Uniqueness of spacelike hypersurfaces of constant mean curvature in cosmological models with certain symmetries

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This talk is based on:
M. Caballero, A. Romero and R. M. Rubio, Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds with a timelike gradient conformal vector field, *Classical and Quantum Gravity* 28 (2011)

UNIVERSIDAD DE CÓRDOBA

What do we call symmetry?

Given a spacetime $(\bar{M}, \bar{g})$, in general relativity, symmetry [1] [2] is usually based on the assumption of the existence of a one-parameter group of transformations generated by a Killing vector field K

$$L_K \bar{g} = 0.$$

The spacetime is called STATIONARY.

Or, more generally, on the existence of a conformal vector field:

$$L_K \bar{g} = 2\rho \bar{g},$$

where ρ is a (smooth) function[3].
The spacetime is called CONFORMALLY STATIONARY, since it is conformal to a stationary one.

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[1] Duggal K L and Sharma R 1999 Symmetries of Spacetimes and Riemannian Manifolds Mathematics and Its Applications vol. 487 (Kluwer Academic Publishers).

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Gradient Conformally Stationary spacetimes

Along this talk K will be a conformal and timelike vectorfield.

In such a case the integral curves of

$$Z = \frac{1}{\sqrt{-\bar{g}(K,K)}} K$$

provide a family of privileged observers in spacetime.

From now on

$$h := \sqrt{-\bar{g}(K,K)}$$

and $\mathcal{F}$ will the orthogonal distribution to K.

Is $\mathcal{F}$ integrable?

In general, no.

If K is locally the gradient of a function, YES ↘

Foliation by totally umbilical constant mean curvature spacelike hypersurfaces with $H = \frac{-\rho}{h}$

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IS THE SPACETIME CAUSAL?

In general, no.

If K is GLOBALLY the gradient of some smooth function, YES.

The (noncompact) spacetime admits a global time function $\Longrightarrow$ it is stably causal.

The existence of a gradient conformal vector field has been used to study certain cosmological models[4][5].

Along this talk, K is a timelike gradient conformal vector field $\longrightarrow \bar{M}$ will be called gradient conformally stationary (GCS) spacetime.

Generalized Robertson Walker (GRW) spacetimes are GCS spacetimes:

If $\overline{M} = (I \times F, -dt^2 + f(t)^2 g) \dashrightarrow K = f(t)\partial_t \dashrightarrow$ the leaves of $\mathcal{F}$ are the slices.

[4] Daftardar V and Dadhich N 1994 Gradient conformal Killing Vectors and Exact solutions *Gen. Relat. Gravit.* **26** 859–868.

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Our results in GCS spacetimes

Mean curvature of a compact spacelike hypersurface in a GCS spacetime

Setting: $\overline{M}$ is a GCS spacetime with timelike gradient conformal vectorfield K.

S is a compact spacelike hypersurface.

If ϕ is a potential function of K and N is the normal vectorfield to S in the same orientation of $-K$

$$\Delta\phi_S = n\rho + nH\bar{g}(K, N), \qquad (1)$$

where H is the mean curvature of S and Δ the Laplacian of $\bar{g}_{|S}$.

When K is Killing, $\rho = 0$ and so

Proposition

The only compact CMC spacelike hypersurfaces are the totally geodesic leaves of the foliation $\mathcal{F}$.

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When K is conformal, we consider

$$p_0 \in S \longrightarrow \phi_S(p_0) \text{ local minimum}$$
$$p^0 \in S \longrightarrow \phi_S(p^0) \text{ local maximum}$$

In p_0 and p^0: $\quad \Delta\phi_S = n\rho + nH\bar{g}(K, N) = n\rho + nHh, \quad$ and so:

Theorem

If S has constant mean curvature, then it is bounded by the mean curvatures of the leaves of $\mathcal{F}$ through p_0 and p^0

$$\frac{-\rho(p_0)}{h(p_0)} \leq H \leq \frac{-\rho(p^0)}{h(p^0)} \qquad (2)$$

Corollary

When K is a timelike gradient homothetic vector field K.
The only compact CMC spacelike hypersurfaces on which the length of K, h, is constant are the leaves of $\mathcal{F}$.

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Main result in GCS spacetimes

Theorem

Let $(\overline{M}, \bar{g})$ be a GCS spacetime and let K be a timelike gradient conformal vector field on $\overline{M}$.
Suppose that $(\log h(\gamma(t)))'' \leq 0$ for each integral curve γ of Z,
then the only compact CMC spacelike hypersurfaces are the leaves of $\mathcal{F}$.

Corollary: Alias and Montiel

In a GRW spacetime $\overline{M}$ whose warping function satisfies $(\log f)'' \leq 0$, the only compact CMC spacelike hypersurfaces are the slices.

Sketch of the proof:

- If γ is an integral curve of Z, $\frac{d}{dt}\left(\frac{\rho(\gamma(t))}{h(\gamma(t))}\right) = (\log h(\gamma(t)))''$.
- Since ϕ is a global time function, it is strictly decreasing along γ.
- Using this two facts: $-\frac{\rho}{h}_{|S}$ attains its maximum (resp. minimum) at a local minimum (resp. maximum) of the function ϕ_S.
- From (2), we get $H = -\frac{\rho}{h} \Rightarrow \Delta\phi_S \leq 0$, and so it is constant.

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Einstein GCS spacetimes

It is well-know that $\overline{M} = (I \times F, -dt^2 + f(t)^2 g)$ is Einstein with $\overline{\mathrm{Ric}} = \bar{c}\,\bar{g}$, if and only if (F, g) has constant Ricci curvature c and f satisfies the following differential equations

$$\frac{f''}{f} = \frac{\bar{c}}{n} \qquad \text{and} \qquad \frac{\bar{c}(n-1)}{n} = \frac{c + (n-1)(f')^2}{f^2} \tag{3}$$

The solutions of (3) are[6]

1. $\bar{c} > 0, c > 0 \dashrightarrow f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt}, \quad a > 0, \quad b = \sqrt{\bar{c}/n}$
2. $\bar{c} > 0, c = 0 \dashrightarrow f(t) = ae^{\varepsilon bt}, \quad a > 0, \quad \varepsilon = \pm 1, \quad b = \sqrt{\bar{c}/n}$
3. $\bar{c} > 0, c < 0 \dashrightarrow f(t) = ae^{bt} + \frac{cn}{4a\bar{c}(n-1)}e^{-bt} \quad a \neq 0, \quad b = \sqrt{\bar{c}/n}$
4. $\bar{c} = 0, c = 0 \dashrightarrow f(t) = a, \quad a > 0$
5. $\bar{c} = 0, c < 0 \dashrightarrow f(t) = \varepsilon\sqrt{\frac{-c}{(n-1)}}\,t + a, \quad \varepsilon = \pm 1$
6. $\bar{c} < 0, c < 0 \dashrightarrow f(t) = a_1 \cos(bt) + a_2 \sin(bt), a_1^2 + a_2^2 = cn/\bar{c}(n-1),$

   $b = \sqrt{-\frac{\bar{c}}{n}}$

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Note that, from (3), we have

$$(n-1)(\log f)'' = \frac{c}{f^2}.$$

Therefore, as a direct application of the previous Corollary we get

THEOREM

Every compact CMC spacelike hypersurface in an Einstein GRW spacetime, whose fiber has Ricci curvature $c \leq 0$ (cases 2 to 6), must be a slice.

Now we will deal with the remaining case, in which $(\log f)''(t) > 0$.

If S is a compact CMC spacelike hypersurface

$$\frac{-f'(t(p^0))}{f(t(p^0))} \leq H \leq \frac{-f'(t(p_0))}{f(t(p_0))}$$

where $p^0 = \max t(S)$ and $p_0 = \min t(S)$.

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where $p^0 = \max t(S)$ and $p_0 = \min t(S)$.

On the other hand, from the expression of f,

$$-\sqrt{\bar{c}/n} \leq \frac{f'(t)}{f(t)} \leq \sqrt{\bar{c}/n}.$$

THEOREM

If $\overline{M}$ is an Einstein GRW spacetime of the first type, there is no compact CMC spacelike hypersurface in $\overline{M}$ with mean curvature $H \geq \sqrt{\bar{c}/n}$ or $H \leq -\sqrt{\bar{c}/n}$, being $\bar{Ric} = \bar{c}\bar{g}$.
Even more, any compact CMC spacelike hypersurface in $\overline{M}$ with mean curvature $H \in]-\sqrt{\bar{c}/n}, \sqrt{\bar{c}/n}[$, intersects the only slice in $\overline{M}$ with mean curvature H.

In the De Sitter spacetime, $\mathbb{R} \times_{\cosh t} \mathbb{S}^n$, there exists a compact CMC spacelike hypersurface with mean curvature $H^2 \leq 1$, which is not a slice.
CONCLUSION: The previous theorem is the best possible result for case 1.

On the other hand, from the expression of f,

$$-\sqrt{\bar{c}/n} \le \frac{f'(t)}{f(t)} \le \sqrt{\bar{c}/n}.$$

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Compact CMC spacelike hypersurfaces in an Einstein GRW spacetimes

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Thank you!!

Global structure of GCS spacetimes

Theorem

A Lorentzian manifold $(\overline{M}, \bar{g})$ admits a global decomposition as a GRW spacetime if and only if it is a GCS spacetime with a timelike gradient conformal vector field, K, such that the flow of its normalized vector field, Z, is well defined and onto in a domain $I \times \mathcal{L}$, for some interval $I \subseteq \mathbb{R}$ and some leaf, $\mathcal{L}$, of the orthogonal foliation $\mathcal{F}_K$ to K.

Theorem

Let $(\overline{M}, \bar{g})$ be a GCS spacetime and let K be a timelike gradient conformal vector field on $\overline{M}$. Suppose that the leaves of $\mathcal{F}_K$ are compact. Then $(\overline{M}, \bar{g})$ admits a global decomposition as a GRW spacetime.