ann.* **314** (1999), 529–553.
- L. J. Alías and S. Montiel, Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized Robertson–Walker spacetimes. *Differential Geometry* (Valencia, 2001), 59–69.
- M. Caballero, A. Romero, and R. M. Rubio, Constant mean curvature spacelike surfaces in three-dimensional generalized Robertson-Walker spacetimes, *Lett. Math. Phys.* **93** (2010), no. 1, 8–105.

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Consider a spacelike hypersurface $f : \Sigma^n \to M^{n+1}$. We let $A : T\Sigma \to T\Sigma$ denote the second fundamental form of the immersion. Its eigenvalues $k_1, ..., k_n$ are the principal curvatures of the hypersurface. Their elementary symmetric functions

$$S_k = \sum_{i_1<..<i_k} k_{i_1} \cdots k_{i_k}, \qquad k = 1, ..., n, \qquad S_0 = 1,$$

define the k-mean curvatures of the immersion via the formula

$$\binom{n}{k} H_k = (-1)^k S_k.$$

Higher order mean curvatures

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define the k-mean curvatures of the immersion via the formula

$$\binom{n}{k} H_k = (-1)^k S_k.$$

Thus $H_1 = -1/n \mathrm{Tr}(A) = H$ is the mean curvature and

$$n(n-1)H_2 = \overline{S} - S + 2\overline{\mathrm{Ric}}(N, N),$$

where S and $\overline{S}$ are, respectively, the scalar curvature of Σ and M^{n+1} and $\overline{\mathrm{Ric}}$ is the Ricci tensor of the generalized Robertson-Walker spacetime.

Newton operators

Newton operators

The classical Newton transformations associated to the immersion are defined inductively by

$$P_0 = I, \qquad P_k = \binom{n}{k} H_k I + A P_{k-1},$$

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It follows by the definition that the operator L_k is elliptic if and only if P_k is positive definite.

Observe for further use that if $H_2 > 0$ on Σ then the operator P_1 is positive definite. Moreover, it can be proved that in case $3 \leq k \leq n$, if there exists an elliptic point on Σ, with respect to an appropriate choice of the Gauss map N, and $H_k > 0$, then each P_j is positive definite, for all $1 \leq j \leq k-1$.

Analytical tools

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Given a compact Riemannian manifold Σ, for any function $u \in C^2(\Sigma)$ there exists a point $p_{\max} \in \Sigma$ with the properties

$$\text{(i) } u(p_{\max}) = \max_{\Sigma} u, \ \text{(ii) } \|\nabla u(p_{\max})\| = 0, \ \text{(iii) } \Delta u(p_{\max}) \leq 0$$

Equivalently, there exists a point $p_{\min} \in \Sigma$ with the properties

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where L is any elliptic operator.

Omori-Yau maximum principle for the Laplacian

[1] Maximum principles on Riemannian manifolds and applications, *Mem. Amer. Math. Soc.* 174 (2005), no. 822, x+99.

Omori-Yau maximum principle for the Laplacian

Definition (Pigola, Rigoli, Setti[1])

Let Σ be a Riemannian manifold. The Omori-Yau maximum principle is said to hold on Σ for the Laplace operator if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

$$\text{(i) } u(p_j) > u^* - \frac{1}{j}, \ \text{(ii) } \|\nabla u(p_j)\| < \frac{1}{j}, \ \text{(iii) } \Delta u(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$. Analogously, for any $u \in C^2(\Sigma)$ with $u_* = \inf_\Sigma u > -\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ such that

$$\text{(i) } u(p_j) < u_* + \frac{1}{j}, \ \text{(ii) } \|\nabla u(p_j)\| < \frac{1}{j}, \ \text{(iii) } \Delta u(p_j) > -\frac{1}{j}$$

for every $j \in \mathbb{N}$.

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Definition (Alias, I., Rigoli[2])

Let Σ be a Riemannian manifold and let $L = \mathrm{Tr}(P \circ \mathrm{hess})$ be an elliptic operator. The Omori-Yau maximum principle is said to hold on Σ for the operator L if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup_\Sigma u < +\infty$, there exists a sequence $\{p_j\}_{j\in\mathbb{N}} \subset \Sigma$ with the properties

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for every $j \in \mathbb{N}$.

[2] Hypersurfaces of constant higher order mean curvature in warped product spaces, Preprint.

Omori-Yau maximum principle for trace operators

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Lemma

Let $(\Sigma, \langle , \rangle)$ *be a complete Riemannian manifold with sectional curvature bounded from below. Then, the OY maximum principle holds on* Σ *for any elliptic operator* $L = \mathrm{Tr}(P \circ \mathrm{hess})$ *with* $\sup_{\Sigma} \mathrm{Tr} P < +\infty$*.*

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Corollary

Let $-I \times_\rho \mathbb{P}^n$ *be a generalized Robertson-Walker spacetime with warping function satisfying* $(\log \rho)'' \leq 0$ *and Riemannian fiber* $\mathbb{P}^n$ *having sectional curvature bounded from below. Let* $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ *be a complete spacelike hypersurface contained in a slab and assume that either*

1. H_2 *is positive, or*
2. H_k *is positive (with* $k \geq 3$*) and there exists an elliptic point in* Σ.

If $\sup_\Sigma |H_1| < +\infty$, *then the sectional curvature of* Σ *is bounded from below and the OY maximum principle holds on* Σ *for every elliptic operator* L *as above.*

Main result

Theorem

Let $-I \times_\rho \mathbb{P}^n$ be a generalized Robertson-Walker spacetime whose warping function satisfies $(\log \rho)'' \leq 0$, with equality only at isolated points, and suppose that $\mathbb{P}^n$ has sectional curvature bounded from below. Let $f : \Sigma^n \to -I \times_\rho \mathbb{P}^n$ be a complete spacelike hypersurface contained in a slab and assume that either

1. *H_2 is a positive constant, or*
2. *H_k is constant (with $k \geq 3$) and there exists an elliptic point in Σ.*

If $\sup_\Sigma |H_1| < +\infty$, then $f(\Sigma)$ is a slice.

Proof

First, we introduce the function $h = \pi_I \circ f$, which is called the *height function* and we observe that we are done if we show that the function h is constant.

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We denote by σ a primitive of the warping function and by Θ the *angle function* which is defined by

$$\Theta := \langle N, T \rangle .$$

The angle function Θ is globally defined and satisfies $\Theta \leq -1 < 0$.

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We denote by σ a primitive of the warping function and by Θ the *angle function* which is defined by

$$\Theta := \langle N, T \rangle .$$

The angle function Θ is globally defined and satisfies $\Theta \leq -1 < 0$.
Observe that the assumptions (1) and (2) imply that the operators L_j are elliptic for all $1 \leq j \leq k-1$, $k \geq 2$, and that all the j-mean curvatures are strictly positive, $1 \leq j \leq k-1$. Using the Omori-Yau maximum principle for the Laplacian applied to the equation

$$\Delta h = -(\log \rho)'(h)(n + \|\nabla h\|^2) - n\Theta H_1$$

it is not difficult to show that, for the chosen orientation

$$(\log \rho)'(h)\Theta < 0.$$

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Let us focus first on the case $k = 2$ and consider the operator

$$\mathcal{L} = -\frac{1}{\Theta}\frac{c_1}{c_0}(\log\rho)'(h)\Delta + L_1 = \mathrm{Tr}(\mathcal{P}\circ \mathrm{hess}),$$

where

$$\mathcal{P} = -(n-1)\frac{(\log\rho)'(h)}{\Theta}I + P_1 = (n-1)\left|\frac{(\log\rho)'(h)}{\Theta}\right|I + P_1$$

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is positive definite. Since $|1/\Theta| \leq 1$, then $\sup_\Sigma |(\log\rho)'(h)| < +\infty$ and $\sup_\Sigma H_1 < +\infty$,

$$\mathrm{Tr}\mathcal{P} = c_1\left(\left|\frac{(\log\rho(h))'}{\Theta}\right| + H_1\right) < +\infty.$$

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$$\mathrm{Tr}\mathcal{P} = c_1\left(\left|\frac{(\log\rho(h))'}{\Theta}\right| + H_1\right) < +\infty.$$

Hence $\mathcal{L}$ is an elliptic operator and the trace of $\mathcal{P}$ is bounded from above. We can then apply the Omori-Yau maximum principle for trace operators.

Proof

Since $h^* < +\infty$ there exists a sequence $\{p_j\} \subset \Sigma$ such that

$$\lim_{j \to +\infty} (\sigma \circ h)(p_j) = (\sigma \circ h)^* = \sigma(h^*),$$

$$\|\nabla(\sigma \circ h)(p_j)\| = \rho(h(p_j))\|\nabla h(p_j)\| < \frac{1}{j},$$

$$\mathcal{L}(\sigma \circ h)(p_j) < \frac{1}{j}.$$

Using

$$\mathcal{L}\sigma(h) = -\frac{c_1}{\Theta}\rho(h)(-(\log \rho)'(h)^2 + \Theta^2 H_2),$$

evaluated at p_j, taking the limit for $j \to +\infty$ and observing that $\Theta(p_j) \to \operatorname{sgn}\Theta = \pm 1$ as $j \to +\infty$, we find

$$0 \geq \operatorname{sgn}\Theta((\log \rho)'(h^*)^2 - H_2).$$

Proof

On the other hand, since h is bounded from below, we can find a sequence $\{q_j\} \subset \Sigma$ such that

$$\lim_{j\to+\infty}(\sigma \circ h)(q_j) = (\sigma \circ h)_* = \sigma(h_*),$$
$$\|\nabla(\sigma \circ h)(q_j)\| = \rho(h(q_j))\|\nabla h(q_j)\| < \frac{1}{j},$$
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Hence, proceeding as above we find

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Hence, proceeding as above we find

$$0 \leq \operatorname{sgn}\Theta((\log\rho)'(h_*)^2 - H_2).$$

Summarizing

$$\operatorname{sgn}\Theta(\log\rho)'(h^*)^2 \leq \operatorname{sgn}\Theta H_2 \leq \operatorname{sgn}\Theta(\log\rho)'(h_*)^2.$$

Taking into account that $(\log\rho)'(h)\Theta < 0$ and that $(\log\rho)'$ is a decreasing function, we get $h^* = h_* = constant$.

Proof

For the general case $k \geq 3$, consider the operator

$$\mathcal{L} = \mathrm{Tr}\Big(\Big[\sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i}\Big(-\frac{(\log\rho)'(h)}{\Theta}\Big)^{k-1-i} P_i\Big] \circ \mathrm{hess}\Big)$$
$$= \sum_{i=0}^{k-1} \frac{c_{k-1}}{c_i}\Big|\frac{(\log\rho)'(h)}{\Theta}\Big|^{k-1-i} L_i.$$

Since $\mathcal{L}$ is a positive linear combination of the L_i's, it is elliptic.

Proof

Furthermore, since, by the Newton inequalities

$$H_j \leq H_1^j < +\infty,$$

each H_j is bounded from above.

$$\begin{aligned} \mathrm{Tr}(\mathcal{P}) =& c_{k-1} \sum_{i=0}^{k-1} \left| \frac{(\log \rho)'(h)}{\Theta} \right|^{k-1-i} H_i \\ \leq & c_{k-1} \sum_{i=0}^{k-1} \sup_{\Sigma} |(\log \rho)'(h)|^{k-1-i} \sup_{\Sigma} H_1^i < +\infty \end{aligned}$$

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It is not difficult to prove that

$$\mathcal{L}\sigma(h) = \frac{c_{k-1}}{(-\Theta)^{k-1}}\rho(h)(-((\log\rho)'(h))^k + (-1)^k\Theta^k H_k).$$

We can then apply the Omori-Yau maximum principle to the operator $\mathcal{L}$ and conclude as in case $k = 2$.

On the compact case

Assume that Σ is a compact spacelike hypersurface. As we have seen, in this case the Omori–Yau maximum principle can be replaced by the properties of the maximum or minimum of a certain function, which always hold on a compact Riemannian manifold, without any assumption on the sectional curvature.

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Assume that Σ is a compact spacelike hypersurface. As we have seen, in this case the Omori–Yau maximum principle can be replaced by the properties of the maximum or minimum of a certain function, which always hold on a compact Riemannian manifold, without any assumption on the sectional curvature.

Summarizing we obtain the following

Theorem

Let $-I \times_\rho \mathbb{P}^n$ be a spatially closed generalized Robertson-Walker spacetime with warping function satisfying $(\log \rho)'' \leq 0$. The only compact spacelike hypersurfaces satisfying either

1. *H_2 is a positive constant, or*
2. *H_k is constant (with $k \geq 3$) and there exists an elliptic point in Σ*

are spacelike slices.

Thank you!