UNIVERSIDAD DE MÁLAGA

The c-boundary of spacetimes and its related boundaries in Differential Geometry

Jónatan Herrera Fernández

(joint work with José Luis Flores and Miguel Sánchez)

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INDEX

J.L. Flores, J. Herrera, M. Sánchez, On the final definition of the causal boundary and its relations with the conformal boundary, *Adv. Theor. Math. Phys.*, to appear.

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Attach an ideal boundary for any spacetime in such a way:

- Any inextendible timelike curve has an endpoint in the boundary.
- The boundary extends naturally the causal and topological structures.

Different solutions:

- Conformal boundary: Common in Relativity.
- Causal boundary: Consistent and totally justified definition. (J.L. Flores, J.H., M. Sánchez '10)

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Intuitive Ideas

- Initial GKP construction ('72) for **strongly causal** spacetimes:
  - For any inextendible future or past timelike curves, attach an ideal point.
  - Two inextendible future (past) timelike curves are attached to the same ideal point if they have the same past (resp. future).
  - The future (past) causal boundary is the set of future (past) ideal points.
  - The (total) causal boundary is defined as the union of the future and past causal boundaries under "certain identifications".

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Future Causal Boundary

- Past Set: $P \subset V$ such that $I^-[P] = P$.
- IP: Past set which cannot be expressed by the union of two proper past sets.
- PIP: IP, $P \subset V$ such that $P = I^-(p)$ for $p \in V$.
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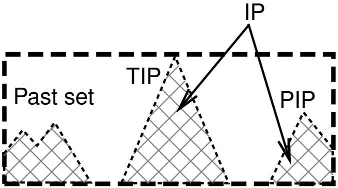

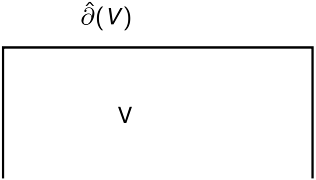

Past Causal Boundary

- Future Set: $F \subset V$ such that $I^+[F] = F$.
- IF: Future set which cannot be expressed by the union of two proper past sets.
- PIF: IP, $F \subset V$ such that $F = I^+(p)$ for $p \in V$.
- TIF: IP, $F \subset V$ such that $F \neq I^+(p)\ \forall\ p \in V$.
- We have the following identifications:

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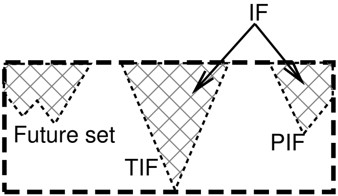
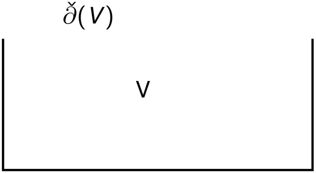


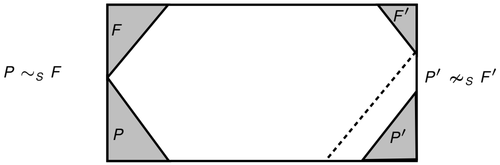


Relating boundaries

S-relation (Szabados, '88)

$P \sim_S F$ if

$$\left\{ \begin{array}{l} P \text{ is a maximal IP in } \downarrow F := I^-[\{q \in V : q \ll p, \forall p \in F\}] \\ F \text{ is a maximal IF in } \uparrow P := I^+[\{p \in V : q \ll p, \forall q \in P\}] \end{array} \right|$$



Observe $I^-(p) \sim_S I^+(p)$ for any $p \in V$.

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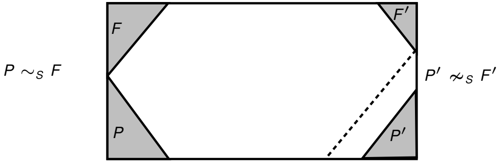

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Point set structure

c-boundary and c-completion (Marolf-Ross, '03)

$$\overline{V} := \{(P,F) \in \left(\hat{V} \cup \emptyset\right) \times \left(\check{V} \cup \emptyset\right), P \sim_S F\}$$

$$\partial V := \overline{V} \setminus V$$

$(P_1, \emptyset)$

F_0

P_1

(P_0, F_0)

p $(I^-(p), I^+(p))$

F_2

P_0

$(\emptyset, F_2)$

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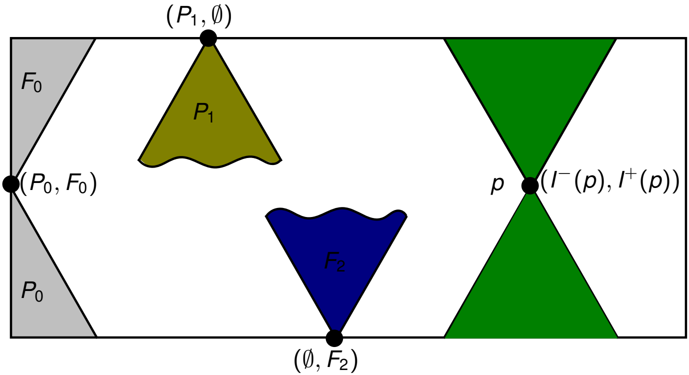

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Properties

- The computation is systematic.
- It is the maximal possible choice.
- It is unique.

C-completion as a chronological set

Extended Chronological relation

$$(P, F) \ll (P', F') \iff F \cap P' \neq \emptyset.$$

F'

$P' \cap F$

F

P'

P

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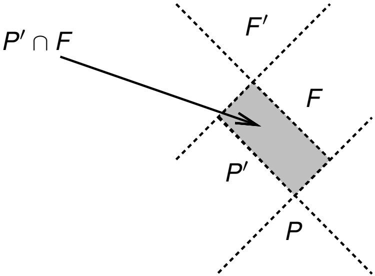

C-completion as a chronological set

Extended Chronological relation

$$(P, F) \overline{\ll} (P', F') \iff F \cap P' \neq \emptyset.$$

Properties

- Preserves the chronological relation in V.
- Does not include new (and spurious) relations.
- Is the minimal admissible.

C-completion as a topological structure

Chronological Limit (Harris, '01 ; Flores, '06)

$$(P,F) \in L(\{(P_n,F_n)\}) \Leftrightarrow \begin{cases} P \in \hat{L}(P_n) \text{ if } P \neq \emptyset \\ F \in \check{L}(F_n) \text{ if } F \neq \emptyset \end{cases}$$

where

$$P \in \hat{L}(P_n) \quad \Leftrightarrow \quad \begin{cases} P \subset LI(P_n) \\ P \text{ is a maximal } IP \text{ in } LS(P_n) \end{cases}$$

Chronological topology

A set $C \subset \overline{V}$ is closed if, and only if, $L(\sigma) \in C$ for any sequence $\sigma \subset C$.

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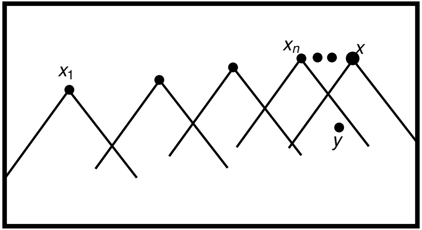

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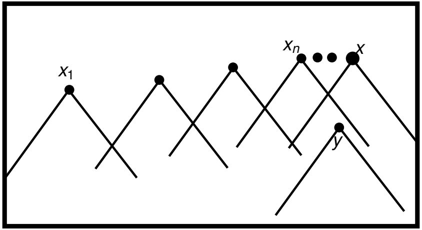

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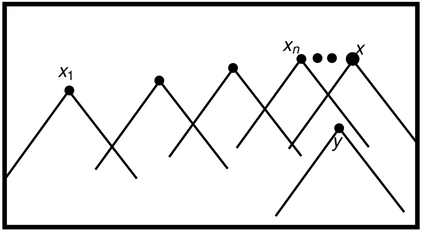

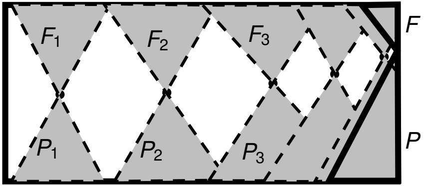

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Intrinsic properties

This definition satisfies:

- Completeness: Any timelike curve has an endpoint.
- V is an open and dense set of $\overline{V}$.
- Future and past sets are open.
- ... and many more properties of uniqueness and consistency (J.L. Flores, J.H., M. Sánchez '10).

  In particular, the consistency with the conformal boundary.

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Extrinsic properties

Assume the existence of an open conformal embedding

$$i : V \to V_0,$$

where V_0 is an (aphysical) spacetime.

Then,

- Define the conformal completion (under the conformal embedding i) as $\overline{V}_i = \overline{i(V)}$.
- $\overline{V}_i$ is endowed with a chronological and topological structure induced from V_0.

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Theorem (J.L. Flores, J.H., M. Sánchez '10)

Under the assumption of some general hypotheses of completeness and regularity:

- Any timelike curve in V has an endpoint in V_0.
- The points of the boundary are regularly accessible.

It is obtained that,

$$\overline{V} \equiv \overline{V}_i$$

at the three levels: point-set, chronological and topological.

* - If $p \ll q$, then there exists an open set $q \in U$ such that $p \ll w$ for all $w \in U$.

- If $p \ll q \leq r$, then $p \ll r$.

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In conclusion, under general and clearly delimited hypotheses, **the conformal and the causal completions coincide.**

In particular, one can state and prove the conformal completion rigorously properties claimed in the literature as the following:

Theorem (J.L. Flores, J.H., M. Sánchez '10)

Assume that a spacetime V admits a C^1 chronologically complete conformal boundary $\partial_i V$. Then:

$$V \text{ is globally hyperbolic} \iff \partial_i V \text{ has no timelike points}$$

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J.L. Flores, J. Herrera, M. Sánchez, *Gromov, Cauchy and causal boundaries for Riemannian, Finslerian and Lorentzian manifolds*, preprint 2010. Available at arXiv:1001.3270v3 [math-ph].

Symmetric case

Along this section, we consider a metric space (M, d), where:

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- d is a distance which becomes from a length space.

Remark

For simplicity, M could be considered a manifold and d a distance which becomes from a Riemannian metric.

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Cauchy completion

Cauchy sequence

- $\{x_n\} \subset M$ is a *Cauchy sequence* if $\forall\ \epsilon > 0$ there exists n_0 such that $d(x_n, x_m) < \epsilon$ for all $m, n \geq n_0$.
- $Cau(M, d) \equiv$ space of Cauchy sequences.

Relating Cauchy sequences

Two Cauchy sequences are related, $\{x_n\} \sim \{x'_n\}$, if, and only if, the *alternate sequence* $\{z_n\}$ (where $z_{2n} = x_n$ and $z_{2n-1} = x'_n$) is also a Cauchy sequence.

Cauchy completion

The *Cauchy completion* and *boundary* are defined by:

$$M_C := Cau(M, d)/\sim, \quad \partial_C M := M_C \setminus M.$$

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- The distance d is naturally extended to the Cauchy completion by defining:

$$d([\{x_n\}],[\{y_n\}]) = lim_n d(x_n, y_n)$$

...and so, it is possible to define a topology in the Cauchy completion considering balls:

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Proposition

The space M_C endowed with the topology associated to this balls satisfies:

- Completeness: Any Cauchy sequence in M_C has a limit point.
- $i : M \to M_C$ is an embedding and M is dense in M_C.
- T_2 topological space.

Notice that the Cauchy boundary $\partial_C M$ may be non-locally compact, even for a Riemannian manifold.

Proposition

The space M_C endowed with the topology associated to this balls satisfies:

- Completeness: Any Cauchy sequence in M_C has a limit point.
- $i : M \to M_C$ is an embedding and M is dense in M_C.
- T_2 topological space.

Notice that the Cauchy boundary $\partial_C M$ may be non-locally compact, even for a Riemannian manifold.

Classical Gromov Compactification

- Gromov developed a universal compactification for any complete Riemannian manifold.
- The idea is to embed the space *M* into a “larger” one, the space of Lipschitz functions, ensuring the compactness of the topological closure.
- Our aim is to extend such construction to arbitrary Riemannian manifolds.
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Lipschitz functions

Lipschitz function

- $f : M \to \mathbb{R}$ is a *Lipschitz function* if

$$|f(y) - f(x)| \leq d(x, y) \quad \text{for all } x, y \in M$$

- $\mathcal{L}_1(M, d) \equiv$ space of Lipschitz functions endowed with the pointwise topology.

Quotient space

- Consider the quotient space $\mathcal{L}_1(M, d)/\mathbb{R}$ ($f \sim f'$ iff $f - f' \equiv$ cte).

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The map

$$\begin{array}{rccc} j: & M_C & \to & \mathcal{L}_1(M,d)/\mathbb{R} \\ & x & \to & [-d(\cdot,x)] \end{array}$$

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Gromov Compactification

Definition

The *Gromov compactification* and *boundary* are defined as follows:

$$M_G := \overline{j(M)}, \quad \partial_G M = M_G \setminus M.$$

Properties (J.L. Flores, J.H., M. Sánchez '10)

- M_G and $\partial_G M$ are compact.
- M_G is second countable and Hausdorff topological space.
- $j(M) \subset M_G$ is open and dense.
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- Eberlein and O′Neill developed a compactification for Hadamard manifolds.
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Space of Busemann Functions

Busemann functions

Given any curve $c : [\alpha, \Omega) \to M$ with $|\dot{c}| \leq 1$, the *Busemann function* associated to c is defined as:

$$b_c(\cdot) := lim_{t \nearrow \Omega}(t - d(\cdot, c(t))).$$

Properties

- The limits in previous definition always exist.
- If b_c becomes infinity at some point, then it is constantly equal to ∞.
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Definition

Denote by $B(M)$ the space of finite Busemann functions.

Then, the *Busemann completion as point set* is defined as $M_B := B(M)/\mathbb{R}$.

Remarks

- $M_B \subset M_G$.
- The inclusion may be strict.
- So, M_B endowed with the Gromov topology is *NOT* compact (or complete in any sense).

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Topology on Busemann completion

Chronological operator

For any sequence $\sigma = \{f_n\} \subset B(M)$, define

$$f \in \hat{L}(\sigma) \leftrightarrow \begin{cases} f \leq liminf_n f_n \text{ and} \\ \forall g \in B(M) : f \leq g \leq limsup_n f_n, \text{ it is } g = f. \end{cases}$$

Chronological and Busemann topology

- The *chronological topology on $B(M)$* is the one such that:

$$C \text{ is closed} \iff \hat{L}(\sigma) \in C \text{ for any sequence } \sigma \subset C$$

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Busemann completion

Theorem (J.L. Flores, J.H., M. Sánchez '10)

M_B, endowed with the Busemann topology, satisfies:

(1) It is sequentially compact.

(2) M is naturally embedded as an open dense subset in M_B.

(3) M_B is T_1, and any non-T_2 related points must lie in $\partial_B M$.

(4) $M_C \subset M_B$.
The inclusion is always continuous.

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The Busemann topology is coarser than the Gromov's one.

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General case

Along this section, we consider a *generalized* metric space (M, d), where:

- M is a locally compact topological space.
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Generalized distance

A map $d : M \times M \to \mathbb{R}$ is a *generalized distance* if:

$$\left.\begin{array}{l} (a1)\ d(x,y) \geq 0 \\ (a2)\ d(x,y) = d(y,x) = 0 \iff x = y. \\ (a3)\ d(x,z) \leq d(x,y) + d(y,z). \end{array}\right\} \text{quasi-distance}$$

$(a4)\ lim_n d(x_n, x) = 0 \iff lim_n d(x, x_n) = 0.$

Symmetrized distance

The *symmetrized distance* d^s associated to a quasi-distance d is:

$$d^s(x,y) := \frac{d(x,y) + d(y,x)}{2}.$$

Even if d comes from a length space, d^s may not.

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Topology

- Define the *forward (backward) ball*:

$$B^+(x,r) := \{y \in M : \; d(x,y) < r\}$$
$$(B^-(x,r) := \{y \in M : \; d(y,x) < r\})$$

- The forward and backward balls define naturally two topologies on M, called *forward* and *backward* resp.
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Preliminaries

For simplicity, we can consider that:

- M is a manifold.
- d is a generalized distance which comes from a **Finsler** metric.

Preliminaries

About Finsler metrics:

- Essentially, it is a smooth field of positively homogeneus norms. Recall:
  - *Positively homogeneus* means $F(\lambda v) = \lambda F(v)$ if $\lambda \geq 0$.
- It has associated a *reversed* Finsler metric defined by $F^{rev}(v) = F(-v)$.
- It defines naturally a generalized distance constructed as a length space:

$$d(x, y) = inf_{\sigma \in C(x,y)} \int_0^1 F(\dot{\sigma}(t))dt$$

(where $C(x, y)$ is the set of curves from x to y).

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Cauchy completion

Cauchy sequence

- $\{x_n\} \subset M$ is a *(forward) Cauchy sequence* if $\forall\ \epsilon > 0$ there exists n_0 such that $d(x_n, x_m) < \epsilon$ for all $m \geq n \geq n_0$.
- $Cau^+(M, d) \equiv$ space of (forward) Cauchy sequences.

Relating Cauchy sequences

Two Cauchy sequences are related, $\{x_n\} \sim \{x'_n\}$, if, and only if,

$$lim_n(lim_m d(x_n, x'_m)) = lim_n(lim_m d(x'_n, x_m)) = 0.$$

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The *Cauchy completion* and *boundary* are defined by:

$$M_C^+ := Cau^+(M, d)/\sim, \quad \partial_C^+ M := M_C^+ \setminus M.$$

Cauchy completion

Cauchy sequence

- $\{x_n\} \subset M$ is a *(forward) Cauchy sequence* if $\forall\ \epsilon > 0$ there exists n_0 such that $d(x_n, x_m) < \epsilon$ for all $m \geq n \geq n_0$.
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Relating Cauchy sequences

Two Cauchy sequences are related, $\{x_n\} \sim \{x'_n\}$, if, and only if,

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Topological Structure

Proposition

The map $d_Q : M_C^+ \times M_C^+ \to [0, \infty]$ defined by

$$d_Q([\{x_n\}], [\{y_n\}]) = lim_n \left(lim_m d(x_n, y_m) \right)$$

is a well defined quasi-distance which extends d.

Remark

- Recall that, as d_Q is a quasi-distance, there exist two possible topologies (forward and backward)...
- ...and the **backward** topology will be preferred.

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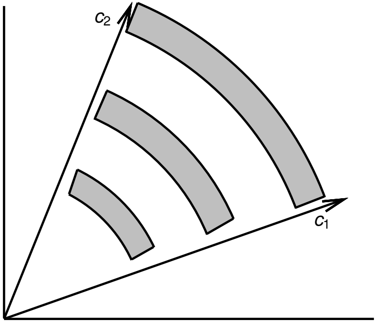
- ...and the **backward** topology will be preferred.

Theorem

The space M_C^+ endowed with the backward topology satisfies:

- Completeness: Any (forward) Cauchy sequence in M_C^+ has limit.
- $i : M \to M_C^+$ is an embedding and M is dense in M_C^+.
- M_C^+ is a T_0 topological space, but not necessarily T_1.

Example



Take $F(v) = \sqrt{g_R + \omega^2} + \omega$ where:

- g_R is a Riemannian metric and
- ω is a one-form.

g_R and ω satisfy:

- c_1 and c_2 has finite length.
- $\omega > 0$ when $\dot{\theta} > 0$.
- When $r \to \infty$, $g_R << |\omega|$.

- So, $d(z_1, z_2) > 0$ and $d(z_2, z_1) = 0$

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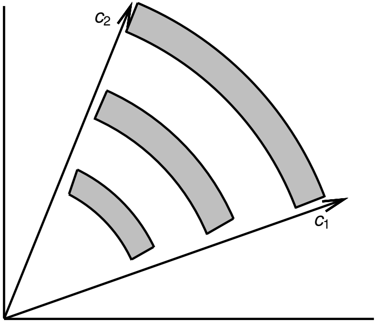

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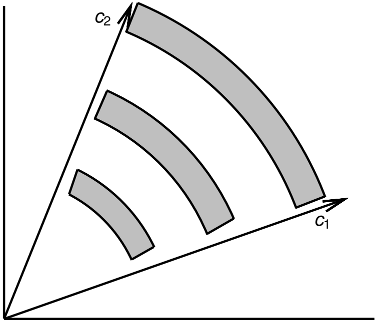

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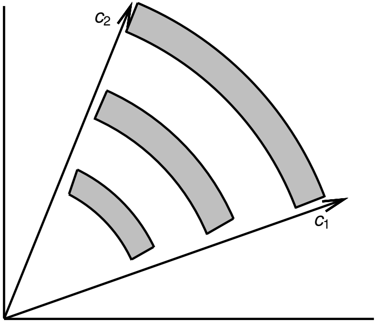

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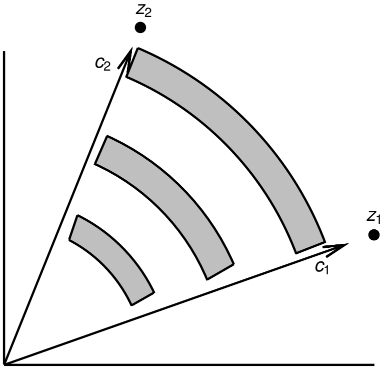

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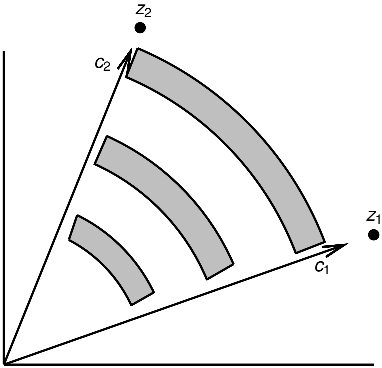

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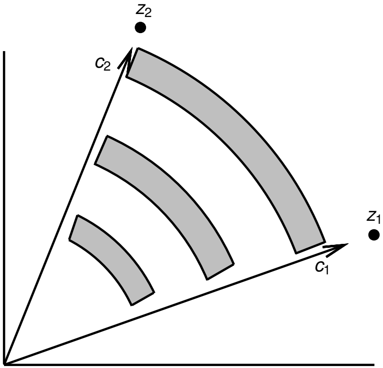

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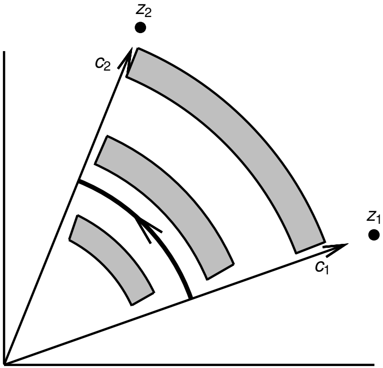

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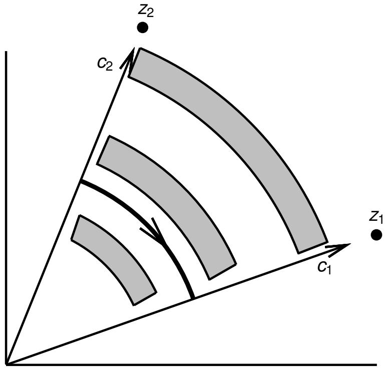

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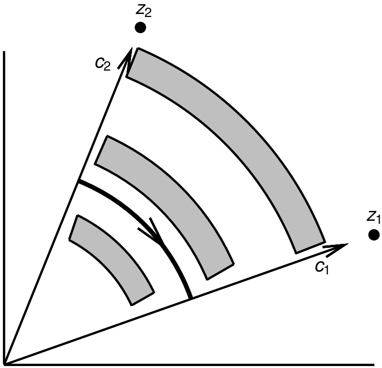

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Cauchy completion

Remarks

- Analogously, we can define the *backward Cauchy completion* M_C^- by changing the order on d.
- Both completions are different in general, and represent the asymmetric character of the generalized distance d.
- In fact,
$$\partial_C^s M = \partial_C^+ M \cap \partial_C^- M$$
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Gromov compactification

Generalized Lipschitz function

- $f : M \to \mathbb{R}$ is a *forward (backward) Lipschitz function* if

$$f(y) - f(x) \leq d(x, y) \quad \text{for all } x, y \in M$$
$$(f(y) - f(x) \leq d(y, x) \quad \text{for all } x, y \in M.)$$

- $\mathcal{L}_1^{\pm}(M, d) \equiv$ space of forward (backward) Lipschitz functions endowed with pointwise topology.

Quotient space

- Consider the quotient space $\mathcal{L}_1^{\pm}(M, d)/\mathbb{R}$ ($f \sim f'$ iff $f - f' \equiv$ cte).

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Theorem

The map

$$\begin{array}{rccc} j^{\pm}: & M_C^{\pm} & \to & \mathcal{L}_1^{\pm}(M,d)/\mathbb{R} \\ & x & \to & [\mp d_Q(\cdot,x)] \end{array}$$

where d_Q is the extension of d to $M_C^{\pm}$, satisfies:

- it is continuous $\iff$ the topology generated by backward (resp. forward) balls is finer than forward (resp. backward) ones.
- It is an embedding if $M_C^{\pm}$ is locally compact and d_Q is a generalized distance.
- In particular, $j^{\pm}|_M$ is always an embedding.

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Gromov Compactification

Definition

The *Gromov compactification* and *boundary* are defined as follows:

$$M_G^{\pm} := \overline{j^{\pm}(M)}, \quad \partial_G^{\pm} M = M_G^{\pm} \setminus M.$$

Properties (J.L. Flores, J.H., M. Sánchez '10)

- $M_G^{\pm}$ and $\partial_G^{\pm} M$ are compact.
- $M_G^{\pm}$ is second countable and Hausdorff topological space.
- $j^{\pm}(M) \subset M_G^{\pm}$ is open and dense.
- $M_C^{\pm} \subset M_G^{\pm}$.
- The topology in $M_C^{\pm}$ is finer than the induced from Gromov construction.
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Busemann completion

Busemann functions

Given any curve $c : [\alpha, \Omega) \to M$ with $F(\dot{c}) \leq 1$ ($F^{rev}(\dot{c}) \leq 1$), the *forward (backward) Busemann function* associated to c is defined as:

$$b_c^+(\cdot) := lim_{t \nearrow \Omega}(t - d(\cdot, c(t)))$$
$$(b_c^-(\cdot) := lim_{t \nearrow \Omega}(-t + d(c(t), \cdot))$$

Properties

- The limits in previous definition always exist.
- If $b_c^\pm$ becomes infinity at some point, then it is constantly equal to ∞.
- It is $b_c^\pm \in \mathcal{L}_1^\pm(M, d)$.

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Busemann completion

Definition

Denote by $B^{\pm}(M)$ the space of finite forward (backward) Busemann functions.

Then, the *forward (backward) Busemann completion as point set* is defined as $M_B^{\pm} := B^{\pm}(M)/\mathbb{R}$.

Topology on Busemann completion

Chronological operator

For any sequence $\sigma = \{f_n\} \subset B^{\pm}(M)$, define

$$f \in \hat{L}(\sigma) \leftrightarrow \begin{cases} f \leq liminf_n f_n \text{ and} \\ \forall g \in B^{\pm}(M) : f \leq g \leq limsup_n f_n, \text{ it is } g = f. \end{cases}$$

Chronological and Busemann topology

- The *chronological topology on* $B^{\pm}(M)$ is the one such that:

$$C \text{ is closed} \iff \hat{L}(\sigma) \in C \text{ for any sequence } \sigma \subset C$$

- The *Busemann topology* is the induced quotient topology on the Busemann completion $M_B^{\pm} = B^{\pm}(M)/\mathbb{R}$.

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Theorem (J.L. Flores, J.H., M. Sánchez '10)

$M_B^{\pm}$, endowed with the Busemann topology, satisfies:

(1) It is sequentially compact.

(2) M is naturally embedded as an open dense subset in $M_B^{\pm}$.

(3) $M_B^{\pm}$ is T_1, and any non-T_2 related points must lie in $\partial_B^{\pm} M$.

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The inclusion is continuous if the backward topology in $M_C^{\pm}$ is finer than the forward one.

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The Busemann topology is coarser than the Gromov's one.

(6) $M_B^{\pm} = M_G^{\pm}$ both, as point set and topologically, if and only if the chronological topology is Hausdorff.

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Busemann completion

Theorem (J.L. Flores, J.H., M. Sánchez '10)

$M_B^{\pm}$, endowed with the Busemann topology, satisfies:

(1) It is sequentially compact.

(2) M is naturally embedded as an open dense subset in $M_B^{\pm}$.

(3) $M_B^{\pm}$ is T_1, and any non-T_2 related points must lie in $\partial_B^{\pm} M$.

(4) $M_C^{\pm} \subset M_B^{\pm}$.
The inclusion is continuous if the backward topology in $M_C^{\pm}$ is finer than the forward one.

(5) $M_B^{\pm} \subset M_G^{\pm}$.
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INDEX

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Preliminaries

Definition (Standard Stationary Spacetime)

(V, g) is a *standard stationary spacetime* if $V = \mathbb{R} \times M$ and

$$g = -dt^2 + \omega \otimes dt + dt \otimes \omega + g_R$$

where ω is a one-form and (M, g_R) a Riemannian manifold.

The causal structure is totally characterized by the following two Finsler metrics of Randers type:

$$F^{\pm}(v) = \sqrt{g_R(v, v) + \omega^2(v)} \pm \omega(v)$$

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In fact,

Proposition

Take $(t_0, x_0), (t_1, x_1) \in V$. Then:

$$(t_0, x_0) \ll (t_1, x_1) \iff d(x_0, x_1) < t_1 - t_0.$$

For a future-directed timelike curve $\gamma(t) = (t, c(t))$ (i.e., $F^+(\dot{c}) < 1$),

$$\begin{aligned} I^-[\gamma] &= \{(t_0, x_0) \in V : (t_0, x_0) \ll \gamma(t) \quad \text{for some } t\} \\ &= \{(t_0, x_0) \in V : (t_0, x_0) \ll (t, c(t)) \quad \text{for some } t\} \\ &= \{(t_0, x_0) \in V : t_0 < t - d(x_0, c(t)) \quad \text{for some } t\} \\ &= \{(t_0, x_0) \in V : t_0 < \lim_t t - d(x_0, c(t))\} \\ &= \{(t_0, x_0) \in V : t_0 < b_c^+(x_0)\} \end{aligned}$$

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Partial boundaries

So,

$$\{\text{Set of IPs }\} \equiv \{\text{ (Forward) Busemann Functions}\} \cup \{\infty\}.$$

In particular,

$$\begin{aligned}\hat{\partial} V &\equiv \left(\partial_B^+ M \times \mathbb{R}\right) \cup \{\infty\} \\ &\equiv \left(\partial_C^+ M \times \mathbb{R}\right) \cup \left(\partial_{\mathcal{B}}^+ M \times \mathbb{R}\right) \cup \{\infty\}\end{aligned}$$

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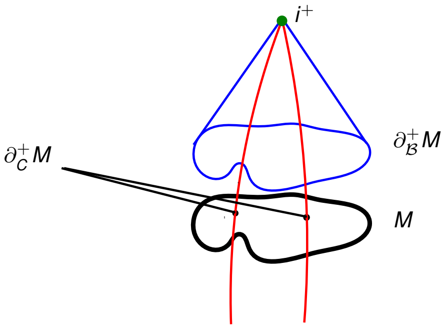
Partial Boundaries

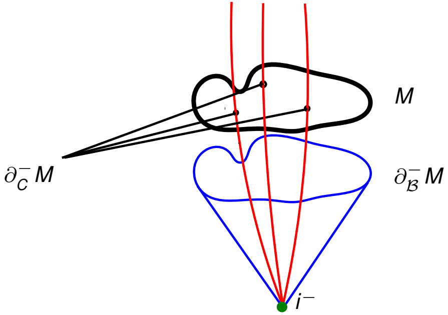
$$\hat{\partial} V = \{\partial^+_C M \times \mathbb{R}\} \cup \{\partial^+_{\mathcal{B}} M \times \mathbb{R}\} \cup i^+$$



Partial Boundaries

And, analogously,

$$\check{\partial} V = \{\partial_{\mathcal{C}}^{-} M \times \mathbb{R}\} \cup \{\partial_{\mathcal{B}}^{-} M \times \mathbb{R}\} \cup i^{-}$$



Main result

Theorem (J.L. Flores, J.H., M. Sánchez '10)

Consider V a standard stationary spacetime such that: (a) the quasi-distance d_Q^+ is a generalized metric, (b) $\partial_C^s M$ is locally compact and (c) the Busemann boundaries $\partial_B^\pm M$ are Hausdorff. Then:

- The preboundaries $\hat{\partial} V$ and $\check{\partial} V$ have the structures of cones with bases $\partial_B^+ M, \partial_B^- M$ and apexes i^+, i^- resp.
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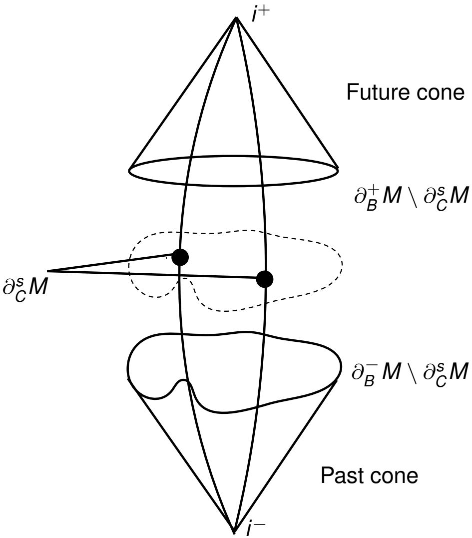
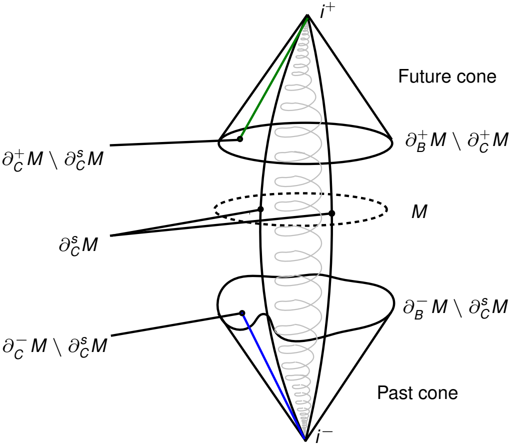


i^+
Future cone
$\partial_B^+ M \setminus \partial_C^s M$
$\partial_C^s M$
$\partial_B^- M \setminus \partial_C^s M$
Past cone
i^-

General case



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Thanks for your attention