Area-Angular momentum inequality in stable marginally trapped surfaces

J.L. Jaramillo[1,2], M. Reiris[1] and S. Dain[3]

1) Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut
Am Mühlenberg 1, D-14476 Golm, Germany

2) Laboratoire Univers et Théories (LUTH), UMR 8102 du C.N.R.S.
Observatoire de Paris, Université Paris Diderot
F-92190 Meudon, France

3) Facultad de Matemática, Astronomía y Física, FaMAF, Universidad Nacional de Córdoba
Instituto de Física Enrique Gaviola, IFEG, CONICET
Córdoba, Argentina

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Scheme

Outline

Motivation I: Mass-Angular Momentum inequality

Bounds on black hole angular momentum

Isolated and stationary black holes, consistent with (weak) cosmic censorship, cannot rotate arbitrarily fast.

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(Strongly asymptotically predictable) Kerr spacetimes (axisymmetric) satisfy

$$\boxed{|J| \leq M^2} \ ,$$

with M the total (ADM) mass and J a Komar (angular momentum) quantity associated with an axial Killing vector η^a and a closed $\mathcal{S}$ surface:

$$J = \frac{1}{16\pi} \int_{\mathcal{S}} \nabla^a \eta^b dS_{ab} \ .$$

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Dynamical case: *gravitational collapse picture*

Inequality $J \leq M^2$ holds for non-stationary, axially symmetric, black holes. (In an Initial Value Problem approach, it holds for vacuum, maximal, asymptotically flat, axisymmetric data [Dain 06, 08...; Chruściel *et al.* 08, 09]).

Motivation I: quasi-local bound on the Angular Momentum

Limits of the Mass-Angular Momentum inequality: M is a global quantity

- A *quasi-local* version desirable to gain insight into gravitational collapse in the presence of matter and/or multiple horizons.
- Difficulties with a quasi-local notion of mass in General Relativity.

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A quasi-local bound for the angular momentum for *black hole horizons*

On *appropriate* closed surfaces in black hole spacetimes, consider

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Previous results

- Stationary axisymmetric black hole spacetime with **matter** *surrounding* the horizon [Ansorg, Cederbaum, Hennig 08, 10, 11].
- Dynamical axisymmetric **vacuum** black hole spacetimes [Dain 10, Aceña, Dain, Gabach-Clément 11; Dain & Reiris 11].

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Requirements on the closed surfaces $\mathcal{S}$

Need of: i) **geometric characterization** of $\mathcal{S}$, and ii) **stability condition**.

- Stationary matter case: *trapping outer, marginally trapped surfaces*.
- Dynamical vacuum case: use of *stable, minimal surfaces* [Dain & Reiris 11].

Motivation II: from *Riemannian* to Lorentzian geometry

Geometric inequalities: shift to a Lorentzian perspective

- This work: adaptation of the proof in [Dain & Reiris 11] for minimal surfaces to marginally (outer) trapped surfaces.
- *Reconciliate* and *extend* the previous results (removal of technical assumptions, and inclusion of matter).
- Gain of geometric insight when employing spacetime (Lorentzian) concepts and tools.

Scheme of the proof in [Dain & Reiris 11]

Two steps in the proof of $A \geq 8\pi|J|$:

1. Lower bound on the area: $A \geq 4\pi \cdot f(\mathcal{M})$.
   - 1.a) Geometric inequality from the stability condition.
   - 1.b) Choice of preferred coordinate system on an axially symmetric sphere: inequality in terms of the action functional $\mathcal{M}$.
2. Upper bound on the angular momentum: $f(\mathcal{M}) \geq 2|J|$.
   Variational tools on the action functional $\mathcal{M}$.

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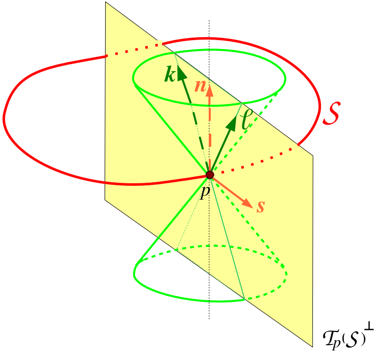
Outline

Notation I: surface $\mathcal{S}$

Let us consider a closed orientable 2-surface $\mathcal{S}$ embedded in a (4-dimensional) spacetime M with metric g_{ab} and Levi-Civita connection ∇_a.

Some elements of $\mathcal{S}$:

- Induced metric on $\mathcal{S}$, as q_{ab}.
- Levi-Civita connection D_a, Ricci scalar ${}^2R$ and area form ϵ_{ab} (area measure dS).
- Normal $T(\mathcal{S})^\perp$ plane spanned by null normals: ℓ^a and k^a. Normalization: $\ell^a k_a = -1$.



Null-rescaling freedom

Null normals ℓ^a and k^a fixed up to a (boost) rescaling freedom (f function on $\mathcal{S}$):

$$\ell'^a = f\ell^a \quad , \quad k'^a = f^{-1}k^a \ .$$

Notation II: some elements of the extrinsic geometry of $\mathcal{S}$

Expansion, shear and normal fundamental 1-form

Expansion $\theta^{(\ell)}$ and shear $\sigma^{(\ell)}_{ab}$ associated with the (*outgoing*) null normal ℓ^a:

$$\theta^{(\ell)} = q^{ab}\nabla_a \ell_b \quad , \quad \sigma^{(\ell)}_{ab} = q^c{}_a q^e{}_b \nabla_c \ell_d - \frac{1}{2}\theta^{(\ell)} q_{ab} \ .$$

Normal fundamental 1-form associated with ℓ^a (connection on $T(\mathcal{S})^\perp$):

$$\Omega^{(\ell)}_a = -k^c q^d{}_a \nabla_d \ell_c \ .$$

Transformation properties under null normal rescaling

Under a rescaling $\ell^a \to f\ell^a$ and $k^a \to f^{-1}k^a$:

$$\theta^{(\ell')} = f\theta^{(\ell)} \ , \ \sigma^{(\ell')}_{ab} = f\sigma^{(\ell)}_{ab} \qquad \Omega^{(\ell')}_a = \Omega^{(\ell)}_a + D_a(\ln f).$$

Definition I: Angular Momentum

Let η^a be an axial Killing vector on $\mathcal{S}$. That is, η^a is a vector field on $\mathcal{S}$ satisfying $\mathcal{L}_\eta q_{ab} = 0$, with closed integral curves, vanishing exactly at two points on $\mathcal{S}$ and normalized such that its integral curves have an affine length of 2π.

Angular momentum on $\mathcal{S}$

Definition 1. We introduce the angular mometum assoaciated with η^a as:

$$J = \frac{1}{8\pi}\int_{\mathcal{S}} \Omega^{(\ell)}_a \eta^a dS \ .$$

Remarks:

- J is independent of the normalization of ℓ^a: use $D_a\eta^a = 0$ and the transformation rule for $\Omega^{(\ell)}_a$.
- If η^a can be extended to a Killing vector in a spacetime neighbourhood of $\mathcal{S}$, then J coincides with the Komar angular momentum.

Definition II: spacetime stability condition for MOTS

Let $\mathcal{S}$ be a *marginally (outer) trapped surface* (MOTS) , i.e. $\theta^{(\ell)} = 0$.

Spacetime stably outermost MOTS

Definition 2. *Given a closed MOTS $\mathcal{S}$ we will refer to it as spacetime stably outermost if there exists an outgoing ($-k^a$-oriented) vector $X^a = \gamma\ell^a - \psi k^a$, with $\gamma \geq 0$ and $\psi > 0$, such that the variation of $\theta^{(\ell)}$ with respect to X^a fulfills*

$$\delta_X \theta^{(\ell)} \geq 0.$$

If, in addition, X^a (in particular γ, ψ) and $\Omega_a^{(\ell)}$ are axisymmetric, we will refer to $\delta_X \theta^{(\ell)} \geq 0$ as an (axisymmetry-compatible) spacetime stably outermost condition.

Remarks [cf. talk by M. Mars]:

- δ_v deformation operator of $\mathcal{S}$ along a normal direction v^a [cf. talk by M. Mars].
- That is, $\mathcal{S}$ is *stably outermost* [cf. talk by M. Mars] along X^a.
- Essentially equivalent to the *outer* of trapping horizon [Hayward 94] (see also [Racz 08]).
- For $f > 0$, it holds $\delta_X \theta^{(\ell')} = f \cdot \delta_X \theta^{(\ell)} > 0$ under null rescaling.

The Result

Area-angular momentum inequality for outermost stably MOTS

Theorem 1. *Given an axisymmetric closed marginally trapped surface $\mathcal{S}$ satisfying the (axisymmetry-compatible) spacetime stably outermost condition, in a spacetime with non-negative cosmological constant and fulfilling the dominant energy condition, it holds the inequality*

$$\boxed{A \geq 8\pi|J|}$$

where A and J are the area and (Komar) angular momentum of $\mathcal{S}$. If equality holds, then i) the geometry of $\mathcal{S}$ is that extreme Kerr throat sphere, and ii) if X^a is spacelike then $\mathcal{S}$ is a section of a non-expanding horizon.

Outline

Sketch in the Initial Data case of [Dain & Reiris 11]

Step 1.a: From a stability condition to a geometric inequality

Consider vacuum data (γ_{ij}, K_{ij}) on a spatial 3-slice Σ, and a (stable) minimal surface $\mathcal{S} \in \Sigma$, with s^i the normal to $\mathcal{S}$ tangent to Σ. The mean curvature H of $\mathcal{S} \in \Sigma$ vanishes and arbitrary deformations of $\mathcal{S}$ increase its area. That is, given flows $F_t : \mathbb{R} \times S^2 \to S$ with $F|_{t=0} = \mathcal{S}$ and $\dot{F}^i|_{t=0} = X^i = \alpha s^i$ then

$$\ddot{A}|_{t=0} = \int_{\mathcal{S}} \alpha \dot{H} dS \geq 0 \quad , \; \forall \alpha \text{ function on } \mathcal{S}.$$

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$$\int_{\mathcal{S}} \left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2 \, {}^2R \right] dS \geq \int_{\mathcal{S}} \alpha^2 \left[\left(K_{ij}s^i\eta^j \right)^2 - K^2 + \text{[Positive terms]} \right], \forall\alpha$$

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Step 1.b: Choice of α; metric and twist potentials σ and ω, and functional $\mathcal{M}$

- Choice of: i) α function and ii) line element ds^2 with σ metric function.
- **In vacuum (!)**, the term $K_{ij} s^i \eta^j$ on Σ can be expressed in terms of a (twist) potential ω on Σ.
- Then $A \geq 4\pi e^{\frac{\mathcal{M}-8}{8}}$, with (on a *round* sphere)

$$\mathcal{M}(\sigma, \omega) = \frac{1}{2\pi} \int_{\mathcal{S}} \left[\left(\frac{d\sigma}{d\theta} \right)^2 + 4\sigma + \frac{1}{\eta^2} \left(\frac{d\omega}{d\theta} \right)^2 \right] dS_0$$

Preliminaries

Explicit assumptions on axisymmetry

- $\mathcal{S}$ closed axisymmetric surface , with axial Killing vector η^a, i.e. $\boxed{\mathcal{L}_\eta q_{ab} = 0}$, with squared-norm $\eta = \eta^a \eta_a$.
- Tetrad $(\xi^a, \eta^a, \ell^a, k^a)$ on $\mathcal{S}$

$$\xi^a \eta_a = \xi^a \ell_a = \xi^a k_a = 0 \quad , \quad \xi^a \xi_a = 1$$

  adapted to axisymmetry $\mathcal{L}_\eta \ell^a = \mathcal{L}_\eta k^a = 0$.
- We write: $q_{ab} = \frac{1}{\eta}\eta_a \eta_b + \xi_a \xi_b$, and

$$\Omega_a^{(\ell)} = \Omega_a^{(\eta)} + \Omega_a^{(\xi)} \qquad \Omega_a^{(\ell)} {\Omega^{(\ell)}}^a = \Omega_a^{(\eta)} {\Omega^{(\eta)}}^a + \Omega_a^{(\xi)} {\Omega^{(\xi)}}^a$$

  with $\Omega_a^{(\eta)} = \eta^b \Omega_b^{(\ell)} \eta_a / \eta$ and $\Omega_a^{(\xi)} = \xi^b \Omega_b^{(\ell)} \xi_a$.
- $\Omega^{(\ell)}$ axisymmetric

$$\boxed{\mathcal{L}_\eta \Omega_a^{(\ell)} = 0}$$

Step 1.a: Geometric inequality from the stability condition.

Lemma 1. *Given a closed marginally trapped surface $\mathcal{S}$ satisfying the spacetime stably outermost condition for an axisymmetric X^a, then for all axisymmetric α it holds*

$$\int_{\mathcal{S}} \left[D_a\alpha D^a\alpha + \frac{1}{2}\alpha^2 \, {}^2\!R \right] dS \geq$$
$$\int_{\mathcal{S}} \left[\alpha^2 \Omega^{(\eta)}_a \Omega^{(\eta)}{}^a + \alpha\beta\sigma^{(\ell)}_{ab}\sigma^{(\ell)}{}^{ab} + G_{ab}\alpha\ell^a(\alpha k^b + \beta\ell^b) \right] dS \ ,$$

where $\beta = \alpha\gamma/\psi$.

Step 1.a: some intermediate steps

Variations of $\theta^{(\ell)}$ along normal directions (here, $\kappa^{(v)} = -v^a k^b \nabla_a \ell_b$):

$$
\begin{aligned}
\delta_{\alpha\ell}\theta^{(\ell)} &= \kappa^{(\alpha\ell)}\theta^{(\ell)} - \alpha\left[\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} + G_{ab}\ell^a k^b + \frac{1}{2}\left(\theta^{(\ell)}\right)^2\right] \\
\delta_{\beta k}\theta^{(\ell)} &= \kappa^{(\beta k)}\theta^{(\ell)} + {}^2\!\Delta\beta - 2\Omega^{(\ell)}_a D^a\beta \\
&\quad +\beta\left[\Omega^{(\ell)}_a\Omega^{(\ell)\,a} - D^a\Omega^{(\ell)}_a - \frac{1}{2}{}^2\!R + G_{ab}k^a\ell^b - \theta^{(\ell)}\theta^{(k)}\right] \ .
\end{aligned}
$$

Making $\theta^{(\ell)} = 0$, we evaluate $\frac{1}{\psi}\delta_X\theta^{(\ell)} = \frac{1}{\psi}\delta_{\gamma\ell-\psi k}\theta^{(\ell)} = \frac{1}{\psi}\left(\delta_{\gamma\ell}\theta^{(\ell)} - \delta_{\psi k}\theta^{(\ell)}\right)$:

$$
\begin{aligned}
\frac{1}{\psi}\delta_X\theta^{(\ell)} \quad & -\frac{\gamma}{\psi}\left[\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} + G_{ab}\ell^a\ell^b\right] \\
& -{}^2\!\Delta\ln\psi - D_a\ln\psi D^a\ln\psi + 2\Omega^{(\ell)}_a D^a\ln\psi \\
& -\left[-D^a\Omega^{(\ell)}_a + \Omega^{(\ell)}_c\Omega^{(\ell)\,c} - \frac{1}{2}{}^2\!R + G_{ab}k^a\ell^b\right]
\end{aligned}
$$

Step 1.a: some intermediate steps

Multiply by α^2, $\beta \equiv \alpha\gamma/\psi$, integrate on $\mathcal{S}$ (by parts) and using $\int_{\mathcal{S}} \frac{\alpha^2}{\psi}\delta_X \theta^{(\ell)} dS \geq 0$

$$
\begin{aligned}
0 \leq \quad & \int_{\mathcal{S}} \alpha\beta \left[-\sigma^{(\ell)}_{ab}\sigma^{(\ell)\,ab} - G_{ab}\ell^a\ell^b\right] dS \\
+ \quad & \int_{\mathcal{S}} \alpha^2 \left[-\Omega^{(\ell)}_a\Omega^{(\ell)\,a} + \frac{1}{2}{}^2R - G_{ab}k^a\ell^b\right] dS \\
+ \quad & \int_{\mathcal{S}} \left[2\alpha D_a \alpha D^a \ln\psi - \alpha^2 D_a \ln\psi D^a \ln\psi\right] dS \\
+ \quad & \int_{\mathcal{S}} \left[2\alpha^2 \Omega^{(\ell)}_a D^a \ln\psi - 2\alpha\Omega^{(\ell)}_a D^a \alpha\right] dS
\end{aligned}
$$

Step 1.a: some intermediate steps

From the axisymmetry of α and ψ, ${\Omega^{(\eta)}}^a D_a\alpha = {\Omega^{(\eta)}}^a D_a\psi = 0$

$$\begin{aligned} 0 \leq \quad & \int_{\mathcal{S}} \alpha\beta \left[-\sigma^{(\ell)}_{ab} {\sigma^{(\ell)}}^{ab} - G_{ab}\ell^a\ell^b \right] dS \\ + \quad & \int_{\mathcal{S}} \alpha^2 \left[-\Omega^{(\eta)}_a {\Omega^{(\eta)}}^a + \frac{1}{2}{}^2\!R - G_{ab}k^a\ell^b \right] dS \\ + \quad & \int_{\mathcal{S}} \left[2(D^a\alpha)(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a) \right. \\ & \left. -(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a)(\alpha D^a \ln\psi - \alpha{\Omega^{(\xi)}}^a) \right] dS \end{aligned}$$

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+ \quad & \int_{\mathcal{S}} \alpha^2 \left[-\Omega^{(\eta)}_a {\Omega^{(\eta)}}^a + \frac{1}{2}{}^2R - G_{ab}k^a\ell^b \right] dS \\
+ \quad & \int_{\mathcal{S}} \Big[2(D^a\alpha)(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a) \\
& -(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a)(\alpha D^a \ln\psi - \alpha{\Omega^{(\xi)}}^a) \Big] dS
\end{aligned}
$$

Using (a Young's) inequality (in particular, $X^2 + Y^2 \geq 2X\cdot Y$)

$$
D^a\alpha D_a\alpha \geq 2D^a\alpha(\alpha D_a \ln\psi - \alpha\Omega^{(\xi)}_a) - |\alpha D\ln\psi - \alpha\Omega^{(\xi)}|^2
$$

Step 1.a: some intermediate steps

And the inequality in Lemma 1 follows

$$
\begin{aligned}
0 \leq & \int_{\mathcal{S}} \alpha\beta \left[-\sigma^{(\ell)}_{ab} \sigma^{(\ell)\,ab} - G_{ab}\ell^a \ell^b \right] dS \\
+ & \int_{\mathcal{S}} \alpha^2 \left[-\Omega^{(\eta)}_a \Omega^{(\eta)\,a} + \frac{1}{2} {}^2\!R - G_{ab} k^a \ell^b \right] dS \\
+ & \int_{\mathcal{S}} D^a \alpha D_a \alpha \, dS
\end{aligned}
$$

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And the inequality in Lemma 1 follows

$$\int_{\mathcal{S}} \left[D^a \alpha D_a \alpha + \frac{1}{2} {}^2R \right] dS \geq$$
$$\int_{\mathcal{S}} \left[\alpha^2 \Omega^{(\eta)}_a \Omega^{(\eta)^a} + \alpha\beta \sigma^{(\ell)}_{ab} \sigma^{(\ell)^{ab}} + G_{ab} \alpha \ell^a (\alpha k^b + \beta \ell^b) \right] dS \ , \forall \alpha$$

Remarks:

- Spacetime expression in which the geometric meaning of each term in the inequality is apparent.
- The assumption of non-negative cosmological constant $\Lambda \geq 0$ and the dominant energy condition, under assumption of Einstein equations $G_{ab} + \Lambda g_{ab} = 8\pi T_{ab}$, plays the role of the *maximal slicing* condition in the initial data discussion.
- The role of dynamical gravitational degrees of freedom (positive-definite shear squared term) is identified and isolated.

Step 1.a: some intermediate steps

For Theorem 1, we disregard the second and third contributions on the right-hand-side:

$$\int_{\mathcal{S}} \left[D^a \alpha D_a \alpha + \frac{1}{2} {}^2R \right] dS \geq \int_{\mathcal{S}} \alpha^2 \Omega^{(\eta)}_a \Omega^{(\eta)^a} \, dS, \quad \forall \alpha \text{ function on } \mathcal{S}..$$

Strategy

As in [Dain & Reiris 11], obtain $A \geq 4\pi \cdot f(\mathcal{M})$ by evaluating the geometric inequality above with:

- A particular coordinate system on $\mathcal{S}$.
- A particular choice of α.
- Identification of a potential $\bar{\omega}$ on $\mathcal{S}$ for the right-hand-side term.

Step 1.b: Choice of preferred coordinate system on $\mathcal{S}$

Sphericity of $\mathcal{S}$

Under the conditions of Th. 1, $\mathcal{S} = S^2$ follows from Lemma 1 (for non-vanishing angular momentum), by choosing a constant α and using Gauss-Bonnet theorem.

Choice of line element on $\mathcal{S}$

On an axisymmetric sphere, a coordinate system can always be found such that

$$ds^2 = q_{ab}dx^a dx^b = e^{\sigma}\left(e^{2q}d\theta^2 + \sin^2\theta d\varphi^2\right) ,$$

with σ and q functions on θ satisfying $\sigma + q = c$, where c is a constant.
We can write $dS = e^c dS_0$, with $dS_0 = \sin\theta d\theta d\varphi$.

Evaluation of Lemma's 1 inequality: choice of α

Following [Dain & Reiris 11] we choose:

$$\boxed{\alpha = e^{c-\sigma/2}}$$

Step 1.b: Left-hand-side of Lemma's 1 inequality

Evaluation of the Left-hand-side of Lemma's 1 inequality results in

$$\int_{\mathcal{S}} \left[D_a \alpha D^a \alpha + \frac{1}{2} \alpha^2 \ {}^2R \right] dS$$
$$= e^c \left[4\pi(c+1) - \int_{\mathcal{S}} \left(\sigma + \frac{1}{4} \left(\frac{d\sigma}{d\theta} \right)^2 \right) dS_0 \right] \quad .$$

Note that $A = 4\pi e^c$.

Step 1.b: Right-hand-side of Lemma's 1 inequality

Potential $\bar{\omega}$ on $\mathcal{S}$

A potential $\bar{\omega}$ for the angular momentum exists **always** on $\mathcal{S}$. In addition, *in vacuum* it coincides with the twist potential ω.

Due to the S^2 topology of $\mathcal{S}$, the form $\Omega_a^{(\ell)}$ can be purely written in terms of a divergence-free and a exact 1-form. In particular (with $\eta = \eta^a\eta_a = e^\sigma \sin^2\theta$)

$$\Omega_a^{(\ell)} = \frac{1}{2\eta}\epsilon_{ab}D^b\bar{\omega} + D_a\lambda \ ,$$

with $\bar{\omega}$ and λ axially symmetric and fixed up to a constant (note $\Omega_a^{(\eta)} = \frac{1}{2\eta}\epsilon_{ab}D^b\bar{\omega}$ is the divergence-free part and $\Omega_a^{(\xi)}$ is the exact part). Morevover, J is read from the values of $\bar{\omega}$ at the poles:

$$\Omega_a^{(\ell)}\eta^a = \frac{1}{2\eta^{1/2}}\xi^a D_a\bar{\omega} \quad , \quad J = \frac{1}{8}\int_0^\pi \partial_\theta\bar{\omega}\, d\theta = \frac{1}{8}\left(\bar{\omega}(\pi) - \bar{\omega}(0)\right) \ .$$

In terms of the potential $\bar{\omega}$, the right-hand-side in Lemma's 1 inequality writes:

$$\int_{\mathcal{S}} \alpha^2 \Omega_a^{(\eta)}\Omega^{(\eta)a} dS = \int_{\mathcal{S}} \frac{\alpha^2}{4\eta^2} D_a\bar{\omega}D^a\bar{\omega}\, dS = e^c \int_{\mathcal{S}} \frac{1}{4\eta^2}\left(\frac{d\bar{\omega}}{d\theta}\right)^2 dS_0$$

Step 1.b: Right-hand-side of Lemma's 1 inequality

Inserting the obtained expressions into Lemma's 1 inequality, we get

$$A \geq 4\pi e^{\frac{\mathcal{M}-8}{8}} \quad ,$$

with the variational functional

$$\mathcal{M}(\sigma, \bar{\omega}) = \frac{1}{2\pi} \int_{\mathcal{S}} \left[\left(\frac{d\sigma}{d\theta} \right)^2 + 4\sigma + \frac{1}{\eta^2} \left(\frac{d\bar{\omega}}{d\theta} \right)^2 \right] dS_0 \ .$$

At this point we match exactly the proof in [Dain & Reiris 11]. In particular, it holds the upper bound in [Aceña, Dain, Gabach-Clément 11] for J

$$e^{\frac{\mathcal{M}-8}{8}} \geq 2|J| \quad ,$$

and the area-angular momentum inequality follows.

Rigidity

Rigidity part of the result:

If equality, $A = 8\pi|J|$, holds:

- All terms in Lemma's 1 inequality vanish.
- The intrinsic geometry of $\mathcal{S}$ is that of an *extreme Kerr throat sphere* [Dain 10].
- If X^a in the *stably outermost condition* is spacelike, then $\sigma^{(\ell)}_{ab}$ vanishes and $\mathcal{S}$ is a section of a non-expanding null hypersurface foliated by marginally trapped surfaces (*instantaneous* Non-Expanding Isolated Horizon [Ashtekar *et al.* ...]).

Outline

Conclusions

Result

We have shown that axisymmetric stable marginally trapped surfaces (in particular, *apparent horizons*) satisfy the inequality $A \geq 8\pi|J|$ in generically dynamical, non-necessarily axisymmetric, spacetimes with ordinary matter that can extend to the horizon.

Comments

Two key ingredients enabling i) the shift from the initial data discussion to a (purely quasi-local) (spacetime) **Lorentzian (!) result** and ii) the **inclusion of matter**:

- The *stably outermost condition* for MOTS leads to a spacetime geometric inequality (Lemma 1) where:
  - Spacetime interpretation of each term in the right-hand-side is transparent.
  - The global sign is controlled by standard physical assumptions on the matter energy content, relaxing the maximal slicing hypothesis in previous results.
- The quadratic term controlling the angular momentum is expressed in terms of a potential $\bar{\omega}$ living solely on the sphere $\mathcal{S}$ (even in the non-vacuum case), matching with the key variational functional $\mathcal{M}$.

On *averaged* outermost stably condition for MOTS

Inequality in Lemma 1 does not require a point-like stability condition. It makes sense to consider (weaker) *averaged* stability conditions for MOTS.

A first posibility: (Dipole) weight-averaged stably outermost condition

Definition 3. *Given a closed MOTS $\mathcal{S}$ we will refer to it as (dipole)* weight-averaged stably outermost *if there exists an outgoing ($-k^a$-oriented) vector $x^a = \gamma \ell^a - k^a$, with $\gamma \geq 0$ such that, for all functions $\alpha > 0$ on $\mathcal{S}$, the variations of $\theta^{(\ell)}$ with respect to $X^a = \alpha x^a$ fulfill the integral condition*

$$\int_{\mathcal{S}} (X^a \ell_a) \delta_X \theta^{(\ell)} dS \geq 0 \quad .$$

Since $\alpha = e^{c-\sigma/2} > 0$, it suffices for deriving the present result.

On *averaged* outermost stably condition for MOTS

Inequality in Lemma 1 does not require a point-like stability condition. It makes sense to consider (weaker) *averaged* stability conditions for MOTS.

More generally: (n-moment) weight-averaged stably outermost condition

Definition 3b. *Given a closed MOTS $\mathcal{S}$ we will refer to it as (2n-moment)* weight-averaged stably outermost *if there exists an outgoing ($-k^a$-oriented) vector $x^a = \gamma\ell^a - k^a$, with $\gamma \geq 0$ such that, for all functions $\alpha > 0$ on $\mathcal{S}$, the variations of $\theta^{(\ell)}$ with respect to $X^a = \alpha x^a$ fulfill the integral condition*

$$\int_{\mathcal{S}} (X^a \ell_a)^n \, \delta_X \theta^{(\ell)} dS \geq 0 \quad .$$

Note that $(X^a\ell_a) = \alpha = \mathrm{const}$ provides an *averaged stably outermost* condition.

On *averaged* outermost stably condition for MOTS

Inequality in Lemma 1 does not require a point-like stability condition. It makes sense to consider (weaker) *averaged* stability conditions for MOTS.

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Note that $(X^a \ell_a) = \alpha = \mathrm{const}$ provides an *averaged stably outermost* condition.

Any interest on these averaged stably conditions? *To be assessed...*

General Relativity remarks

Some remarks on physical applications

- (Axially symmetric) **Quasi-local models of BH horizons** satisfy $A \geq 8\pi|J|$: Outer Trapping Horizons, $\delta_k \theta^{(\ell)} < 0$ [Hayward 94] and, in particular, Dynamical Horizons [cf. Ashtekar & Krishnan 04].
- Quasi-local characterization of **Black Hole (sub)extremality** [cf. Booth & Fairhurst 08].
- **Positivity of surface gravity**. Inequality $A \geq 8\pi|J|$ is equivalent to the non-negativity of Dynamical and Isolated Horizon surface gravities [cf. Ashtekar & Krishnan 04]: $\kappa = \frac{A^2-(8\pi J)^2}{[...]^2}$.
  Consistency test of their associated *first law of black hole thermodynamics* [cf. Ashtekar & Krishnan 04].
- Curious timing with *Numerical Relativity* work: violations of $A \geq 8\pi|J|$ found in simulation of astrophysical models *violating the null energy condition* [Bode *et al.* 11]: