Stability of marginally outer trapped surfaces and applications

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9 September 2011

Outline

Introduction

- Black holes are among the most important objects in general relativity.
- They are conjectured to be the generic end-state of a gravitational collapse: Weak cosmic censorship.
- Black holes, by definition require global in time information on the spacetime $\rightarrow$ Concept of little use from an evolutionary point of view.
- Need for a replacement that can be defined at each instant of time.
- Important for (among others):
  - Numerical evolutions: Need of tracking the surface of the “black holes”.
  - Tool for addressing weak cosmic censorship: A good quasi-local replacement should approximate the event horizon at late times (if it forms).

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Marginally outer trapped surfaces (MOTS) are widely believed to be good replacements of black holes.

- Confirmation requires a proper understanding of these objects.
- Some progress has been made in the last years.

In this talk I will discuss the concept of stability of MOTS and some of its consequences.

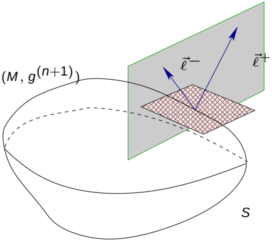
Geometry of spacelike surfaces

Surface S: closed (i.e. compact, without boundary), orientable, spacelike codimension-two embedded submanifold of $(M, g^{(n+1)})$ ($n \geq 3$).

- Scalar product with $g^{(n+1)}$ denoted by $<\, ,\, >$.
- NS admits a basis of future null normals $\vec{\ell}^{+}$ and $\vec{\ell}^{-}$ satisfying $< \vec{\ell}^{+}, \vec{\ell}^{-} >= -2$. Unique up to boosts $\vec{\ell}^{\pm} \longrightarrow F^{\pm 1}\vec{\ell}^{\pm}$, $F > 0$.

Notation:

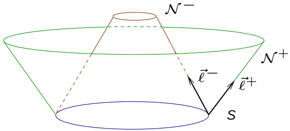
- Induced metric on S: h.
- Second fundamental form vector: $\vec{\chi}(\vec{X}, \vec{Y}) = -(\nabla_{\vec{X}} \vec{Y})^{\perp}$, $\vec{X}, \vec{Y}$ tangent to S.
- Null extrinsic curvatures $\chi_{\pm}(\vec{X}, \vec{Y}) \equiv < \vec{\chi}(\vec{X}, \vec{Y}), \vec{\ell}^{\pm} >$.
- Mean curvature vector: $\vec{H} = \mathrm{tr}_S \vec{\chi}$.
- Null expansions: $\theta_{\pm} =< \vec{H}, \vec{\ell}^{\pm} >$



- Connection on the normal bundle: $\boldsymbol{s}(\vec{X}) \equiv -\frac{1}{2} < \vec{\ell}^{-}, \nabla_{\vec{X}} \vec{\ell}^{+} >$.

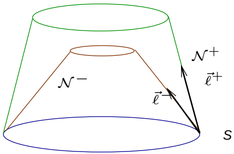
Marginally outer trapped surfaces

- $\mathcal{N}^+$ and $\mathcal{N}^-$: null hypersurfaces generated by pulses of light starting on S along $\vec{\ell}^+$ and $\vec{\ell}^-$.



Behaviour of area of future light fronts:

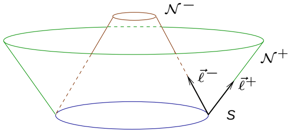
- "Normally" (weak gravitational field), one of the areas increases and the other decreases.
- In strong gravitational fields: Both areas decrease.



First variation of area: $\delta_{\vec{\xi}}|S| = \int_S < \vec{H}, \vec{\xi} > \boldsymbol{\eta_S}$

Convergence in both directions $\Longleftrightarrow$ $\vec{H}$ future timelike.

Marginally outer trapped surfaces



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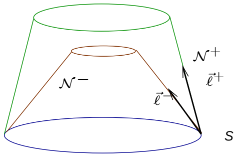

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Convergence in both directions $\Longleftrightarrow$ $\vec{H}$ future timelike.

- If there exists a notion of "exterior", non-increase of area along the exterior null cone also signals a strong gravitational field.

Definition (Marginally outer trapped surface (MOTS))

A MOTS is a surface with stationary area w.r.t. variations along $\mathcal{N}^+$ or along $\mathcal{N}^-$ $\Longleftrightarrow$ $\vec{H} \propto \vec{\ell}^+$ or $\vec{H} \propto \vec{\ell}^-$ everywhere $\Longleftrightarrow$ $\theta_+ = 0$ or $\theta_- = 0$ everywhere.

- Physically useful when an "exterior" $\vec{\ell}^+$ is well-defined and $\vec{H} \propto \vec{\ell}^+$.

Stability of MOTS

- MOTS are critical points of area: expect similarities with minimal surfaces.
- For minimal hypersurfaces: The second variation of area defines the stability operator L_0: $\delta^2_{\psi\vec{m}}|S| = \int_S \psi L_0 \psi \boldsymbol{\eta_S}$.
- However, 2^{nd} variation of area for MOTS:

$$\delta^2_{\psi\vec{\ell}^+}|S| = -\int_S \psi^2 \left(\mathrm{Ein}(\vec{\ell}^+, \vec{\ell}^+) + \mathrm{tr}_h(\chi^2_+) \right) \boldsymbol{\eta_S}$$

- No differential operator on ψ and always non-positive if the null energy condition (NEC) is satisfied ($\mathrm{Ein}(\vec{\ell}, \vec{\ell}) \geq 0, \forall \vec{\ell}$ null).
- For minimal sufaces: information on second variation of area, essentially equivalent to first variation of mean curvature.

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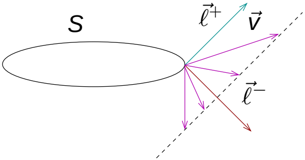
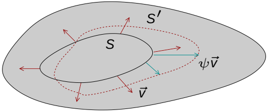


- Fix a section of the bundle of normal directions nowhere tangent to $\vec{\ell}^+$.
- Given $\vec{\ell}^+$, this section is uniquely represented by a vector field $\vec{v} \in \mathfrak{X}(S)^\perp$ satisfying:

$$\vec{v} = -\frac{1}{2}\vec{\ell}^- + V\vec{\ell}^+, \quad V \in C^\infty(S, \mathbb{R})$$

- Define $\vec{v}^\star = \frac{1}{2}\vec{\ell}^- + V\vec{\ell}^+$, $\quad < \vec{v}^\star, \vec{v} >= 0, \quad < \vec{v}^\star, \vec{v}^\star >= - < \vec{v}, \vec{v} >$.

Stability operator



- For any function ψ on S, we can calculate the first order variation $\delta_{\psi\vec{v}}\theta_+$ along $\psi\vec{v}$.
- Defines the stability operator L_v: $\boldsymbol{L_v\psi \equiv \delta_{\psi\vec{v}}\theta_+}$.

Explicitly:

$$L_v\psi = -\Delta_h\psi + 2\mathbf{s}(\nabla_h\psi) + \left(\frac{R(h)}{2} - \mathrm{Ein}(\vec{\ell}^+, \vec{v}^\star) - V\mathrm{tr}_h(\chi_+^2) - ||\mathbf{s}||_h^2 + \mathrm{div}_h\mathbf{s}\right)\psi,$$

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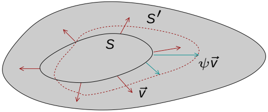

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- Similar role for MOTS as the stability operator L_0 for minimal surfaces.

Properties:

- Elliptic operator, not self-adjoint in general.
- There exists a unique principal eigenvalue λ_v.
  - Eigenvalue with smallest real part, always real, eigenspace is one-dimensional, eigenfunctions have constant sign.

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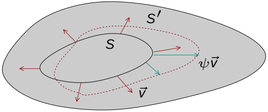

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- There exists a unique principal eigenvalue λ_v.
  - Eigenvalue with smallest real part, always real, eigenspace is one-dimensional, eigenfunctions have constant sign.
- Characterizations of the principal eigenvalue (Donsker & Varadhan)

$$\lambda_v = \inf_{\mu_S}\sup_{\psi}\int_S \psi^{-1}L_v\psi\mu_S, \qquad \lambda_v = \sup_{\psi}\inf_{x\in S}\psi^{-1}(x)L_v\psi(x),$$

$\mu_S \in \{$probability measures on $S\}$ $\quad \psi \in \{$smooth positive functions on $S\}$.

Properties of the principal eigenvalue

- Perform the Hodge decomposition $\boldsymbol{s} = df + \boldsymbol{z}$ ($\mathrm{div}_h \boldsymbol{z} = 0$).

Lemma (Andersson, M., Simon)

Let $L_v\psi = -\Delta_h\psi + 2\mathbf{s}(\nabla_h\psi) + c\psi$. The principal eigenvalue λ_v admits the Rayleigh-Ritz type characterization

$$\lambda_v = \inf_u \int_S \left(||\nabla_h u||_h^2 + Qu^2 - ||d\omega_u + \boldsymbol{z}||_h^2 u^2 \right) \boldsymbol{\eta}_S, \quad u \in C^\infty(S, \mathbb{R}^+), ||u||_{L^2} = 1,$$

where $Q \equiv c + ||\mathbf{s}||_h^2 - \mathrm{div}_h \mathbf{s}$ and ω_u is the unique solution of

$$-\Delta_h \omega_u - 2u^{-2} < \nabla_h \omega_u, \nabla_h u >= 2u^{-2} \boldsymbol{z}\left(\nabla_h u\right), \quad \int_S u^2 \omega_u \boldsymbol{\eta}_S = 0.$$

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- If $\boldsymbol{s}$ is a gradient, $\omega_u = 0, \forall u \longrightarrow$ standard Rayleigh-Ritz quotient.
- Define the self-adjoint operators

$$L_s = -\Delta_h + Q, \qquad \text{principal eigenvalue } \lambda_s$$
$$L_z = -\Delta_h + Q - ||\boldsymbol{z}||^2_h, \qquad \text{principal eigenvalue } \lambda_z$$

Then, $\boldsymbol{\lambda_s \geq \lambda_v \geq \lambda_z}$.

Stable and strictly stable MOTS

Definition

- *A MOTS S is called strictly stable along $\vec{v}$ if $\lambda_v > 0$.*
- *A MOTS S is called stable along $\vec{v}$ if $\lambda_v \geq 0$.*
- *A MOTS S is called unstable along $\vec{v}$ if $\lambda_v < 0$.*

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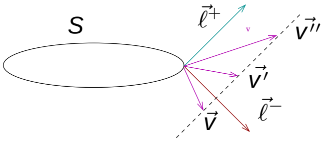
- Let $W \equiv \mathrm{tr}_h(\chi_+^2) + \mathrm{Ein}(\vec{\ell}^+, \vec{\ell}^+)$. The eigenvalues λ_v, $\lambda_{v'}$ satisfy the estimates

$$\lambda_{v'} + \inf_S[(V' - V)W] \leq \lambda_v \leq \lambda_{v'} + \sup_S[(V' - V)W].$$

  - If $W \equiv 0$, λ_v is independent of direction.
  - Under NEC, $\lambda_{v''} \leq \lambda_{v'} \leq \lambda_v$
  - Under NEC and $W \not\equiv 0$, $\lambda_{v''} < 0$ for some direction sufficiently close to $\vec{\ell}^+$.



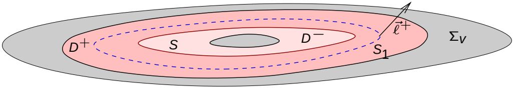
- Under NEC, a MOTS which is stable with respect to a spacelike direction, it is also stable with respect to the null direction $-\vec{\ell}^-$.

Stable, locally outermost and strictly stable MOTS

- The stability operator is related to barrier properties of MOTS.
- Fix $\vec{v}$. Let Σ_v be a hypersurface containing S and tangent to $\vec{v}$.

Definition

S is called locally outermost in Σ_v if $\exists$ a two-sided neighbourhood $D^+ \cup D^- \subset \Sigma_v$ of S such that the exterior D^+ contains no surface S_1satisfying: (i) $S \cup S_1$ bound a domain in D^+, (ii) $\theta_+[S_1] \leq 0$ w.r.t. the outer normal.



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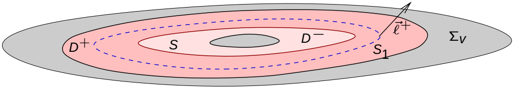

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Lemma (Andersson, M., Simon)

A MOTS S is stable along $\vec{v}$ iff there exists an outward variation $\psi\vec{v}$ ($\psi \geq 0$, $\psi \not\equiv 0$) such that $\delta_{\psi\vec{v}}\theta_+ \geq 0$. S is strictly stable iff there exist an outward variation such that $\delta_{\psi\vec{v}}\theta_+ > 0$.

- Consequence:

Strictly stable along $\vec{v}$ $\Longrightarrow$ Locally outermost in Σ_v $\Longrightarrow$ Stable along $\vec{v}$

Topology of locally outermost MOTS

- Recall that a spacetime $(M, g^{(n+1)})$ satisfies the Dominant Energy Condition (DEC) if $-\mathrm{Ein}(\vec{u}, \cdot)$ is future causal $\forall \vec{u}$ future causal.

Theorem (Galloway & Schoen; Galloway '06)

Let $(M, g^{(n+1)})$ be a spacetime satisfying DEC. If S is a locally outermost MOTS for some $\Sigma_v \subset M$ with $\vec{v}$ spacelike, then S is of positive Yamabe type (i.e. admits a metric of positive constant curvature).

- Main analytic ingredient:

$$\lambda_s = \frac{\int_S \psi L_s \psi \eta_S}{\int_S \psi^2 \eta_S} \geq \lambda_v \geq 0.$$

- Particular case: Two dimensional locally outermost MOTS in spacetimes satisfying DEC are of spherical topology.

MOTS and symmetries

- The stability operator is written in terms of the intrinsic geometry of S.
- Sometimes not the most convenient expression. Example
  - Local groups of isometries transform MOTS into MOTS $\rightarrow$ $\delta_{\vec{\xi}}\theta_+ = 0$ on any MOTS along a Killing field $\vec{\xi}$.
  - If $\vec{\xi} = F\vec{v}$ with $< \vec{\ell}^+, \vec{v} >= 1$ it follows, $L_v F = 0$.
- Property not at all obvious from the explicit expression of $L_v \longrightarrow$ Need for an alternative expression of L_v adapted to spacetime information.

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Lemma (Carrasco, M.)

Let $\vec{\xi}$ be a vector field in a spacetime $(M, g^{(n+1)})$. Define the deformation tensor $a^{\xi} \equiv \mathcal{L}_{\vec{\xi}}\, g^{(n+1)}$. Let S be a MOTS with embedding Φ_S, then

$$\delta_{\vec{\xi}}\theta_+ = -\frac{1}{4}\theta_- a^{\xi}(\vec{\ell}^+, \vec{\ell}^+) - tr_h \left(a^{\xi,S} \cdot \chi_+ \right) + h^{\alpha\beta}\ell^{+\gamma} \left[\frac{1}{2}\nabla_\gamma a^{\xi}_{\alpha\beta} - \nabla_\alpha a^{\xi}_{\gamma\beta} \right] \Big|_S .$$

where $a^{\xi,S} = \Phi_S^\star(a^{\xi})$ and $h^{\alpha\beta}$ is the projector tangent to S.

- For a Killing $\vec{\xi}$, $\delta_{\vec{\xi}}\theta_+ = 0$ follows immediately.

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- For a Killing $\vec{\xi}$, $\delta_{\vec{\xi}}\theta_+ = 0$ follows immediately.
- This expression allows to relate MOTS and symmetries: Example

MOTS and symmetries (II)

Theorem (Carrasco, M. '09)

Let S be a stable MOTS with respect to a direction $\vec{v}$. Suppose that $(M, g^{(n+1)})$ admits a conformal Killing vector $\vec{\xi}$, i.e. $\mathcal{L}_{\vec{\xi}}\, g^{(n+1)} = 2\phi g^{(n+1)}$ (including homotheties $\phi = C$, and isometries $\phi = 0$).

(i) *If $0 \not\equiv 2\vec{\ell}^{+}(\phi) - < \vec{\xi}, \vec{v}^{\star} > W|_S \leq 0$, then $< \vec{\xi}, \vec{\ell}^{+} > |_S < 0$.*

(ii) *If S is strictly stable and $2\vec{\ell}^{+}(\phi) - < \vec{\xi}, \vec{v}^{\star} > W|_S \leq 0$ then $< \vec{\xi}, \vec{\ell}^{+} > |_S \leq 0$ and vanishes at one point only if it vanishes everywhere.*

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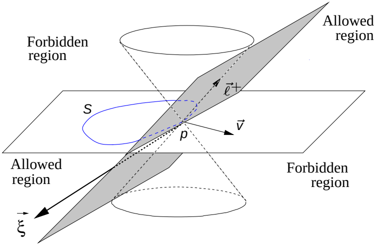
- For Killing vectors or homotheties in spacetimes satisfying the NEC.



Axially symmetric MOTS and angular momentum

Definition

A MOTS S is axially symmetric if there exists a vector $\vec{\eta} \in \mathfrak{X}(S)$ satisfying

1. $\mathcal{L}_{\vec{\eta}} h = 0$.
2. $\mathcal{L}_{\vec{\eta}} \boldsymbol{s} = 0$, *for some choice of basis* $\{\vec{\ell}^{+}, \vec{\ell}^{-}\}$.
3. $\vec{\eta}$ *commutes with the stability operator* L_v *for some choice of* $\vec{v}$.

For axially symmetric MOTS, we always restrict $\{\vec{\ell}^{+}, \vec{\ell}^{-}\}$, to satisfy 2.

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Angular momentum of an axially symmetric MOTS: $J = \frac{1}{8\pi} \int_S \boldsymbol{s}(\vec{\eta}) \boldsymbol{\eta}_S$.

- The definition is independent of the choice of basis $\{\vec{\ell}^{+}, \vec{\ell}^{-}\}$ (satisfying 2.)

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Theorem (Jaramillo, Reiris, Daín, 2011)

Let S be an axially symmetric, two-dimensional MOTS, stable with respect to the null direction $-\vec{\ell}^-$. Then

$$|\boldsymbol{S}| \geq \boldsymbol{8\pi} |\boldsymbol{J}|.$$

MOTS and Killing horizons

- Killing horizons are special null hypersurfaces closely related to MOTS.
- What are the properties of MOTS lying on Killing horizons?

We adopt the following definition:

Definition (Null hypersurface)

Let S be a closed, oriented, $(n-1)$-manifold. A null hypersurface $\mathcal{H}_\xi$ is a manifold with topology $S \times \mathbb{R}$, and an embedding $\Phi : \mathcal{H}_\xi \longrightarrow (M, g^{(n+1)})$ such that $g_{\mathcal{H}_\xi} \equiv \Phi^\star(g^{(n+1)})$ is degenerate along the $\mathbb{R}$ factor.

MOTS and Killing horizons

- Killing horizons are special null hypersurfaces closely related to MOTS.
- What are the properties of MOTS lying on Killing horizons?

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Definition (Killing horizon)

Let $(M, g^{(n+1)})$ with a Killing vector $\vec{\xi}$. A Killing horizon of $\vec{\xi}$ is a null embedded hypersurface $\mathcal{H}_\xi$ such that $\forall p \in \mathcal{H}_\xi$, $\xi|_p \neq 0$, future null, and tangent to $\mathcal{H}_\xi$.

Definition (Surface gravity)

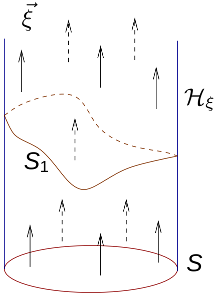
The surface gravity κ_ξ of a Killing horizon $\mathcal{H}_\xi$ is a function $\mathcal{H}_\xi \to \mathbb{R}$ defined by

$$\nabla_{g^{(n+1)}} \lambda|_p = 2\kappa_\xi \vec{\xi}|_p, \quad \forall p \in \mathcal{H}_\xi, \qquad \text{where} \quad \lambda \equiv - < \vec{\xi}, \vec{\xi} > .$$

Killing horizons

Well-known properties of Killing horizons:

- Let $\pi : \mathcal{H}_\xi \simeq S \times \mathbb{R} \to S$ the projection onto S.
- There exists a Riemannian metric h on S such that $g_{\mathcal{H}_\xi} = \pi^\star(h)$.
- $\mathrm{Ein}(\vec{\xi}, \vec{\xi}) = 0$ on $\mathcal{H}_\xi$.
- Let S_1 be any section of $(\mathcal{H}_\xi, S, \pi)$ and h_1 the induced metric.
- (S_1, h_1) is isometric to (S, h) (with isometry π).



- S_1 is a MOTS w.r.t the null direction $\vec{\xi}$.
- S_1 satisfies $\chi^\xi \equiv < \vec{\chi}, \vec{\xi} > = 0$.

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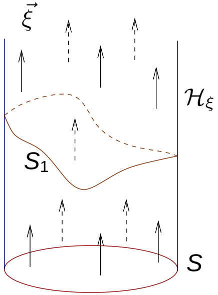

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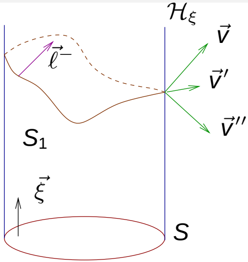
- S_1 is a MOTS w.r.t the null direction $\vec{\xi}$.
- S_1 satisfies $\chi^\xi \equiv < \vec{\chi}, \vec{\xi} > = 0$.
- If $(M, g^{(n+1)})$ satisfies the DEC then κ_ξ is constant on $\mathcal{H}_\xi$.
- If the Killing vector is integrable ($\boldsymbol{\xi} \wedge d\boldsymbol{\xi} = 0$), κ_ξ is also constant on $\mathcal{H}_\xi$.
- We assume from now on that the surface gravity is constant.

Definition

A Killing horizon is degenerate if $\kappa_\xi \equiv 0$ and non-degenerate if $\kappa_\xi \neq 0$.

Stability operator of MOTS in Killing horizons

- Since $\text{Ein}(\vec{\xi}, \vec{\xi}) + tr_h(\chi_\xi^2) = 0$ on a Killing horizon, the principal eigenvalue of S_1 is independent of $\vec{v}$.
- On S_1 select the null, future directed normal vector $\vec{\ell}^-$ satisfying $< \vec{\ell}^-, \vec{\xi} >= -2$.



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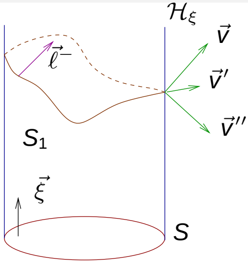

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Lemma (Stability operator of a section of a Killing horizon)

Let $\mathcal{H}_\xi$ a Killing horizon and S_1 a section on $\mathcal{H}_\xi$. The stability operator of S_1 is

$$L_{S_1}(\psi) \equiv \delta_{-\frac{1}{2}\psi\vec{\ell}^-} \equiv -div_h\left(u\nabla_h\left(\frac{\psi}{u}\right)\right) + 2\boldsymbol{z}(\nabla_h\psi) + \frac{1}{2}\kappa_\xi\psi,$$

where $u, \boldsymbol{z}$ are defined by the Hodge decomposition $\boldsymbol{s} = \frac{1}{2u}du + \boldsymbol{z}$ of the normal connection $\boldsymbol{s}$ in the basis $\{\vec{\ell}^-, \vec{\xi}\}$.

Stability of Killing horizons

- Natural question: Does the stability operator depend on the section S_1?
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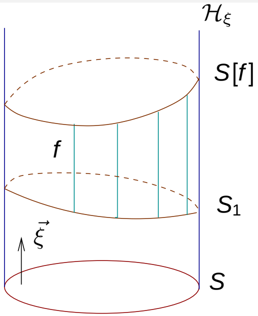
- Let $\pi_f : S[f] \to S_1$ be the natural projection.



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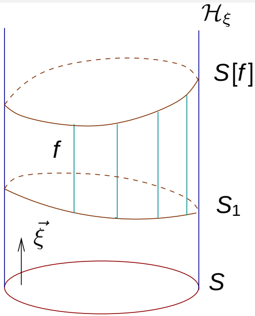

Proposition (Dependence of stability operator on the section)

Assume that κ_ξ is constant. The stability operator of $S[f]$ is related to the stability operator of S_1 by

$$L_{S[f]}\left(\psi \circ \pi_f\right) = e^{-\kappa_\xi f} L_{S_1}\left(e^{\kappa_\xi f}\psi\right) \circ \pi_f, \qquad \forall \psi \in C^2(S_1, \mathbb{R}).$$

- Corollary: the principal eigenvalue is independent of the section.
- The stability eigenvalue $\lambda_{\mathcal{H}_\xi}$ is a property of the Killing horizon $\mathcal{H}_\xi$.

Static and axially symmetric Killing horizons

Definition

- *A Killing horizon $\mathcal{H}_\xi$ is* static *if the defining Killing vector $\vec{\xi}$ is integrable.*
- *A Killing horizon is* axially symmetric *if there exists a section S_1 which is an axially symmetric MOTS.*

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- The divergence-free part $\boldsymbol{z}$ of the connection $\boldsymbol{s}$ of the normal bundle is independent of the section.
- Let S be simply connected and $\mathcal{H}_\xi \simeq S \times \mathbb{R}$. If $\mathcal{H}_\xi$ is either static or axially symmetric and 3-dimensional then the principal eigenfunction ϕ of any section necessarily satisfies $\boldsymbol{z}(\nabla_{h_1}\phi) = 0$.
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Proposition

Let S be simply connected and $\mathcal{H}_\xi \simeq S \times \mathbb{R}$. Assume that $\mathcal{H}_\xi$ is either static or 3-dimensional and axially symmetric. If $\mathcal{H}_\xi$ is degenerate (i.e. with vanishing surface gravity) then it is marginally stable (i.e. $\lambda_{\mathcal{H}_\xi} = 0$)

- What about non-degenerate horizons?

Area-angular momentum inequality for Killing horizons

Theorem (Ansorg, Pfister; Hennig, Ansorg, Cederbaum '08)

Let $\mathcal{H}_\xi$ be an axially symmetric Killing horizon of topology $\mathbb{S}^2 \times \mathbb{R}$.

- *If $\mathcal{H}_\xi$ is degenerate then $|S| = 8\pi|J|$.*
- *If $\mathcal{H}_\xi$ is non-degenerate and satisfies $\int_{S_1} u\theta_- \eta_{S_1} > 0$ then $|S| > 8\pi|J|$.*

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Corollary 1

Non-degenerate, three-dimensional, stable axially symmetric Killing horizons satisfy the strict inequality $|S| > 8\pi|J|$.

Non-existence of two-black hole configurations

Theorem (Hennig, Neugebauer '11)

No stationary and axially symmetric, vacuum, four-dimensional, black hole spacetime can have an event horizon with two connected components provided each connected component is either

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Corollary 2

There exists no stationary and axially symmetric, vacuum, four-dimensional black hole spacetime with an event horizon consisting of two connected components.

MOTS and 3-slices

- MOTS are particularly relevant in the context of 3+1 decomposition of a spacetime.
- Consider an initial data set $(\Sigma, g, K, \rho, \boldsymbol{J})$ (Σ possibly with boundary):

$$R(\gamma) - ||K||_g^2 + (\mathrm{tr}_g K)^2 = 2\rho, \qquad \mathrm{div}_g(K - \mathrm{tr}_g K\, g) = -\boldsymbol{J}.$$

- Assume (Σ, g, K) asymptotically flat: $\Sigma = \mathcal{K} \cup (\mathbb{R}^n \setminus B_R)$ with $\mathcal{K}$ compact and $g \to g_E$, $K \to 0$ at infinity (at an appropriate rate).

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- A surface $S \subset \mathrm{int}(\Sigma)$ is **bounding** if $\Sigma \setminus S$ is not connected.
- The **exterior region** $\Omega^+(S)$ is the connected component of $\Sigma \setminus S$ containing the asymptotically flat end.
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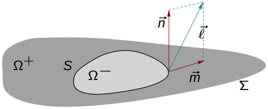
- Canonical choice: The outer normal $\vec{m}$ points towards Ω^+.
- If $(\Sigma, g, K, \rho, \boldsymbol{J})$ is embedded in a spacetime:
  - $\vec{n}$: Future directed unit normal of $\Sigma \subset (M, g^{(n+1)})$
  - Outer null direction $\vec{\ell}^+ = \vec{n} + \vec{m}$
  - $\theta_\pm = \pm H + \mathrm{tr}_S K$, H mean curvature along $\vec{m}$.



MOTS and the future trapped region

- MOTS are borderline objects. Do they arise as interesting boundaries?

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A bounding surface $S \subset int(\Sigma)$ is called a weakly outer trapped surface if $H + tr_S K \leq 0$ and a bounding MOTS if $H + tr_S K = 0$.

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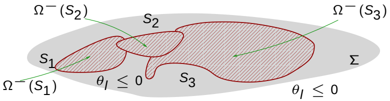

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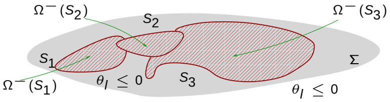

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Theorem (Andersson & Metzger, 2007)

Let $(\Sigma, g, K, \rho, \boldsymbol{J})$ be 3-dimensional and asymptotically flat. Then $\partial^t \mathcal{T}^+$ is either empty or a finite union of disjoint stable MOTS.

- Similar result hold for the past trapped region: $\mathcal{T}^- = \bigcup_{\theta_-(\partial\Omega^-)\geq 0} \Omega^-.$

MOTS and uniqueness theorems

- The most fundamental issue regarding MOTS is: How do they evolve in spacetime?
- There exist a number of results in this direction (and many open questions).
- We only mention one such result addressing the question:

What happens if the spacetime does not evolve at all?

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What happens if the spacetime does not evolve at all?

- If MOTS are to be quasi-local, dynamical, replacements of black holes, in a non-evolving situation they should coincide.
- However, stationary black holes satisfy uniqueness theorems.

Natural question:

Do stationary spacetimes containing MOTS also satisfy uniqueness theorems?

Uniqueness theorem for static spacetimes with MOTS

- Result in the static case:

Theorem (Carrasco, M. '11)

Let (Σ, g, K) be an asymptotically flat initial data set embedded in a vacuum spacetime admitting a static Killing vector $\vec{\xi}$.
If $\mathcal{T}^+$ is not empty, then $\Sigma \setminus \mathcal{T}^+$ can be isometrically embedded into the Schwarzschild spacetime of mass m for some $m > 0$, provided:

(i) *The first homology group $H_1(\Sigma \setminus \mathcal{T}^+, \mathbb{Z}_2)$ is trivial.*

(ii) *$\mathcal{T}^-$ is non-empty and satisfies $\mathcal{T}^- \subset \mathcal{T}^+$.*

(iii) *Each arc-connected component of $\partial^t \Sigma^{ext}$ is topologically closed, where Σ^{ext} is the connected component of $\{\vec{\xi}$ timelike $\} \cap \Sigma$ containing the asymptotically flat end.*

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- Not fully satisfactory because of conditions (i)-(iii) (required by the method of proof, not by any essential reason).
- Problem: Prove the theorem after dropping conditions (i)-(iii):

Work in progress with Martín Reiris (AEI, Potsdam)