Calabi-Bernstein results and parabolicity of maximal surfaces in Lorentzian product spaces

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The results we are going to introduce are contained in the following references

⋆ **A. L. Albujer**, *New examples of entire maximal graphs in* $\mathbb{H}^2 \times \mathbb{R}_1$, Differential Geometry and its Applications 26 (2008), 456–462.

⋆ **A. L. Albujer** and **L. J. Alías**, *A local estimate for maximal surfaces in Lorentzian product spaces*, Matemática Contemporânea 34 (2008), 1–10.

⋆ **A. L. Albujer** and **L. J. Alías**, *Calabi-Bernstein results for maximal surfaces in Lorentzian product spaces*, Journal of Geometry and Physics 59 (2009), 620–631.

⋆ **A. L. Albujer** and **L. J. Alías**, *Parabolicity of maximal surfaces in Lorentzian product spaces*, Mathematische Zeitschrift 267 (2011), 453–464.

Calabi-Bernstein Theorem (1970)

Parametric version

The only complete maximal surfaces in $\mathbb{R}^3_1$ are the spacelike planes.

Non-parametric version

The only entire maximal graphs in $\mathbb{R}^3_1$ are the spacelike planes. That is, the only entire solutions to the maximal surfaces equation

$$\mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right)=0, \quad |Du|^2<1$$

on the Euclidean plane $\mathbb{R}^2$ are affine functions.

- ⋆ Cheng and Yau (1976) extended this result to $\mathbb{R}^{n+1}_1$.
- ⋆ Since then, many authors have approached this theorem from different points of view: McNertey (1980), Kobayashi (1983), Estudillo and Romero (1991, '92 and '94), Romero (1996), Alías and Palmer (2001), Romero and Rubio (2010), Caballero, Romero and Rubio (2010).

Introduction

In this lecture...

... we extend the Calabi-Bernstein theorem to more general ambient spaces: Lorentzian product spaces.

For doing it:

- ⋆ We introduce global Calabi-Bernstein type results, both in parametric and non-parametric versions.
- ⋆ We introduce local results with global Calabi-Bernstein type consequences.

Preliminaries

The ambient space

- Let $(M^2, \langle,\rangle_M)$ be a connected Riemannian surface.
- Consider the product manifold $M^2 \times \mathbb{R}$ endowed with the metric

$$\langle,\rangle = \langle,\rangle_M - dt^2.$$

- We will denote this Lorentzian manifold by $M^2 \times \mathbb{R}_1$.

- Let $\varphi : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ a connected spacelike surface.
- $\partial_t = (\partial/\partial_t)_{(x,t)}$ is a unitary timelike vector field globally defined on $M^2 \times \mathbb{R}_1$:
  - $M^2 \times \mathbb{R}_1$ is time-orientable.
  - There exists a unique unitary timelike normal vector field $N \in \mathcal{X}^{\perp}(\Sigma)$ globally defined on Σ such that

$$\langle N, \partial_t \rangle \leq -1.$$

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$$\Theta = \langle N, \partial_t \rangle.$$

Preliminaries

- A spacelike surface is said to be maximal if its mean curvature vanishes, $H = 0$, being $H = -\frac{1}{2}\mathrm{tr}(A)$ where $A = A_N$ is the shape operator of Σ.
- The Gaussian curvature of a maximal surface is

$$K = \kappa_M \Theta^2 + \frac{1}{2}\|A\|^2,$$

where κ_M is the Gaussian curvature of M along Σ.

- The height function of Σ, $h : \Sigma \longrightarrow \mathbb{R}$, is defined by

$$h = \pi_{\mathbb{R}} \circ \varphi.$$

⋆ It follows that $\|\nabla h\|^2 = \Theta^2 - 1$.

- A spacelike surface Σ is said to be a slice if its height function is constant. Equivalently, if $\Theta = -1$. In this case

$$M^2_{t_0} = M^2 \times \{t_0\}, \ \ t_0 \in \mathbb{R}.$$

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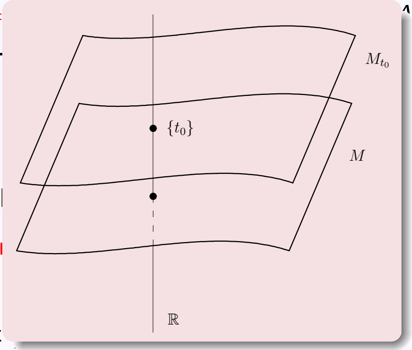

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Parametric version of a Calabi-Bernstein result

- As a first topological result, we have the following lemma:

Lemma 1

Let M^2 be a Riemannian surface. If $M^2 \times \mathbb{R}_1$ admits a complete spacelike surface $\varphi : \Sigma^2 \to M^2 \times \mathbb{R}_1$, then M is necessarily complete and the projection $\Pi = \pi_M \circ \varphi : \Sigma \to M$ is a covering map.

- ⋆ This follows form the fact that Π is a local diffeomorphism between the Riemannian surfaces Σ and M which increases the distance.
- ⋆ As a consequence, we observe that if M^2 is simply connected, then every complete spacelike surface in $M^2 \times \mathbb{R}_1$ is an entire spacelike graph.

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  ⋆ This follows form the fact that Π is a local diffeomorphism between the Riemannian surfaces Σ and M which increases the distance.

  ⋆ As a consequence, we observe that if M^2 is simply connected, then every complete spacelike surface in $M^2 \times \mathbb{R}_1$ is an entire spacelike graph.

- Every maximal surface satisfies $\Delta h = 0$. Therefore, if Σ is compact, then it is necessarily a slice:

Proposition 1

Let M^2 be a Riemannian surface (necessarily compact). Then the only compact maximal surfaces in $M^2 \times \mathbb{R}_1$ are the slices $M \times \{t_0\}$, $t_0 \in \mathbb{R}$.

Parametric version of a Calabi-Bernstein result

- Under completeness assumption, we have the following parametric version of a Calabi-Bernstein result in $M^2 \times \mathbb{R}_1$,

Theorem 1

Let M^2 be a Riemannian surface (necessarily complete) with non-negative Gaussian curvature, $K_M \geq 0$. Then every complete maximal surface Σ^2 in $M^2 \times \mathbb{R}_1$ is totally geodesic. In addition, if $K_M > 0$ at some point of M, then Σ is a slice $M \times \{t_0\}$, $t_0 \in \mathbb{R}$.

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- The assumptions on K_M are necessary.

  First, observe that is $M^2 = \mathbb{R}^2$, the $M^2 \times \mathbb{R}_1 = \mathbb{R}^3_1$ is the 3-dimensional Lorentz-Minkowski space, and any spacelike affine plane $\mathbb{R}^3_1$ which is not horizontal determines a complete totally geodesic surface which is not a slice.

  The assumption $K_M \geq 0$ is also necessary, as shown by the fact that there are examples of complete maximal surfaces in $\mathbb{H}^2 \times \mathbb{R}_1$ which are not totally geodesic.

Parametric version of a Calabi-Bernstein result

Parabolicity

- A Riemannian surface (without boundary) is said to be parabolic if every non-positive subharmonic function on the surface must be constant.
- **Parabolicity criterium (Ahlfors and Blanc-Fiala-Huber)**

  Every complete Riemannian surface with non-negative Gaussian curvature is parabolic.

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Proof of Theorem 1

- ⋆ Σ maximal $+\ \kappa_M \geq 0 \Rightarrow K = \kappa_M \Theta^2 + \frac{1}{2}\|A\|^2 \geq 0 \Rightarrow \Sigma$ is parabolic.
- ⋆ Consider $\Phi(\Theta) = \frac{1}{\Theta} < 0$. We can compute

$$\Delta\left(\frac{1}{\Theta}\right) = -\frac{\Delta\Theta}{\Theta^2} + \frac{2\|\nabla\Theta\|^2}{\Theta^3} = -\frac{1}{\Theta}\left(\kappa_M(\Theta^2-1) + \frac{\|A\|^2}{\Theta^2}\right) \geq 0$$

$\rightsquigarrow$ $1/\Theta$ is constant $\rightsquigarrow$ $\Theta = \Theta_0$ y $\Delta\Phi(\Theta) = 0$.

- ⋆ Therefore, Σ is totally geodesic, and if there exists a point $q \in \Sigma$ such that $\kappa_M(q) > 0$ then Σ is a slice.

Entire maximal graphs

- Let $\Omega \subseteq M$ be a connected domain. Every function $u \in \mathcal{C}^\infty(\Omega)$ defines a graph over Ω given by
$$\Sigma(u) = \{(x, u(x)) : x \in \Omega\} \subset M^2 \times \mathbb{R}_1.$$
A graph is said to be entire if $\Omega = M$.

- The metric induced on Ω from the metric on the ambient space via $\Sigma(u)$ is given by
$$\langle , \rangle = \langle , \rangle_M - du^2.$$

- A graph is a spacelike slice in $M^2 \times \mathbb{R}_1$ if and only if $|Du|^2 < 1$.

- $\Sigma(u)$ is a maximal graph if and only if the function u satisfies

$$\mathrm{Maximal}[u] = \mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du|^2 < 1.$$

Here Div and Du stand for the divergence operator and the gradient of u in Ω. And $|\cdot|$ denotes the norm of a vector field with respect to $\langle , \rangle_M$.

Entire maximal graphs

- As a consequence of Lemma 1, if M is a simply connected complete surface, then every complete spacelike surface in $M^2 \times \mathbb{R}_1$ is an entire spacelike graph.
- However, not every entire spacelike graph $M^2 \times \mathbb{R}_1$ is necessarily complete.
- Thus, Theorem 1 does not imply in principle that, under the same hypothesis on M, any entire graph in $M^2 \times \mathbb{R}_1$ must be totally geodesic.
- This is true for entire maximal graphs in $\mathbb{R}^3_1$ (Calabi, 1970) and for entire maximal graphs in Robertson-Walker spaces of the form $\mathbb{R}^2 \times_\varrho \mathbb{R}_1$ (Latorre and Romero, 2001) and generalized Robertson-Walker spaces of the form $M^2 \times_\varrho \mathbb{R}_1$, under certain assumptions on the warping function ϱ and on the Gaussian curvature of the fiber (Caballero, Romero and Rubio, 2010).
- It is also true for entire maximal graphs in $M^2 \times \mathbb{R}_1$ as a consequence of the following theorem:

Non-parametric version of a Calabi-Bernstein result

Theorem 2

Let M^2 be a Riemannian surface with non-negative Gaussian curvature. Then any maximal surface Σ^2 in $M^2 \times \mathbb{R}_1$ which is complete with respect to the metric induced from the Riemannian product $M^2 \times \mathbb{R}$ is totally geodesic. In addition, if $K_M > 0$ at some point on Σ, then M is necessarily complete and Σ is a slice $M \times \{t_0\}$.

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- If M is complete and $\varphi : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ is a spacelike surface properly immersed in $M^2 \times \mathbb{R}_1$ the metric induced on Σ from the Riemannian product $M^2 \times \mathbb{R}$ is complete. Therefore,

Corollary 1

Let M^2 be a complete Riemannian surface with non-negative Gaussian curvature. Then any maximal surface Σ^2 properly immersed into $M^2 \times \mathbb{R}_1$ is totally geodesic. In addition, if $K_M > 0$ at some point on Σ, then M is necessarily complete and Σ is a slice $M \times \{t_0\}$.

Non-parametric version of a Calabi-Bernstein result

- Corollary 1 can be applied when M is complete and $\Sigma \subset M^2 \times \mathbb{R}_1$ is a closed embedded maximal surface, so for entire maximal graphs.
- This allows us to give the following non-parametric version of the Calabi-Bernstein theorem,

Theorem 3

Let M^2 be a complete Riemannian surface with non-negative Gaussian curvature. Then any entire maximal graph $\Sigma(u)$ in $M^2 \times \mathbb{R}_1$ is totally geodesic. In addition, if M is not flat, then u is constant; in other words, the only entire solutions to the maximal surface equation

$$\mathrm{Maximal}[u] = \mathrm{Div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du|^2 < 1.$$

are the constant functions.

Proof of Theorem 2

- The proof follows the ideas introduced by Chern (1969) in his proof of the classical Bernstein theorem, and adapted later by Romero (1996) to maximal surfaces in $\mathbb{R}^3_1$.

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⋆ Let Σ be a maximal surface in $M^2 \times \mathbb{R}_1$. We will denote by $g = \langle , \rangle$ the Riemannian metric induced on Σ from $M^2 \times \mathbb{R}_1$.

⋆ Since $1 - \Theta \geq 2 > 0$, we may introduce on Σ the conformal metric $\hat{g} = (1 - \Theta)^2 g$.

- We will see first that $(\Sigma, \hat{g})$ is a parabolic surface.

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⋆ It is well known that

$$(1 - \Theta)^2 \hat{K}_\Sigma = K_\Sigma - \Delta \log (1 - \Theta),$$

being $\hat{K}_\Sigma$ and K_Σ the Gaussian curvatures of $(\Sigma, \hat{g})$ and (Σ, g).

⋆ From the expressions for $\Delta\Theta$ and $\|\nabla\Theta\|^2$,

$$\Delta \log (1 - \Theta) = \frac{1}{2}\|A\|^2 + \Theta(\Theta + 1)\kappa_M$$

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⋆ From the expressions for $\Delta\Theta$ and $\|\nabla\Theta\|^2$,

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being $\hat{K}_\Sigma$ and K_Σ the Gaussian curvatures of $(\Sigma, \hat{g})$ and (Σ, g).

⋆ Therefore,

$$K_\Sigma - \Delta \log(1 - \Theta) = -\Theta \kappa_M \geq 0.$$

and $\hat{K}_\Sigma \geq 0$ on Σ.

Proof of Theorem 2

⋆ Now, we will see that $(\Sigma, \hat{g})$ is complete.

Non-parametric version of a Calabi-Bernstein result

Proof of Theorem 2

- Now, we will see that $(\Sigma, \hat{g})$ is complete.
- Let g' denote the Riemannian metric induced on Σ from the Riemannian product $M^2 \times \mathbb{R}$. It can be seen that

$$\hat{g}(X,X) \geq \frac{1}{2}(g(X,X) + g'(X,X)) \geq \frac{1}{2}g'(X,X),$$

  for every $X \in \mathcal{X}(\Sigma)$.
- Since g' is complete on Σ, then $\hat{g}$ is also complete.

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  for every $X \in \mathcal{X}(\Sigma)$.
- Since g' is complete on Σ, then $\hat{g}$ is also complete.
- Summing up, $(\Sigma^2, \hat{g})$ is a complete Riemannian surface with $\hat{K}_\Sigma \geq 0$ and hence $(\Sigma^2, \hat{g})$ is parabolic.

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- Since g' is complete on Σ, then $\hat{g}$ is also complete.
- Summing up, $(\Sigma^2, \hat{g})$ is a complete Riemannian surface with $\hat{K}_\Sigma \geq 0$ and hence $(\Sigma^2, \hat{g})$ is parabolic.
- When $n = 2$, the Laplacian operators Δ and $\hat{\Delta}$ on Σ with respect to g and $\hat{g}$ are related by $\Delta = (1 - \Theta)^2 \hat{\Delta}$, so that parabolicity is preserved under conformal changes of metric.
- Therefore, (Σ^2, g) is also parabolic and the proof follows as in Theorem 1 since $1/\Theta$ is a negative subharmonic function on (Σ^2, g).

A duality result between minimal and maximal graphs

- Calabi (1970) first observed that there exists a duality between minimal graphs in the Euclidean space $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ and maximal graphs in the Lorentz-Minkowski space $\mathbb{R}^3_1 = \mathbb{R}^2 \times \mathbb{R}_1$.
- The same duality holds in general when considering a Riemannian product space $M \times \mathbb{R}$ and the corresponding Lorentzian product space $M \times \mathbb{R}_1$.
- Let $\Omega \subseteq M$ be a connected domain. A function $u \in \mathcal{C}^\infty(\Omega)$ defines a minimal graph $\Sigma(u)$ in $M \times \mathbb{R}$ ($H = 0$) if and only if u satisfies the following partial differential equation on Ω,

$$\mathrm{Minimal}[u] = \mathrm{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

A duality result between minimal and maximal graphs

Theorem 4

Let $\Omega \subseteq M^2$ be a simply connected domain of a Riemannian surface M^2. There exists a non-trivial solution u to the minimal surface equation on Ω,

$$\text{Minimal}[u] = 0,$$

if and only if there exists a non-trivial solution w to the maximal surface equation on Ω,

$$\text{Maximal}[w] = 0, \quad |Dw|^2 < 1.$$

- By a non-trivial solution we mean a solution defining a non-totally geodesic graph.
- Non-trivial solutions to either the minimal or maximal surface equation correspond to non-totally geodesic either minimal or maximal graphs.

Proof of Theorem 4

- The proof of the theorem follows the ideas of the proof of Alías and Palmer (2001) when $M = \mathbb{R}^2$.
- Since Ω is a simply connected domain of a Riemannian surface, it is endowed with a globally defined area form $d\Omega$ and an almost complex structure J.
- For every $X \in \mathcal{X}(M)$,

$$\operatorname{Div} X d\Omega = d\omega_{JX},$$

where $\omega_{JX}(Y) = \langle JX, Y\rangle_M$.

- Consider $u \in \mathcal{C}^\infty(\Omega)$ a non-trivial solution of

$$\text{Minimal}[u] = \operatorname{Div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0.$$

- Then, the 1-form ω_{JU} is closed on Ω, where

$$U = \frac{Du}{\sqrt{1+|Du|^2}}.$$

Proof of Theorem 4

- Since Ω is simply connected, $JU = Dw$ for a certain function w on Ω.
- Using that J is an isometry, we have

$$|Dw|^2 = \frac{|Du|^2}{1 + |Du|^2} < 1,$$

  so w defines a spacelike graph over Ω.
- Then, $JW = D(-u)$, where

$$W = \frac{Dw}{\sqrt{1 - |Dw|^2}}.$$

- Therefore, ω_{JW} is closed on Ω and, equivalently, $\text{Maximal}[w] = 0$.
- The relations $JU = Dw$ and $JW = D(-u)$ imply that u is non-trivial if and only if w is non-trivial .
- A similar argument proves the converse.

Counterexamples when $M^2 = \mathbb{H}^2$

- Consider the half-plane model of the hyperbolic plane $(\mathbb{H}^2, \langle , \rangle_{\mathbb{H}^2})$,

$$\mathbb{H}^2 = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}, \qquad \langle , \rangle_{\mathbb{H}^2} = \frac{1}{x_2^2}(dx_1^2 + dx_2^2).$$

Example 1

An entire maximal graph in $\mathbb{H}^2 \times \mathbb{R}_1$ which is complete.

- The function

$$u(x_1, x_2) = \log(x_1^2 + x_2^2)$$

defines a non-trivial entire minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ (Montaldo and Onnis, 2007).

- Thus, from our duality result there exists $w \in \mathcal{C}^\infty(\mathbb{H}^2)$ which determines a non-totally geodesic entire maximal graph in $\mathbb{H}^2 \times \mathbb{R}_1$.
- The assumption $K_M \geq 0$ in Theorem 3 is necessary.

Counterexamples when $M^2 = \mathbb{H}^2$

- Let us see that the induced metric on $\mathbb{H}^2$ via the maximal graph is complete, that is, the entire maximal graph determined by w is complete.
- Let g denote this metric, then $g(,) = \langle , \rangle_{\mathbb{H}^2} - dw^2$. That is, if $X \in \mathcal{X}(\mathbb{H}^2)$

$$g(X,X) = \langle X, X\rangle_{\mathbb{H}^2} - X(w)^2 = \langle X, X\rangle_{\mathbb{H}^2} - \langle X, Dw\rangle^2_{\mathbb{H}^2}.$$

By Cauchy-Schwarz inequality, $g(X,X) \geq \langle X, X\rangle_{\mathbb{H}^2}(1 - |Dw|^2)$. From the relation between Du and Dw,

$$1 - |Dw|^2 = \frac{1}{1 + x_2^2|D_o u|_o^2} \geq \frac{1}{5}.$$

So, finally $g(,) \geq \frac{1}{5}\langle , \rangle_{\mathbb{H}^2}$, and the metric g is complete on $\mathbb{H}^2$.
- The assumption $K_M \geq 0$ in Theorem 1 is also necessary.

Counterexamples when $M^2 = \mathbb{H}^2$

- Let us … graph is complet… w is complet…
- Let g d… is, if $X \in \mathcal{X}$(…

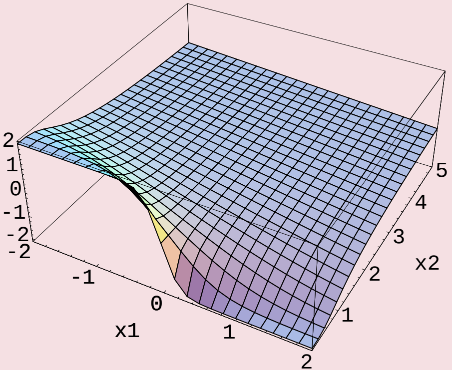
$$g \ldots \; v\rangle^2_{\mathbb{H}^2}.$$

By Cau… $Dw|^2)$.

From t…

So, fina… $\mathbb{H}^2$.

$$w(x_1, x_2) = i\frac{2}{\sqrt{5}}F\left(\operatorname{arc\,sen}\left(i\frac{x_1}{x_2}\right), \frac{1}{\sqrt{5}}\right)$$



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Counterexamples when $M^2 = \mathbb{H}^2$

Example 2

An entire maximal graph in $\mathbb{H}^2 \times \mathbb{R}_1$ which is not complete.

- The function
$$u(x_1, x_2) = \frac{x_1}{x_1^2 + x_2^2}$$
also determines a non-trivial entire minimal graph in $\mathbb{H}^2 \times \mathbb{R}$ (Montaldo and Onnis, 2007).
- Then, there exists $w \in \mathcal{C}^\infty(\mathbb{H}^2)$ which defines another non-totally geodesic entire maximal graph in $\mathbb{H}^2 \times \mathbb{R}_1$.
- Denote the graph by $\Sigma(w)$. This example is not complete.
- Let $\alpha : (0,1) \to \Sigma(w)$ be the divergent curve in $\Sigma(w)$,
$$\alpha(s) = (0, s, w(0,s)).$$
- Then
$$\int_0^1 \|\alpha'(s)\| ds = \int_0^1 \frac{ds}{\sqrt{1+s^2}} = \operatorname{arcsinh}(1) = \log(1+\sqrt{2}).$$
- α has finite length $\rightsquigarrow$ $\Sigma(w)$ is not complete.

Example 2

An entire maximal graph in $\mathbb{H}^2 \times \mathbb{R}_1$ which is not complete.

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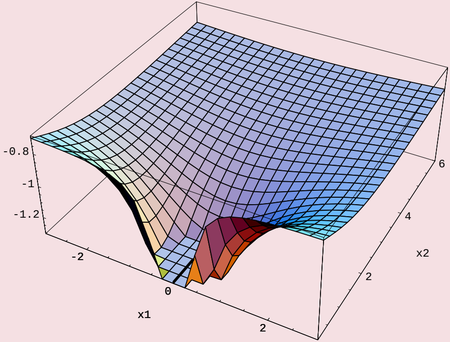
- Then, t ... n-totally geodesi
- Denote ...
- Let α :
- Then

$\int_0$ ... $- \sqrt{2})$.

- α has f

$$w(x_1, x_2) = \log\left(\frac{x_1^2 + x_2^2}{2\left(x_2 + \sqrt{x_2^2 + (x_1^2 + x_2^2)^2}\right)}\right)$$



- A Riemannian surface Σ with non empty boundary, $\partial\Sigma \neq \emptyset$, is relatively parabolic if every bounded harmonic function on Σ is determined by its boundary values.
- In this terms, a Riemannian surface Σ without boundary is parabolic if and only if for every non-empty open subset $O \subset \Sigma$ with non empty boundary, $\Sigma \setminus O$ is relatively parabolic.
- The following result provides us with very useful parabolicity test:

Lemma 2

Let Σ^2 be a Riemannian surface with non empty boundary. If Σ^2 admits a proper continuous function $\psi : \Sigma \rightarrow \mathbb{R}$ which is eventually positive and superharmonic, then Σ^2 is relatively parabolic.

- Fernández and López proved that properly immersed maximal surfaces with non-empty boundary in $\mathbb{R}^3_1$ are parabolic if the Lorentzian norm on the maximal surface in $\mathbb{R}^3_1$ is eventually positive and proper.
- We extend here that result to the case of maximal surfaces in $M^2 \times \mathbb{R}_1$.

Theorem 5

Let M^2 be a complete Riemannian surface with non-negative Gaussian curvature. Consider Σ a maximal surface in $M^2 \times \mathbb{R}_1$ with non-empty boundary, $\partial\Sigma \neq \emptyset$, and assume that the function $\phi : \Sigma \to \mathbb{R}$ defined by

$$\phi(p) = r^2(p) - h^2(p), \quad \text{where } r(p) = \mathrm{dist}_M(\Pi(p), x_0),$$

(x_0 a fixed point) is eventually positive and proper. Then Σ is relatively parabolic.

- Let $a > 1$, and consider the compact set

$$K = \{p \in \Sigma : \phi(p) \leq a\}.$$

- Since relative parabolicity is not affected by adding or removing compact subsets, then Σ is relatively parabolic if and only if $\Sigma' = \Sigma \setminus K$ is relatively parabolic.
- $\log \phi : \Sigma' \rightarrow \mathbb{R}$ is a proper positive function on Σ'; in order to prove that Σ' is relatively parabolic it suffices to see that $\log \phi$ is superharmonic.
- A heavy computation yields

$$\Delta\phi = 2(r\hat{\Delta}\hat{r} + 1) + 2\langle N^*, \hat{\nabla}\hat{r}\rangle_M^2 + 2r\hat{\Delta}\hat{r}\langle N^*, \tau\rangle_M^2 - 2(\Theta^2 - 1).$$

- As $K_M \geq 0$, the Laplacian comparison theorem yields $\hat{\Delta}\hat{r} \leq 1/\hat{r}$, so that

$$r\hat{\Delta}\hat{r} \leq 1 \quad \text{on } \Sigma'.$$

Proof of Theorem 5

- Using this in the previous expression for $\Delta\phi$, we get

$$\frac{1}{2}\Delta\phi \leq 2 + \|N^*\|^2 - (\Theta^2 - 1) = 2.$$

- On the other hand, we also have that

$$\begin{aligned} \|\nabla\phi\|^2 &= 4r^2\|\nabla r\|^2 - 8rh\langle\nabla r, \nabla h\rangle + 4h^2\|\nabla h\|^2 \\ &= 4\phi + 4(r\langle\bar{\nabla}\bar{r}, N\rangle + h\Theta)^2 \geq 4\phi. \end{aligned}$$

- It follows from here that

$$\Delta \log\phi = \frac{1}{\phi^2}(\phi\Delta\phi - \|\nabla\phi\|^2) \leq 0,$$

  which means that $\log\phi$ is a proper positive superharmonic function on Σ'.
- Therefore, Σ' is relatively parabolic, and Σ is also relatively parabolic. This finishes the proof of Theorem 5.

Some applications of Theorem 5

Corollary 2

Let M^2 be a complete Riemannian surface with non-negative Gaussian curvature. Then every proper maximal immersion $\varphi : \Sigma^2 \rightarrow M^2 \times \mathbb{R}_1$ with non-empty boundary satisfying

$$f(\Sigma) \subset \mathcal{W}_a = \{(x,t) \in M^2 \times \mathbb{R}_1 : |t| \leq a\hat{r}(x)\}$$

for some $0 < a < 1$, is relatively parabolic.

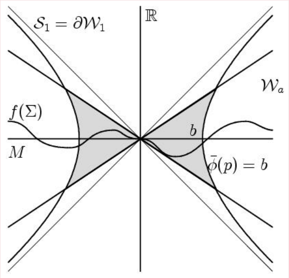
Proof

- $\star$ $f(\Sigma) \subset \mathcal{W}_a$

  $\rightsquigarrow$ ϕ is eventually positive.

- $\star$ $\phi^{-1}([0,b])$ is compact $\forall b > 0$

  $\rightsquigarrow$ ϕ es proper.



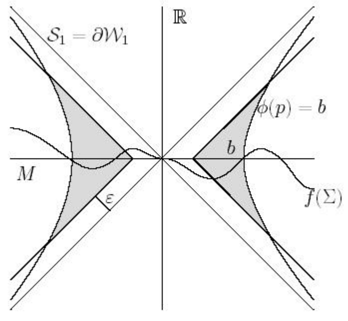
Some applications of Theorem 5

Proposition 2

Let M^2 be a complete Riemannian surface and let $\Sigma(u)$ be a spacelike graph over a domain $\Omega \subset M^2$ which is starlike with respect to some point $x_0 \in \text{int}(\Omega)$. Then the function $\phi = r^2 - h^2$ is eventually positive and proper over $\Sigma(u)$.

Proof

- ⋆ We may assume $u(x_0) = 0$.
- ⋆ $f(\Sigma(u)) - \{(x_0, 0)\} \subset \text{int}(\mathcal{W}_1) = \{(x,t) \in M^2 \times \mathbb{R}_1 : \hat{r}^2(x) - t^2 > 0\}$.
  $\rightsquigarrow \phi \geq 0$.
- ⋆ $\exists \varepsilon > 0$ t. q. $\text{dist}_+(f(p), S_1) > \varepsilon$ eventually in $\Sigma(u)$ (dist_+ is the distance associated to the metric on $M \times \mathbb{R}$).
- ⋆ $\phi^{-1}([0, b])$ is compact $\forall b > 0$
  $\rightsquigarrow \phi$ is proper.



Some applications of Theorem 5

- As a final application of Theorem 5, we can give an alternative proof of the non-parametric version of our Calabi-Bernstein result in Theorem 3:

Theorem 3

Let M^2 be a complete Riemannian surface with non-negative Gaussian curvature, $K_M \geq 0$. Then every entire maximal graph $\Sigma(u)$ in $M^2 \times \mathbb{R}_1$ is totally geodesic. Moreover, if $K_M > 0$ at some point of M, then u is constant.

Proof

- ⋆ Apply Proposition 2 to the case where $\Omega = M^2$ is a complete Riemannian surface.
- ⋆ Then we conclude that $\Sigma(u)$ is parabolic (as a surface without boundary).
- ⋆ The proof then follows as in Theorem 1.

A local estimate for maximal surfaces in $M^2 \times \mathbb{R}_1$

- Alías and Palmer (2001) introduced a local integral inequality for the Gaussian curvature of a maximal surface in $\mathbb{R}^3_1$, giving a new proof of Calabi-Bernstein theorem in $\mathbb{R}^3_1$.
- This can be extended to maximal surfaces in $M^2 \times \mathbb{R}_1$.

Theorem 6

Let M^2 be an analytic Riemannian surface with $K_M \geq 0$, and let Σ be a maximal surface $M^2 \times \mathbb{R}_1$. Let $p \in \Sigma$, and $R > 0$ be such that the geodesic disc of radius R about p satisfies $D(p, R) \subset\subset \Sigma$. Then for all $0 < r < R$

$$0 \leq \int_{D(p,r)} \|A\|^2 \mathrm{d}\Sigma \leq c_r \frac{L(r)}{r \log(R/r)},$$

where $L(r)$ denotes the length of $\partial D(p, r)$, and

$$c_r = \frac{\pi^2(1+\alpha_r^2)^2}{4\alpha_r \arctan \alpha_r} > 0.$$

Here $\alpha_r = \sup_{D(p,r)} \cosh\theta \geq 1$, where θ denotes the hyperbolic angle between N and T along Σ.

A local estimate for maximal surfaces in $M^2 \times \mathbb{R}_1$

- When Σ is complete, the local integral inequality of Theorem 6 implies Theorem 1.
- Actually, if Σ is complete, for a fixed $p \in \Sigma$ and a fixed r, $R \to \infty$, so

$$\int_{D(p,r)} \|A\|^2 d\Sigma = 0.$$

- Then $\|A\|^2 = 0$, that is Σ is totally geodesic

  The proof of Theorem 6 is based on the following local intrinsic estimate.

Lemma 3 (Alías and Palmer, 1999)

Let Σ be an analytic Riemannian surface with $K_\Sigma \geq 0$. Let $\psi \in \mathcal{C}^\infty(\Sigma)$ which satisfies

$$\psi \Delta \psi \geq 0$$

on Σ. Then for $0 < r < R$

$$\int_{D_r} \psi \Delta \psi \leq \frac{2L(r)}{r \log(R/r)} \sup_{D_R} \psi^2,$$

where p is a fixed point in Σ and $D_r \subset D_R \subset\subset \Sigma$.

Proof of Theorem 6

- Since M is analytic and Σ is locally given by the maximal surface equation, $(\Sigma, \langle , \rangle)$ is also analytic. And $K_\Sigma \geq 0$ from Gauss equation.
- We can apply the lemma to an appropriate smooth function. Consider $\psi = \arctan \Theta$.
- We can compute

$$\Delta\psi = \frac{\Delta\Theta}{1+\Theta^2} - \frac{2\Theta\|\nabla\Theta\|^2}{(1+\Theta^2)^2} = \frac{2\Theta}{(1+\Theta^2)^2}\|A\|^2 + \frac{(\Theta^2-1)\Theta}{1+\Theta^2}\kappa_M,$$

and

$$\psi\Delta\psi \geq \frac{2\alpha_r \arctan\alpha_r}{(1+\alpha_r^2)^2}\|A\|^2 \geq 0 \quad \text{on} \quad D(p,r).$$

- Integrating this inequality over $D(p,r)$ and using the lemma, we get

$$0 \leq \frac{2\alpha_r \arctan\alpha_r}{(1+\alpha_r^2)^2}\int_{D(p,r)}\|A\|^2 d\Sigma \leq \int_{D(p,r)}\psi\Delta\psi \leq \frac{\pi^2}{2}\frac{L(r)}{r\log(R/r)}.$$

- The theorem is proved.

THAT IS ALL...

THANKS A LOT FOR YOUR ATTENTION !!

MUCHAS GRACIAS POR VUESTRA ATENCIÓN !