Singularities of the asymptotic completion of developable Möbius strips

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Let U be an open domain in Euclidean two-space $\boldsymbol{R}^2$ and $f : U \longrightarrow \boldsymbol{R}^3$ a C^∞ map. A point $p \in U$ is called a *singular point* of f if the Jacobi matrix of f is of rank less than 2 at p. Let

$$F(s,u) = \gamma(s) + u\xi(s) \qquad (|u| < \epsilon)$$

be a ruled Möbius strip immersed in $\boldsymbol{R}^3$, where $\epsilon > 0$, $\gamma(s)$ is a generating curve and $\xi(s)$ is a ruling vector field of F. Then, the C^∞ map

$$\tilde{F}(s,u) = \gamma(s) + u\xi(s) \qquad (u \in \boldsymbol{R})$$

is called the *asymptotic completion* (or *a-completion*) of the immersed strip F. It is well-known that complete and flat (i.e. zero Gaussian curvature) surfaces immersed in $\boldsymbol{R}^3$ are cylindrical. This fact implies that the a-completion of a developable Möbius strip (i.e. a flat ruled Möbius strip) must have singular points. Since the most generic singular points appeared on developable surfaces are cuspidal edge singularities (cf. [1]), we are interested in how often singular points other than cuspidal edge singularities appear on the a-completion of a developable Möbius strip. We have the following result:

Proposition. *The asymptotic completion of a developable Möbius strip has at least one singular point other than cuspidal edge singularities.*

A developable Möbius strip which contains a closed geodesic is called a *rectifying Möbius strip*. Roughly speaking, a rectifying strip can be constructed from an isometric deformation of a rectangular domain on a plane. We also prove the following assertion:

Theorem. *The asymptotic completion of a rectifying Möbius strip has at least three singular points other than cuspidal edge singularities.*

These lower bounds of the numbers of non-cuspidal-edge singularities in the proposition and the theorem are both sharp.

References

[1] S. Murata and M. Umehara, *Flat surfaces with singularities in Euclidean 3-space*, J. Diff. Geom. **82** (2009), 279–316.