Beltrami-Meusnier Formulas of Semi Euclidean Space E_v^{n+1}

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Abstract. In this paper, we study the sectional curvatures of the generalized semi ruled surfaces in semi Euclidean space E_v^{n+1}. The first fundamental form and the metric coefficients of the generalized semi ruled surfaces are calculated and in these regards, Riemannian-Christoffel curvatures are obtained by the help of Christoffel Symbols. So, the curvatures of arbitrary non-degenerate tangential sections of the generalized semi ruled surface is investigated. In addition to this, the relations between the sectional curvatures are obtained. These are called semi-Euclidean Beltrami-Meusnier formulas.

Key words: *Sectional Curvature, Ruled surface, Beltrami-Meusnier Formula.*

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1.Preliminaries

Semi Euclidean space E_v^{n+1} is an Euclidean space provided with the metric tensor

$$\left\langle \vec{X},\vec{Y} \right\rangle = -\sum_{i=1}^{v} x_i y_i + \sum_{i=v+1}^{n+1} x_i y_i \tag{1.1}$$

where $\overrightarrow{X} = (x_1,...,x_n,x_{n+1})$ and $\vec{Y} = (y_1,...,y_n,y_{n+1})$. Especially, if $v=0$, then E_0^{n+1} is called Euclidean space, if $v=1$ $(n \geq 2)$ then E_v^{n+1} is called as Minkowski $n-$ space, [10]. Since $\langle \ , \ \rangle$ is an indefinite metric, recall that a vector $\overrightarrow{X} \in E_v^{n+1}$ can have one of the three causal characters; it can be spacelike if $\left\langle \vec{X},\vec{X} \right\rangle > 0$ or $\vec{X} = 0$, timelike if $\langle \vec{X}, \vec{X} \rangle < 0$ and null (lightlike) if $\langle \vec{X}, \vec{X} \rangle = 0$ and $\vec{X} \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(t) \subset E_v^{n+1}$ can locally be spacelike, timelike or null (lightlike) if all of its velocity vectors $\dot{\alpha}(t)$ are respectively spacelike, timelike or null (lightlike), [10]. The norm of $\overrightarrow{X} \in E_v^{n+1}$ is given as $\left\|\overrightarrow{X}\right\| = \sqrt{\left|\langle \overrightarrow{X}, \overrightarrow{X} \rangle\right|}$.

Let the set of all timelike vectors in E_v^{n+1} be Γ, we call

$$C\left(\overrightarrow{X}\right) = \left\{\overrightarrow{Y} \in \Gamma \middle| \ \langle \overrightarrow{Y}, \overrightarrow{X} \rangle < 0\right\} \tag{1.2}$$

for $\overrightarrow{X} \in \Gamma$, as time-cone of semi-Euclidean space E_v^{n+1} including vector $\overrightarrow{X}$, [10].

If the timelike vectors $\vec{X}$ and $\vec{Y}$ are in the same time-cone in E_v^{n+1}, then there is a unique non-negative real number such that

$$\langle \vec{X}, \vec{Y} \rangle = -\left\|\vec{X}\right\|\left\|\vec{Y}\right\| \cosh\theta \tag{1.3}$$

where the number θ is called an angle between the timelike vectors, [10].

If $\overrightarrow{X}$ and $\overrightarrow{Y}$ are spacelike vectors in E_v^{n+1} that span a timelike subspace, there is unique positive real number such that

$$\langle \overrightarrow{X}, \overrightarrow{Y} \rangle = \left\|\overrightarrow{X}\right\|\left\|\overrightarrow{Y}\right\| \cosh\theta \tag{1.4}$$

where the semi-Euclidean angle between $\overrightarrow{X}$ and $\overrightarrow{Y}$ is defined with θ, [11].

Suppose that $\overrightarrow{X}$ is spacelike vector and $\vec{Y}$ is timelike vector in E_v^{n+1}, Then, there is unique non-positive real number of such that

$$\left|\langle \overrightarrow{X}, \overrightarrow{Y} \rangle\right| = \left\|\overrightarrow{X}\right\|\left\|\overrightarrow{Y}\right\| \sinh\theta \tag{1.5}$$

where θ is a semi-Euclidean angle between the vectors $\overrightarrow{X}$ and $\overrightarrow{Y}$, [11].

A two-dimensional subspace Π of the tangent space $T_p(M)$ is called a tangent plane to M at p. For tangent vectors v, w define $Q(\vec{v}, \vec{w}) = \langle \vec{v}, \vec{v} \rangle \langle \vec{w}, \vec{w} \rangle - \langle \vec{v}, \vec{w} \rangle^2$. Tangent plane Π is non-degenerate if and only if $Q(\vec{v}, \vec{w}) \neq 0$ for one-hence-basis $\vec{v}, \vec{w}$ for Π. The absolute value $\left|Q(\vec{v}, \vec{w})\right|$ is the square of the area of the parallelogram with sides $\vec{v}$ and $\vec{w}$. $Q(\vec{v}, \vec{w})$ is positive if $g|\Pi$ is definite, negative if it is indefinite (use an orthonormal basis), [10].

Lemma 1.1. Let Π be a non-degenerate tangent plane to M to p. The number

$$K(\vec{v},\vec{w}) = \frac{\langle R_{\vec{v}\vec{w}}\vec{v},\vec{w}\rangle}{Q(\vec{v},\vec{w})} \tag{1.6}$$

is independent of the choice of basis $\vec{v},\vec{w}$ for Π, and is called the sectional curvature $K(\Pi)$ of Π, [10].

2. Generalized Semi-Ruled Surfaces in E_v^{n+1}

Let $\{e_1(t), e_2(t), ..., e_k(t)\}$, $k<n$, be an orthonormal vector field which is defined at the each point $\alpha(t)$ of a non-null curve in $(n+1)-$dimensional semi-Euclidean space E_v^{n+1}. This system is denoted by $E_{k,\mu}(t)$ and is given by

$$E_{k,\mu}(t) = Sp\{e_1(t), e_2(t), ..., e_k(t)\} \quad , \quad 0 \le \mu \le v,$$

where

$$\langle e_i(t), e_j(t)\rangle = \varepsilon_i \delta_{ij} \quad , \quad \varepsilon_i = \begin{cases} 1 \ , & 1 \le i \le k-\mu \\ -1 \ , & k-\mu+1 \le i \le k. \end{cases}$$

For $\mu \ge 1$, it is clear that the subspace $E_{k,\mu}(t)$ has number of μ timelike vectors. $E_{k,0}(t) = E_k(t)$ is an Euclidean subspace if there is no timelike vector in $E_{k,\mu}(t)$, that is $\mu = 0$. If there is one timelike vector (that is, $\mu = 1$), then $E_{k,1}(t)$ is Minkowski subspace. Thus, in semi-Euclidean space $E_v^{n+1}(t)$ a semi-ruled surface (generalized semi-ruled surface) is given parametrically by

$$\varphi(t, u_1, u_2, ..., u_k) = \alpha(t) + \sum_{i=1}^{k} u_i e_i(t) \tag{2.1}$$

and denoted by M, [3].

Here, the subspaces $E_{k,\mu}(t)$ and the curve α are called the generating spaces and base curve, respectively. We assume that

$$\left\{ \dot{\alpha}(t) + \sum_{i=1}^{k} u_i \dot{e}_i(t), e_1(t), ..., e_k(t) \right\} \tag{2.2}$$

is linearly independent and $\varphi(t, u_1, u_2, ..., u_k)$ is $(k+1)-$dimensional semi-ruled surface. The subspace

$$A(t) = Sp\left\{e_1(t),...,e_k(t),\dot{e}_1(t),...,\dot{e}_k(t)\right\} \qquad (2.3)$$

is called asymptotic bundle of M with respect to $E_{k,\mu}(t)$ such that $\dot{e}_i(t)$ is the derivative of the vector field of $e_i(t)$, $1 \le i \le k$, [3].

If $\dim A(t) = k+m, 0 \le m \le k$, then there exists an orthonormal basis $\{e_1(t),...,e_k(t),a_{k+1}(t),...,a_{k+m}(t)\}$ of $A(t)$ containing $E_{k,\mu}(t)$.

Also, for the orthonormal base $\{e_1(t),e_2(t),...,e_k(t)\}$, there are following equations

$$\begin{aligned} \dot{e}_i &= \sum_{j=1}^{k} \alpha_{ij}\, e_j + \varepsilon_{k+i}\, \kappa_i\, a_{k+i} \quad , \quad 1 \le i \le m \\ \dot{e}_h &= \sum_{j=1}^{h} \alpha_{hj}\, e_j \quad , \quad m+1 \le h \le k \end{aligned} \qquad (2.4)$$

where

$$\varepsilon_{ij}\alpha_{ij} = -\alpha_{ji} \;\; , \;\; \varepsilon_j = \langle e_j, e_j \rangle \;\; , \;\; \varepsilon_{ij} = \varepsilon_i \varepsilon_j \qquad (2.5)$$

and for $r \le \mu$,

$$\begin{aligned} &\kappa_1 > \kappa_2 > ... > \kappa_{m-r} > 0 \\ &\kappa_{m-r+1} < \kappa_{m-r+2} < ... < \kappa_m < 0, \end{aligned} \qquad (2.6)$$

[3]. The subspace

$$T(t) = Sp\left\{e_1(t),...,e_k(t),\dot{e}_1(t),...,\dot{e}_k(t),\dot{\alpha}(t)\right\} \qquad (2.7)$$

is called tangential subbundle of M with respect to $E_{k,\mu}(t)$. Thus,

$$k+m \le \dim T(t) \le k+m+1 \quad , \quad 0 \le m \le k.$$

Suppose that $\dim T(t) = k+m,\ 0 \le m \le k.$ Then $\{e_1,...,e_k,a_{k+1},...,a_{k+m}\}$ is an orthonormal basis both of asymptotic bundle $A(t)$ and tangential bundle $T(t)$. This means that $A(t)$ is coincident with $T(t)$. Assume that, for all t, $\dim T(t) = k+m+1,\ 0 \le m \le k.$ Thus, one can find an orthonormal basis for $T(t)$ as $\{e_1,...,e_k,a_{k+1},...,a_{k+m},a_{k+m+1}\}$. In the case of $\dim T(t) = k+m+1,$ $(k+1)-$ dimensional semi-ruled surface M has a $(k-m)-$dimensional tangent subspace called central space of M at the each point $\alpha(t)$ and is denoted by $Z_{k-m,r}(t) \subset E_{k,\mu}(t)$. While the semi-subspace $Z_{k-m,r}(t)$ is moving through the base curve α of M, it generates a $(k-m+1)-$dimensional ruled surface contained by M which is called as $(k-m+1)-$dimensional central ruled surface. This surface is denoted by Ω. Moreover, the central surface Ω is also, a semi-ruled surface because $Z_{k-m,r}(t)$ is a semi-subspace, [3].

We assume that the base curve of M is also the base curve of $\Omega \subset M$. Therefore, we have

$$\dot{\alpha}(t) = \sum_{\nu=1}^{k} \zeta_\nu e_\nu + \sum_{h=1}^{m} \eta_h a_{k+h} + \eta_{m+1} a_{k+m+1} \quad , \quad \eta_{m+1} \neq 0. \tag{2.8}$$

The tangential space $T(t)$ of M is perpendicular to the asymptotic bundle $A(t)$ at the central points. If we consider the equation (2.8) at the central point of central ruled surface $\Omega \subset M$, then we get

$$u_\sigma = 0 \quad , \quad 1 \leq \sigma \leq m. \tag{2.9}$$

Thus, the canonical base of the tangential bundle of semi ruled surface M is

$$\left\{ \sum_{i=1}^{k} \left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji} \right) e_i + \sum_{\sigma=1}^{m} \varepsilon_{k+\sigma} u_\sigma \kappa_\sigma a_{k+\sigma} + \eta_{m+1} a_{k+m+1}, e_1, e_2, \ldots, e_k \right\}. \tag{2.10}$$

Now, we can evaluate the first fundamental form of M and the metric coefficients with respect to this canonical base. In conventional notation we choose $u_0 = t$ and calculate the metric coefficients of M as follows

$$\begin{aligned}
g_{00} &= \langle \varphi_t , \varphi_t \rangle = \sum_{i=1}^{k} \varepsilon_i \left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji} \right)^2 + \sum_{\sigma=1}^{m} \varepsilon_{k+\sigma} \left(u_\sigma \kappa_\sigma \right)^2 + \varepsilon_{k+m+1} \left(\eta_{m+1} \right)^2, \\
g_{i0} &= \langle \varphi_{u_i}, \varphi_t \rangle = \varepsilon_i \left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji} \right) \quad , \quad 1 \leq i \leq k, \\
g_{ij} &= \langle \varphi_{u_j}, \varphi_{u_j} \rangle = \varepsilon_i \delta_{ij} \quad , \quad 1 \leq i, j \leq k.
\end{aligned} \tag{2.11}$$

Therefore, the determinant of the matrix of the first fundamental form of M is as follows

$$g = \det\left[g_{ab} \right] = \varepsilon \left(\sum_{\sigma=1}^{m} \varepsilon_{k+\sigma} \left(u_\sigma \kappa_\sigma \right)^2 + \varepsilon_{k+m+1} \left(\eta_{m+1} \right)^2 \right) \quad , \quad 1 \leq a, b \leq k \tag{2.12}$$

where $\varepsilon = \varepsilon_1.\varepsilon_2. \dots .\varepsilon_k$. Since g is non-degenerate at the each point of $T_M(p)$ the matrix $[g_{ab}]$ is invertible and the inverse matrix is denoted by $[g^{ab}]$. The coefficients of the inverse matrix $[g^{ab}]$ are as follows

$$
\begin{aligned}
g^{00} &= \varepsilon g^{-1}, \\
g^{i0} &= -\varepsilon g^{-1}\left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji}\right) \qquad , \quad 1 \le i \le k, \\
g^{i\lambda} &= g^{-1}\left(\left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji}\right)\left(\zeta_\lambda + \sum_{j=1}^{k} u_j \alpha_{j\lambda}\right)\varepsilon + \delta_{i\lambda}\varepsilon_i g\right) \qquad , \quad 1 \le i, \lambda \le k.
\end{aligned}
\tag{2.13}
$$

In [3], the Koszul equation is

$$
\Gamma_{ij}^{k} = \frac{1}{2}\sum_{m} g^{km}\left[\frac{\partial g_{jm}}{\partial x_i} + \frac{\partial g_{im}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_m}\right]. \tag{2.14}
$$

If we substitute the equations (2.11) and (2.13) into the equation (2.14), then we reach the Christoffel symbols for $1 \le i, j, \lambda \le k$, as follows;

$$
\begin{aligned}
\Gamma_{00}^{0} &= \frac{1}{2g}\left[\frac{\partial g}{\partial u_0} + \sum_{i=1}^{k}\left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji}\right)\frac{\partial g}{\partial u_i}\right], \\
\Gamma_{00}^{\lambda} &= -\frac{1}{2g}\left(\zeta_\lambda + \sum_{j=1}^{k} u_j \alpha_{j\lambda}\right)\left(\frac{\partial g}{\partial u_0} + \sum_{i=1}^{k}\left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji}\right)\frac{\partial g}{\partial u_i}\right) \\
&\quad + \left(\left(\dot{\zeta}_\lambda + \sum_{j=1}^{k} u_j \dot{\alpha}_{j\lambda}\right) + \sum_{i=1}^{k}\left(\zeta_i + \sum_{j=1}^{k} u_j \alpha_{ji}\right)\alpha_{i\lambda} - \varepsilon\varepsilon_\lambda \frac{1}{2}\frac{\partial g}{\partial u_\lambda}\right), \\
\Gamma_{ij}^{0} &= \Gamma_{ji}^{0} = 0, \\
\Gamma_{ij}^{\lambda} &= \Gamma_{ji}^{\lambda} = 0, \\
\Gamma_{\lambda 0}^{0} &= \Gamma_{0\lambda}^{0} = \frac{1}{2g}\frac{\partial g}{\partial u_\lambda}, \\
\Gamma_{i0}^{\lambda} &= \Gamma_{0i}^{\lambda} = \frac{1}{2g}\left[-\left(\zeta_\lambda + \sum_{j=1}^{k} u_j \alpha_{j\lambda}\right)\frac{\partial g}{\partial u_i} + 2g\left(\alpha_{i\lambda}\right)\right].
\end{aligned}
\tag{2.15}
$$

where $\varepsilon = \varepsilon_1.\varepsilon_2.\varepsilon_3. \dots .\varepsilon_k$ and $\varepsilon_i = \mp 1$.

The Riemannian-Christoffel curvature tensor of the semi-ruled surface M is given by

$$
R_{hlij} = \sum_{r=0}^{k} g_{rh}\left(\frac{\partial}{\partial u_i}\Gamma_{jl}^{r} - \frac{\partial}{\partial u_j}\Gamma_{il}^{r} - \sum_{s=0}^{k}\Gamma_{il}^{s}\Gamma_{js}^{r} + \sum_{s=0}^{k}\Gamma_{jl}^{s}\Gamma_{is}^{r}\right). \tag{2.16}
$$

Considering the equations (2.11) and (2.15), the Riemannian-Christoffel curvature tensors are

$$R_{0000} = R_{i000} = R_{ij00} = 0 \quad , \quad 1 \le i, j \le k, \tag{2.17}$$

$$R_{0hij} = 0 \qquad , \qquad 1 \le i, j, h \le k, \tag{2.18}$$

$$R_{lhij} = 0 \qquad , \qquad 1 \le i, j, h, l \le k, \tag{2.19}$$

$$R_{i0j0} = \varepsilon \left(-\frac{1}{2} \frac{\partial^2 g}{\partial u_i \partial u_j} + \frac{1}{4g} \frac{\partial g}{\partial u_i} \frac{\partial g}{\partial u_j} \right). \tag{2.20}$$

A normal tangent vector n of the semi-ruled surface M orthogonal to $E_{k,\mu}(t)$ at the each point $\xi(t, u_v)$ is defined to be

$$n = \sum_{\sigma=1}^{m} \varepsilon_{k+\sigma} u_\sigma \kappa_\sigma(t) a_{k+\sigma}(t) + \eta_{m+1} a_{k+m+1}(t) \quad , \quad \eta_{m+1} \ne 0 \tag{2.21}$$

where n is either a timelike or spacelike vector.

Thus, the following theorem can be given related to the principal sectional curvatures at the point $\xi \in M$.

Theorem 2.1. *Let M be a semi-ruled surface with the central ruled surface in E_v^{n+1} and n be a non-null normal tangent vector of M orthogonal to $E_{k,\mu}(t)$. Thus, the i^{th} principal sectional curvature (e_i, n) at the each point $\xi \in M$ is*

$$K_\xi(e_i, n) = \varepsilon_i \left(-\frac{1}{2g} \frac{\partial^2 g}{\partial u_i^2} + \frac{1}{4g^2} \left(\frac{\partial g}{\partial u_i} \right)^2 \right), \quad 1 \le i \le k, \tag{2.22}$$

[1].

Let $e(t)$ be a unit vector in the generating space $E_{k,\mu}(t)$, that is, we write

$$e(t) \in Sp\{e_1(t), \ldots, e_m(t), e_{m+1}(t), \ldots, e_k(t)\} \tag{2.23}$$

and $e(t)$ is either unit spacelike or timelike vector. Now, we investigate these situations, separately.

$i)$ Let $e(t)$ be spacelike vector. In this case, we write

$$e=\sum_{x=1}^{s}\sinh\theta_x e_x+\sum_{y=s+1}^{m}\cosh\theta_y e_y+\sum_{z=m+1}^{m+\mu-s}\sinh\theta_z e_z+\sum_{w=m+1+\mu-s}^{k}\cosh\theta_w e_w \quad (2.24)$$

and

$$-\sum_{x=1}^{s}\sinh^2\theta_x+\sum_{y=s+1}^{m}\cosh^2\theta_y-\sum_{z=m+1}^{m+\mu-s}\sinh^2\theta_z+\sum_{w=m+1+\mu-s}^{k}\cosh^2\theta_w=1 \quad (2.25)$$

where the hyperbolic angles $\theta_1,\theta_2,\ldots,\theta_s,\theta_{s+1},\ldots,\theta_m,\theta_{m+1},\ldots,\theta_k$ are the angles between spacelike unit vector e and the base vectors $e_1,e_2,\ldots,e_s,e_{s+1},\ldots,e_m$, $e_{m+1},\ldots,e_k$, respectively.

$ii)$ Let $e(t)$ be a timelike vector. Therefore, we have

$$e=\sum_{x=1}^{s}\cosh\theta_x e_x+\sum_{y=s+1}^{m}\sinh\theta_y e_y+\sum_{z=m+1}^{m+\mu-s}\cosh\theta_z e_z+\sum_{w=m+1+\mu-s}^{k}\sinh\theta_w e_w \quad (2.26)$$

and

$$-\sum_{x=1}^{s}\cosh^2\theta_x+\sum_{y=s+1}^{m}\sinh^2\theta_y-\sum_{z=m+1}^{m+\mu-s}\cosh^2\theta_z+\sum_{w=m+1+\mu-s}^{k}\sinh^2\theta_w=-1 \quad (2.27)$$

where the hyperbolic angles between the base vectors $e_1,e_2,\ldots,e_s,e_{s+1},\ldots,e_m,e_{m+1},\ldots,e_k$ and timelike unit vector e are $\theta_1,\theta_2,\ldots,\theta_s,\theta_{s+1},\ldots,\theta_m,\theta_{m+1},\ldots,\theta_k$, respectively.

Considering these situations, we give following theorem.

Theorem 2.2. *Let M be a semi-ruled surface with the central ruled surface in E_v^{n+1}. Taking n to be a normal tangential vector orthogonal to $E_{k,\mu}(t)$ of M, there exists the following relation between the sectional curvature of the non-degenerate section (e,n) and the principal sectional curvatures at the point $\zeta\in\Omega\subset M$ as follows:*

$i)$ If the unit vector $e(t)$ is a spacelike vector, then

$$K_\zeta(e,n)=-\sum_{i=1}^{s}\sinh^2\theta_i K_\zeta(e_i,n)+\sum_{j=s+1}^{m}\cosh^2\theta_j K_\zeta(e_j,n), \quad (2.28)$$

ii) If the unit vector $e(t)$ is a timelike vector, then

$$K_\zeta(e,n)=\sum_{i=1}^{s}\cosh^2\theta_i K_\zeta(e_i,n)-\sum_{j=s+1}^{m}\sinh^2\theta_j K_\zeta(e_j,n) \qquad (2.29)$$

where the angles $\theta_1,\theta_2,\ldots,\theta_s,\theta_{s+1},\ldots,\theta_m,\theta_{m+1},\ldots,\theta_k$ are the hyperbolic angles between the unit vector e (spacelike or timelike) and the base vectors $e_1,e_2,\ldots,e_s,e_{s+1},\ldots,e_m,e_{m+1},\ldots,e_k$, respectively, [1].

3. Beltrami-Meusnier Formulas of Semi-Euclidean Space E_v^{n+1}

Let Ω be a central ruled surface of the generalized semi-ruled surface M in E_v^{n+1}. Therefore, any unit vector at the each central point $\zeta\in\Omega$ is defined to be

$$\begin{aligned}a=\lambda_0\frac{n}{\|n\|}+\lambda_1e_1+\ldots+\lambda_{s-1}e_{s-1}+\lambda_se_s+\lambda_{s+1}e_{s+1}+\ldots+\lambda_me_m\\+\lambda_{m+1}e_{m+1}+\ldots+\lambda_{m+\mu-s}e_{m+\mu-s}+\lambda_{m+1+\mu-s}e_{m+1+\mu-s}+\ldots+\lambda_ke_k\end{aligned}\quad,\quad \langle a,a\rangle=\pm1.$$

where a and e_σ, $1\le\sigma\le m$, are linearly independent. The unit vector a is either spacelike or timelike vector. There exist the following four cases depending on whether the normal tangent vector n which is orthogonal to the generating space $E_{k,\mu}(t)$ of M orthogonal to $E_{k,\mu}(t)$ is spacelike or timelike vector.

i) The unit vector a and the unit normal vector n are spacelike.

ii) The unit vector a is a spacelike and the unit normal vector n is timelike.

iii) The unit vector a is a timelike and the unit normal vector n is spacelike.

iv) The unit vector a and the unit normal vector n are timelike.

Now, we investigate these situations, separately.

i) Let the unit vector a and the unit normal vector n be spacelike vectors. In this case, any spacelike vector a at the central point $\xi\in\Omega$ can be written as follows

$$a=\cosh\psi_0\frac{n}{\|n\|}+\sum_{x=1}^{s}\sinh\psi_xe_x+\sum_{y=s+1}^{m}\cosh\psi_ye_y+\sum_{z=m+1}^{m+\mu-s}\sinh\psi_ze_z+\sum_{w=m+1+\mu-s}^{k}\cosh\psi_we_w \qquad (3.1)$$

and

$$-\sum_{x=1}^{s}\sinh^2\theta_x+\sum_{y=s+1}^{m}\cosh^2\theta_y-\sum_{z=m+1}^{m+\mu-s}\sinh^2\theta_z+\sum_{w=m+1+\mu-s}^{k}\cosh^2\theta_w=1$$

where e_s, $1\le s\le m$, is a timelike vector and the angles $\psi_0,\psi_1,\ldots,\psi_s,\ldots,\psi_k$ are the hyperbolic angles between the spacelike unit vector a and the vectors $n,e_1,\ldots,e_s,\ldots,e_k$, respectively.

ii) Let the unit vector a be a spacelike vector and the unit normal vector n be a timelike vector. Suppose that e_s, $1\le s\le m$, is a timelike vector. In this case we write the spacelike unit vector a at the central point $\xi\in\Omega$ as

$$a=\sinh\psi_0\frac{n}{\|n\|}+\sum_{x=1}^{s}\sinh\psi_x e_x+\sum_{y=s+1}^{m}\cosh\psi_y e_y+\sum_{z=m+1}^{m+\mu-s}\sinh\psi_z e_z+\sum_{w=m+1+\mu-s}^{k}\cosh\psi_w e_w \quad (3.2)$$

and

$$-\sum_{x=1}^{s}\sinh^2\theta_x+\sum_{y=s+1}^{m}\cosh^2\theta_y-\sum_{z=m+1}^{m+\mu-s}\sinh^2\theta_z+\sum_{w=m+1+\mu-s}^{k}\cosh^2\theta_w=1.$$

So that the angles $\psi_0,\psi_1,\ldots,\psi_s,\ldots,\psi_k$ are the hyperbolic angles between the spacelike unit vector a and the vectors $n,e_1,\ldots,e_s,\ldots,e_k$, respectively.

iii) Let the unit vector a be a timelike vector and the unit normal vector n be a spacelike vector. As above cases, at the central point $\xi\in\Omega$ any timelike vector a is written as follows

$$a=\sinh\psi_0\frac{n}{\|n\|}+\sum_{x=1}^{s}\cosh\psi_x e_x+\sum_{y=s+1}^{m}\sinh\psi_y e_y+\sum_{z=m+1}^{m+\mu-s}\cosh\psi_z e_z+\sum_{w=m+1+\mu-s}^{k}\sinh\psi_w e_w. \quad (3.3)$$

It is clear that

$$-\sum_{x=1}^{s}\cosh^2\theta_x+\sum_{y=s+1}^{m}\sinh^2\theta_y-\sum_{z=m+1}^{m+\mu-s}\cosh^2\theta_z+\sum_{w=m+1+\mu-s}^{k}\sinh^2\theta_w=-1$$

where the angles $\psi_0,\psi_1,\ldots,\psi_s,\ldots,\psi_k$ are the hyperbolic angles between the timelike unit vector a and the vectors $n,e_1,\ldots,e_s,\ldots,e_k$, respectively. In addition to that, the vectors e_s, $1\le s\le m$, are the timelike vectors.

iv) Let the unit vector a and the unit normal vector n be timelike vectors. Let e_s, $1\le s\le m$, be timelike vectors. In this case, timelike unit vector a can be written as follows

$$a=\cosh\psi_0\frac{n}{\|n\|}+\sum_{x=1}^{s}\cosh\psi_x e_x+\sum_{y=s+1}^{m}\sinh\psi_y e_y+\sum_{z=m+1}^{m+\mu-s}\cosh\psi_z e_z+\sum_{w=m+1+\mu-s}^{k}\sinh\psi_w e_w.\ (3.4)$$

In addition, there is

$$-\sum_{x=1}^{s}\cosh^2\theta_x+\sum_{y=s+1}^{m}\sinh^2\theta_y-\sum_{z=m+1}^{m+\mu-s}\cosh^2\theta_z+\sum_{w=m+1+\mu-s}^{k}\sinh^2\theta_w=-1.$$

where the hyperbolic angles between the timelike unit vector a and the vectors $n,e_1,\ldots,e_s,\ldots,e_k$ are $\psi_0,\psi_1,\ldots,\psi_s,\ldots,\psi_k$, respectively.

Considering equation (1.6), at the point $(\zeta+ue_\sigma)\in M$, the curvature of non-degenerate section (e_σ,a) is

$$K_{\zeta+ue_\sigma}(e_\sigma,a)=\frac{\beta_\sigma\beta_\sigma\lambda_0\lambda_0R_{\sigma0\sigma0}}{\langle e_\sigma,e_\sigma\rangle\langle a,a\rangle-\langle e_\sigma,a\rangle^2}\quad,\quad 1\le\sigma\le m. \tag{3.5}$$

Therefore, taking the cases *i*), *ii*), *iii*) and *iv*) into consideration, we give the following theorem below about the relationship between the curvatures of section (e_x,a), $1\le x\le s$ and the x^{th} principal section (e_x,n).

Theorem 3.1. *Let M be a generalized semi-ruled surface with the central ruled surface in E_v^{n+1} and the unit vector n be a non-null normal tangent vector of M orthogonal to the generating space $E_{k,\mu}(t)$. In this case, the following relations between the curvatures of non-degenerate section (e_x,a), $1\le x\le s$, and non-degenerate the x^{th} principal section (e_x,n) at the point $(\zeta+ue_x)\in M$ exist:*

i) If the unit vector a and the unit normal vector n are spacelike vectors, then

$$\left(1+\sinh^2\psi_x\right)K_{\zeta+ue_x}(e_x,a)=\cosh^2\psi_0K_{\zeta+ue_x}(e_x,n) \tag{3.6}$$

ii) If the unit vector a is a spacelike vector and the unit normal vector n is timelike vector, then

$$\left(1+\sinh^2\psi_x\right)K_{\zeta+ue_x}(e_x,a)=-\sinh^2\psi_0K_{\zeta+ue_x}(e_x,n), \tag{3.7}$$

iii) If the unit vector a is a timelike vector and the unit normal vector n is spacelike vector, then

$$\left(1-\cosh^2\psi_x\right)K_{\zeta+ue_x}(e_x,a)=-\sinh^2\psi_0K_{\zeta+ue_x}(e_x,n), \tag{3.8}$$

iv) *If the unit vector a and the unit normal vector n are timelike vectors, then*

$$\left(1-\cosh^2\psi_x\right)K_{\zeta+ue_x}\left(e_x,a\right)=\cosh^2\psi_0 K_{\zeta+ue_x}\left(e_x,n\right) \qquad (3.9)$$

where the base vector e_x are a timelike vectors and the non-null unit vector a is linearly independent of the timelike base vector e_x at the each point $\xi\in\Omega$. In addition to that, the hyperbolic angles are ψ_0 and ψ_x between a and n, and between a and e_x, respectively.

Proof. Let the base vectors e_x, $1\le x\le s$, be the timelike vectors in the generating space $E_{k,\mu}(t)$ of the generalized semi-ruled surface M in E_v^{n+1} and the non-null unit vector a be linearly independent of the timelike base vectors e_x at the each point $\xi\in\Omega$.

i) Suppose that the unit vector a and the unit normal tangent vector n are spacelike vectors. Let the coordinates of the base vectors e_i, $1\le i\le k$, and the spacelike unit vector a (given by equation (3.1)) be $(\beta_0,\beta_1,\ldots,\beta_i,\ldots,\beta_k)$ ve $(\gamma_0,\gamma_1,\ldots,\gamma_s,\ldots,\gamma_k)$, respectively. In this case, we can write

$$\beta_0=\langle e_i,e_0\rangle=0 \quad , \quad \beta_i=\langle e_i,e_i\rangle=\varepsilon_i \quad , \quad 1\le i\le k ,$$

and

$$\gamma_0=\langle a,e_0\rangle=\frac{\cosh\psi_0}{\|n\|} \quad , \quad \gamma_x=\langle a,e_x\rangle=\sinh\psi_x \quad , \quad 1\le x\le s \quad ,$$
$$\gamma_y=\langle a,e_y\rangle=\cosh\psi_y \quad , \quad s+1\le y\le m.$$

Substituting the last equations and the equation (2.20) into the equation (3.5), we find

$$K_{\zeta+ue_x}\left(e_x,a\right)=\frac{\varepsilon_x^2\dfrac{\cosh^2\psi_0}{\|n\|^2}\,\varepsilon\left(-\dfrac{1}{2}\dfrac{\partial^2 g}{\partial u_x^2}+\dfrac{1}{4g}\left(\dfrac{\partial g}{\partial u_x}\right)^2\right)}{\varepsilon_x-\sinh^2\psi_x} \qquad (3.10)$$

where $\varepsilon=\varepsilon_1.\varepsilon_2.\ldots.\varepsilon_k$. Since $\|n\|^2=\varepsilon g$ and $\varepsilon_x=\langle e_x,e_x\rangle=-1$, we get

$$\left(1+\sinh^2\psi_x\right)K_{\zeta+ue_x}\left(e_x,a\right)=-\cosh^2\psi_0\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_x^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_x}\right)^2\right). \qquad (3.11)$$

From the equations (2.22) and (3.11), we find the relation between the curvature of non-degenerate section (e_x, a) and non-degenerate the x^{th} principal section (e_x, n) as

$$\left(1+\sinh^2\psi_x\right)K_{\zeta+ue_x}\left(e_x,a\right)=-\cosh^2\psi_0\varepsilon_x K_{\zeta+ue_x}\left(e_x,n\right).$$

To prove the $ii)$, $iii)$ and $iv)$, in a similar manner using the equations (3.2), (3.3), (3.4) and (2.20), (3.25), we reach the relations (3.7), (3.8) and (3.9), respectively. These complete the proof.

Now, considering the cases $i)$, $ii)$, $iii)$ and $iv)$ for $s+1\le y\le m$, we give the following theorem related to the relation between the curvatures of section (e_y, a) and the y^{th} principal section (e_y, n).

Theorem 3.2. *Let M be a generalized semi-ruled surface with the central ruled surface in E_v^{n+1} and the unit vector n be a non-null normal vector which is orthogonal to the generating space $E_{k,\mu}(t)$. In addition to that, the base vectors e_y, $s+1\le y\le m$, are spacelike vectors and the non-null unit vector a is linearly independent of the spacelike base vectors e_y at the point $\xi\in\Omega$. Therefore, the following relationship between the curvatures of non-degenerate section (e_y, a) and non-degenerate the y^{th} principal section (e_y, n) at the point $(\zeta+ue_x)\in M$ as follows:*

$i)$ Let the unit vector a and the unit normal vector n be spacelike vectors, then

$$\left(1-\cosh^2\psi_y\right)K_{\zeta+ue_y}\left(e_y,a\right)=\cosh^2\psi_0 K_{\zeta+ue_y}\left(e_y,n\right), \tag{3.12}$$

$ii)$ Let the unit vector a be a spacelike vector and the unit normal vector n be a timelike vector, therefore

$$\left(1-\cosh^2\psi_y\right)K_{\zeta+ue_y}\left(e_y,a\right)=-\sinh^2\psi_0 K_{\zeta+ue_y}\left(e_y,n\right), \tag{3.13}$$

$iii)$ Let the unit vector a be a timelike vector and the unit normal vector n be a spacelike vector, in this case

$$\left(1+\sinh^2\psi_y\right)K_{\zeta+ue_y}\left(e_y,a\right)=-\sinh^2\psi_0 K_{\zeta+ue_y}\left(e_y,n\right), \tag{3.14}$$

$iv)$ Let the unit vector a and the unit normal vector n be timelike vectors, then

$$\left(1+\sinh^2\psi_\sigma\right)K_{\zeta+ue_\sigma}\left(e_\sigma,a\right)=\cosh^2\psi_0 K_{\zeta+ue_\sigma}\left(e_\sigma,n\right), \tag{3.15}$$

where the angles ψ_0 and ψ_y represent the hyperbolic angles a and n, and between a and e_y, respectively.

Proof. Let the base vectors e_y, $s+1 \le y \le m$, be the spacelike vectors in $E_{k,\mu}(t)$ and the non-null unit vector a be linearly independent of the spacelike base vectors e_y at the each point $\xi \in \Omega$.

i) Assume that the unit vector a and the unit normal tangent vector n are spacelike vectors. If the coordinates of the base vectors e_i, $1 \le i \le k$ and spacelike unit vector a, which is given by equation (3.1), are $(\beta_0, \beta_1, \dots, \beta_i, \dots, \beta_k)$ and $(\gamma_0, \gamma_1, \dots, \gamma_s, \dots, \gamma_k)$, respectively, then we have

$$\beta_0 = \langle e_i, e_0 \rangle = 0 \quad , \quad \beta_i = \langle e_i, e_i \rangle = \varepsilon_i \quad , \quad 1 \le i \le k \ ,$$

and

$$\gamma_0 = \langle a, e_0 \rangle = \frac{\cosh \psi_0}{\|n\|} \quad , \quad \gamma_x = \langle a, e_x \rangle = \sinh \psi_x \quad , \quad 1 \le x \le s \quad ,$$
$$\gamma_y = \langle a, e_y \rangle = \cosh \psi_y \quad , \quad s+1 \le y \le m.$$

Substituting these equations together with the equation (2.20) into the equation (3.5) and considering that $\|n\|^2 = \varepsilon g$, we reach

$$\left(1-\cosh^2 \psi_y\right) K_{\zeta+ue_y}\left(e_y, a\right) = \cosh^2 \psi_0 \left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_y^2} + \frac{1}{4g^2}\left(\frac{\partial g}{\partial u_y}\right)^2 \right). \qquad (3.16)$$

From the equations (2.22) and (3.16), we get the relation between curvature of non-degenerate section (e_y, a) and non-degenerate the y^{th} principal section (e_y, n), as

$$\left(1-\cosh^2 \psi_y\right) K_{\zeta+ue_y}\left(e_y, a\right) = \cosh^2 \psi_0 \varepsilon_y K_{\zeta+ue_y}\left(e_y, n\right). \qquad (3.17)$$

To prove the *ii*), *iii*) and *iv*) in this theorem, in a similar way using the equations (3.2), (3.3), (3.4) together with the equations (2.20) and (3.25), we find the equations (3.13), (3.14) and (3.15), respectively.

Thus, the proof is completed.

Considering the unit vector e given by the equations (2.24) and (2.26) in $E_{k,\mu}(t)$ and the unit vector a given by the equations (3.1), (3.2), (3.3) and (3.4) at the point

$\xi \in \Omega$, then we give the following theorem according to these situations of the vectors e and a, separately.

Theorem 3.3. *Let M be a generalized semi-ruled surface with the central ruled surface in E_v^{n+1} and the vector n be a non-null normal tangent vector which is orthogonal to the generating space $E_{k,\mu}(t)$. Considering that the non-null vector a which is independent of the non-null unit vector e in $E_{k,\mu}(t)$ at the $\forall \zeta \in \Omega$. There exist the relations between the curvature of non-degenerate section (e,a) and the curvature of non-degenerate section (e,n) as follows:*

i) The vectors n, e and a are spacelike vectors. In this case

$$K_\zeta(e,a) = \frac{\cosh^2 \psi_0}{1 - \langle e,a \rangle^2} K_\zeta(e,n). \tag{3.18}$$

ii) If the vectors n and a are spacelike vectors, the vector e is timelike vector. Then

$$K_\zeta(e,a) = \frac{\cosh^2 \psi_0}{1 + \langle e,a \rangle^2} K_\zeta(e,n). \tag{3.19}$$

iii) If the vectors a and e are spacelike vectors, the vector n is timelike vector. Therefore

$$K_\zeta(e,a) = -\frac{\sinh^2 \psi_0}{1 - \langle e,a \rangle^2} K_\zeta(e,n). \tag{3.20}$$

iv) If the vectors n and e are spacelike vectors, the vector a is timelike vector. Thus,

$$K_\zeta(e,a) = -\frac{\sinh^2 \psi_0}{1 + \langle e,a \rangle^2} K_\zeta(e,n). \tag{3.21}$$

v) If the vectors n and e are spacelike vectors, the vector a is a timelike vector. In this case

$$K_\zeta(e,a) = -\frac{\sinh^2 \psi_0}{1 + \langle e,a \rangle^2} K_\zeta(e,n). \tag{3.22}$$

vi) If the vectors a and e are timelike vectors, the vector n is a spacelike vector. Then

$$K_\zeta(e,a) = -\frac{\sinh^2\psi_0}{1-\langle e,a\rangle^2}K_\zeta(e,n). \qquad (3.23)$$

vii) *If the vectors* a *and* n *are timelike vectors, the vector* e *is a spacelike vector. Therefore*

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0}{1+\langle e,a\rangle^2}K_\zeta(e,n). \qquad (3.24)$$

viii) *If the vectors* a, e *and* n *are timelike vectors. Thus*

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0}{1-\langle e,a\rangle^2}K_\zeta(e,n) \qquad (3.25)$$

where ψ_0 *is the hyperbolic angle between non-null unit vector* a *and non-null normal tangent vector* n.

Proof. Assume that the non-null unit vector e which is given by (2.24) and (2.26) in the generating space $E_{k,\mu}(t)$ and the non-null vector a given by the equations (3.1), (3.2), (3.3) and (3.4) which is linearly independent with the vector e at the point $\xi\in\Omega$. In addition to this, suppose that the non-null vector n is orthogonal to $E_{k,\mu}(t)$.

Considering the equation (1.6), the sectional curvature at the point $\xi\in\Omega$ is given by

$$K_\zeta(e,a) = \frac{\sum_{x=1}^{s}\beta_x\beta_x\lambda_0\lambda_0R_{x0x0} + \sum_{y=s+1}^{m}\beta_y\beta_y\lambda_0\lambda_0R_{y0y0}}{\langle e,e\rangle\langle a,a\rangle-\langle e,a\rangle^2}. \qquad (3.26)$$

i) First of all, let the vectors n, e and a be spacelike vectors. If the coordinates of e given by equation (2.24) and the tangent vector a given by equation (3.1) are $(\beta_0,\beta_1,\ldots,\beta_s,\ldots,\beta_k)$ and $(\gamma_0,\gamma_1,\ldots,\gamma_s,\ldots,\gamma_k)$, respectively. Then we write

$$\begin{aligned}\beta_0 &= \langle e,e_0\rangle = 0,\\ \beta_x &= \langle e,e_x\rangle = \sinh\theta_x \quad , \quad 1\le x\le s\ ,\\ \beta_y &= \langle e,e_y\rangle = \cosh\theta_y \quad , \quad s+1\le y\le m.\end{aligned} \qquad (3.27)$$

and

$$\gamma_0 = \langle a, e_0 \rangle = \frac{\cosh\psi_0}{\|n\|}$$
$$\gamma_x = \langle a, e_x \rangle = \sinh\psi_x \quad , \quad 1 \leq x \leq s \quad , \tag{3.28}$$
$$\gamma_y = \langle a, e_y \rangle = \cosh\psi_y \quad , \quad s+1 \leq y \leq m.$$

Substituiting the equations (3.27), (3.28) and (2.20) into equation (3.26), we find

$$K_\zeta(e,a) = \frac{\frac{\cosh^2\psi_0}{\|n\|^2}\left(\sum_{x=1}^{s}\sinh^2\theta_x\left[\varepsilon\left(-\frac{1}{2}\frac{\partial^2 g}{\partial u_x^2}+\frac{1}{4g}\left(\frac{\partial g}{\partial u_x}\right)^2\right)\right]+\sum_{y=s+1}^{m}\cosh^2\theta_y\left[\varepsilon\left(-\frac{1}{2}\frac{\partial^2 g}{\partial u_y^2}+\frac{1}{4g}\left(\frac{\partial g}{\partial u_y}\right)^2\right)\right]\right)}{1-\langle e,a\rangle^2} \tag{3.29}$$

where $\varepsilon = \varepsilon_1.\varepsilon_2. \dots .\varepsilon_k$. Using $\|n\|^2 = -g$, from the equation (3.29), we reach

$$K_\zeta(e,a) = \frac{\cosh^2\psi_0\left(\sum_{x=1}^{s}\sinh^2\theta_x\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_x^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_x}\right)^2\right)+\sum_{y=s+1}^{m}\cosh^2\theta_y\left(-\frac{1}{2g}\frac{\partial^2 g}{\partial u_y^2}+\frac{1}{4g^2}\left(\frac{\partial g}{\partial u_y}\right)^2\right)\right)}{1-\langle e,a\rangle^2}.$$

Considering $\varepsilon_x = -1$, $\varepsilon_y = 1$ and the equation (2.24) together with the last equation we complete the proof.

By following similar calculations and using the equations (2.28) and (2.29) together with the equations (2.22), (2.26), (3.1), (3.2), (3.3) and (3.4), we complete the proof of the other cases.

These relations given by the equations (3.18)-(3.25) between the curvature of the non-degenerate section (e,a) and the curvature of the non-degenerate section (e,n) are called semi-Euclidean Beltrami Meusnier formulas of generalized semi-ruled surface at the central point $\xi \in \Omega$ in E_v^{n+1}.

4. References

[1] Akyiğit, M., E_v^{n+1}, $(n+1)-$boyutlu yarı Öklid uzayında genelleştirilmiş yarı regle yüzeylerin kesit eğrilikleri, Ph.D Thesis, Sakarya Üniversitesi, Fen Bilimleri Enstitüsü, Sakarya, 2011.

[2] Beem, J. K., Ehrlich, P. E., Easley, K. L., Global Lorentzian Geometry, Marcel Dekker, New York,. 1981.

[3] Ekici, C., Yarı Öklidiyen uzaylarda genelleştirilmiş yarı regle yüzeyler, Ph.D Thesis, Osmangazi Üniversitesi, Fen Bilimleri Enstitüsü, Eskişehir, 1998.

[4] Ekici, C., On the curvatures of $(k+1)-$dimensional semi ruled surfaces in E_v^{n+1}, Mathematical and Computational Applications, **5** (3), 139-148, 2000.

[5] Ersoy S., Tosun M., Lorentzian Beltrami-Euler formula and generalized Lorentzian Lamarle formula in IR_1^n, Le Matematiche, Vol. LXIV –Fasc. I, 25–45, 2009.

[6] Ersoy, S., Tosun M., Sectional curvature of timelike ruled surface Part I: Lorentzian Beltrami-Euler formula, Iranian Journal of Science and Technology, **34** A3, 2010.

[7] Frank, H., Giering, O., Verallgemeinerte regelflachen, Math. Z. 150, 261-271, 1976.

[8] Frank, H., Giering, O., Regelflachen mit zentralchen, Math. Öster. Akad. Wiss., Wien Math-Naturwiss. K1. Abt.II, 187, 139-163, 1978.

[9] Frank, H., Giering, O., Zur schnittkrümmung verallgemeinerter regelflachen, Archiv Der Mathematik, Fasc.1, 32 , 86-90, 1979.

[10] O'Neill, B., Semi-Riemannian Geometry, Academic Press, New York, 1983.

[11] Ratcliffe, J. G., Foundations of Hyperbolic Manifolds, Department Mathematics, Vanderbilt University, 1994.

[12] Thas, C., Properties of ruled surfaces in the Euclidean space E^n, *Bull. Inst. Math. Acad. Sinica,* **6** (1), 133-142, 1978.

[13] Tosun, M., Kuruoğlu, N., On (k+1)-dimensional time-like ruled surface in the Minkowski space IR_1^n, Institute of Mathematics and Computer Sciences, **11** (1), 1-9, 1998.