Biharmonic Submanifolds in Euclidean Spheres

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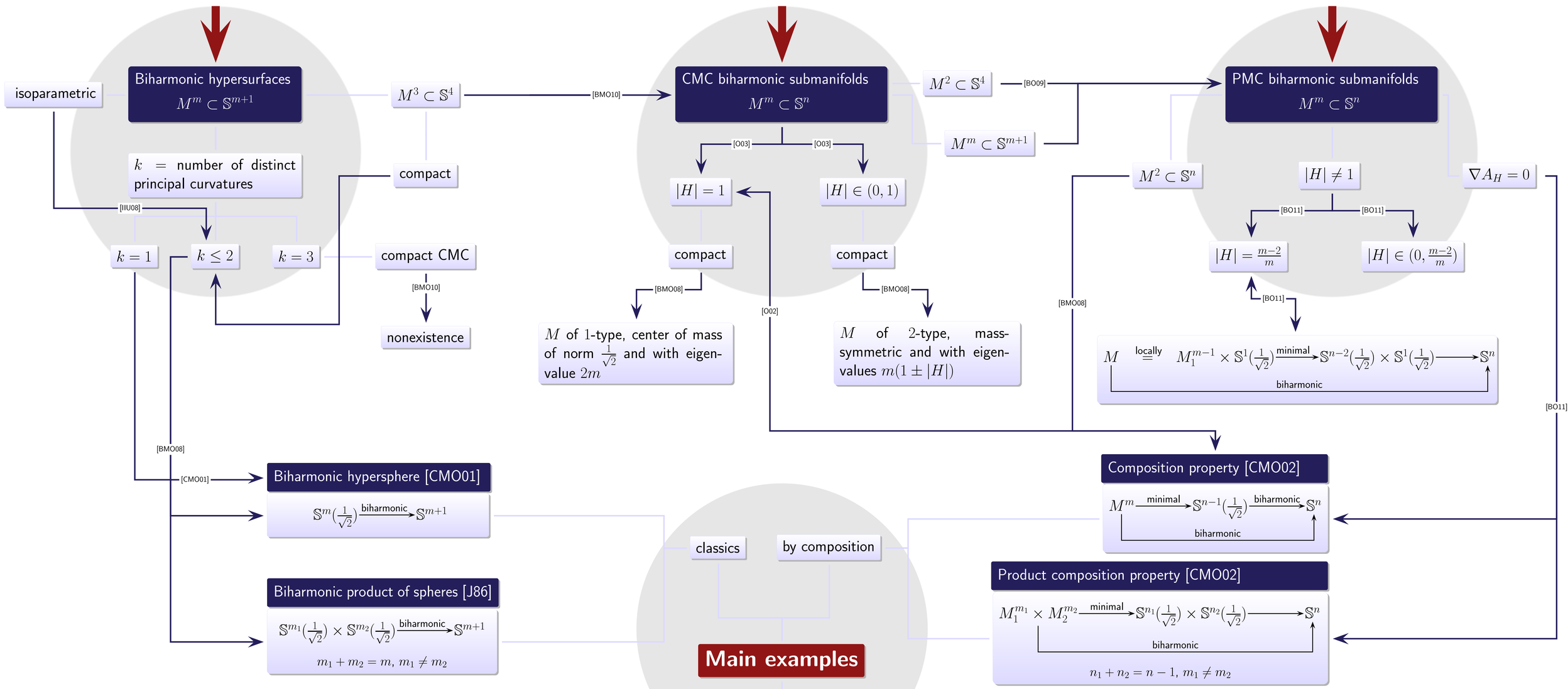

Biharmonic maps

Harmonic maps. A smooth map $\phi:(M,g)\to(N,h)$ between two Riemannian manifolds, M compact, is harmonic if it is a critical point of the energy functional

$$E(\phi)=\frac{1}{2}\int_M |d\phi|^2 v_g.$$

The corresponding Euler-Lagrange equation is ([ES64, EL83])

$$\tau(\phi)=\operatorname{trace}_g \nabla d\phi = 0.$$

Harmonic submanifolds. The inclusion of a submanifold is harmonic if and only if it is minimal, i.e. a critical point of the volume functional.
Biharmonic maps are critical points of the bienergy functional

$$E_2(\phi)=\frac{1}{2}\int_M |\tau(\phi)|^2 v_g,$$

thus solutions of the Euler-Lagrange equation associated to E_2 ([J86]), also called the biharmonic equation,

$$\begin{aligned}\tau_2(\phi) &= -\Delta\tau(\phi) - \operatorname{trace}_g\ R^N(d\phi,\tau(\phi))d\phi\\ &= 0.\end{aligned}$$

$\Delta = -\operatorname{trace}_g\left(\nabla^\phi\nabla^\phi - \nabla^\phi_\nabla\right)$ is the rough Laplacian.

$R^N(X,Y)=[\nabla_X,\nabla_Y]-\nabla_{[X,Y]}$ is the curvature on N.
Remark. (i) When M is not compact, ϕ is said to be biharmonic if it is a solution of the biharmonic equation.
(ii) A harmonic map is biharmonic (it is an absolute minimum of the bienergy). A non-harmonic biharmonic map is called proper biharmonic.

Biharmonic submanifolds. A submanifold is biharmonic if the inclusion map is biharmonic.
A submanifold $i:M\to\mathbb{R}^n$ is biharmonic if and only if it has harmonic mean curvature vector field $\Delta H=0$, or $\Delta^2 i=0$, i.e. it is biharmonic in the sense of Chen ([C91]).

Biharmonic submanifolds in $\mathbb{S}^n$

Generalized Chen's Conjecture. The biharmonic submanifolds in spaces with constant non positive sectional curvature are the minimal ones.
The conjecture, although still open, has been verified in numerous cases ([BMO08, CMO02, C91]). The situation is rather different in the case the ambient space is a sphere.

This poster illustrates the known results on the classification of biharmonic submanifolds in $\mathbb{S}^n$.
The inclusion map of a submanifold $i:M^m\to\mathbb{S}^n$ in the n-dimensional unit Euclidean sphere is biharmonic if and only if ([J86, CMO01])

$$\Delta H = mH.$$

Characterizations. ([O02, BO11])

- A submanifold M^m is biharmonic in $\mathbb{S}^n$ if and only if

$$\begin{cases}\Delta^\perp H + \operatorname{trace} B(\cdot, A_H\cdot) - mH = 0,\\ 4\operatorname{trace} A_{\nabla^\perp_{(\cdot)}H}(\cdot) + m\operatorname{grad}(|H|^2) = 0.\end{cases}$$

  A_H is the Weingarten operator of M associated to H.
  B is the second fundamental form of M.

- A hypersurface with constant mean curvature (CMC) M^m is biharmonic in $\mathbb{S}^{m+1}$ if and only if

$$|A|^2 = m.$$

  A is the shape operator of M in $\mathbb{S}^{m+1}$.

- A submanifold with parallel mean curvature vector field (PMC) M^m is biharmonic in $\mathbb{S}^n$ if and only if

$$\begin{cases}|A_H|^2 = m|H|^2\\ \langle A_H, A_\eta\rangle = 0, \quad \forall \eta\in C(NM), \eta\perp H.\end{cases}$$

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