Hypersurfaces in non-flat Lorentzian space forms satisfying $L_k\psi = A\psi + b$

H. Fabian Ramírez (hectorfabian.ramirez@um.es) **Joint work with Pascual Lucas** (plucas@um.es)

Departamento de Matemáticas, Universidad de Murcia

Abstract

We study hypersurfaces either in the De Sitter space $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ or in the anti De Sitter space $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ whose position vector ψ satisfies the condition $L_k\psi = A\psi + b$, where L_k is the linearized operator of the $(k+1)$th mean curvature of the hypersurface, for a fixed $k = 0, \ldots, n-1$, $A \in \mathbb{R}^{(n+2)\times(n+2)}$ is a constant matrix and $b \in \mathbb{R}^{n+2}$ is a constant vector. For every k, we prove that when A is self-adjoint and $b = 0$, the only hypersurfaces satisfying that condition are hypersurfaces with zero $(k+1)$-th mean curvature and constant k-th mean curvature, open pieces of standard pseudo-Riemannian products in $\mathbb{S}_1^{n+1}$ $\big(\mathbb{S}_1^m(r) \times \mathbb{S}^{n-m}(\sqrt{1-r^2}), \mathbb{H}^m(-r) \times \mathbb{S}^{n-m}(\sqrt{1+r^2}), \mathbb{S}_1^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r), \mathbb{H}^m(-\sqrt{r^2-1}) \times \mathbb{S}^{n-m}(r)\big)$ open pieces of standard pseudo-Riemannian products in $\mathbb{H}_1^{n+1}$ $\big(\mathbb{H}_1^m(-r) \times \mathbb{S}^{n-m}(\sqrt{r^2-1}), \mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}_1^{n-m}(r), \mathbb{S}_1^m(\sqrt{r^2-1}) \times \mathbb{H}^{n-m}(-r), \mathbb{H}^m(-\sqrt{1-r^2}) \times \mathbb{H}^{n-m}(-r)\big)$ and open pieces of a quadratic hypersurfaces $\{x \in \mathbb{M}_c^{n+1} : \langle Rx, x\rangle = d\}$ where R is a self-adjoint constant matrix whose minimal polynomial is $t^2 + at + b$, with $a^2 - 4b \le 0$, and $\mathbb{M}_c^{n+1}$ stands for $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ or $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$. When the k-th mean curvature is constant and b is a non-zero constant vector, we show that the hypersurface is totally umbilical, and then we also obtain a classification result.

1 Preliminaries

Let $\mathbb{R}_q^{n+2}$ be the $(n+2)$-dimensional pseudo-Euclidean space of index $q \ge 1$, whose metric tensor is given by

$$\langle,\rangle = -\sum_{i=1}^{q} dx_i^2 + \sum_{j=q+1}^{n+2} dx_j^2,$$

where $(x_1, \ldots, x_{n+2})$ are the usual rectangular coordinates of $\mathbb{R}^{n+2}$. We will denote by $\mathbb{M}_c^{n+1}$ the De Sitter space $\mathbb{S}_1^{n+1}$ or the anti De Sitter space $\mathbb{H}_1^{n+1}$ according to $c = 1$ or $c = -1$, respectively. We will use $\mathbb{R}_q^{n+2}$ to denote the corresponding pseudo-Euclidean space where $\mathbb{M}_c^{n+1}$ lives, so that $q = 1$ if $c = 1$ and $q = 2$ if $c = -1$. Then its metric is given by $\langle,\rangle = -dx_1^2 + cdx_2^2 + dx_3^2 + \cdots + dx_{n+2}^2$, and we can write

$$\mathbb{M}_c^{n+1} = \{x \in \mathbb{R}_q^{n+2} | -x_1^2 + cx_2^2 + x_3^2 + \cdots + x_{n+2}^2 = c\}$$

It is well known that $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$ and $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$ are Lorentzian totally umbilical hypersurfaces with constant sectional curvature $+1$ and -1, respectively. Consider $\psi : M \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ a connected orientable hypersurface immersed into $\mathbb{M}_c^{n+1}$, with Gauss map N. As is well known, $\langle N, N\rangle = \varepsilon = \pm 1$, according to M is endowed with a Lorentzian ($\varepsilon = 1$) or Riemannian metric ($\varepsilon = -1$).

Let $\mathcal{B} = \{E_1, E_2, \ldots, E_{n+1}\}$ be a (local) frame in $\mathbb{M}_c^{n+1}$. Without loss of generality, we will say that $\mathcal{B}$ is an orthonormal frame if

$$(g_{ij}) = \begin{bmatrix} -1 & & \mathbf{0} \\ & 1 & \\ & & \ddots & \\ \mathbf{0} & & & 1 \end{bmatrix}$$

or pseudo-orthonormal frame if

$$(g_{ij}) = \begin{bmatrix} 0 & -1 & & \mathbf{0} \\ -1 & 0 & & \\ & & 1 & \\ & & & \ddots \\ \mathbf{0} & & & & 1 \end{bmatrix}$$

It is well known that the shape operator S of the hypersurface M can be expressed, in an appropriate frame (orthonormal or pseudo-orthonormal), in one of the following types:

$$\text{I. } S \approx \begin{bmatrix} \kappa_1 & & & \mathbf{0} \\ & \kappa_2 & & \\ & & \ddots & \\ \mathbf{0} & & & \kappa_n \end{bmatrix}; \quad \text{II. } S \approx \left[\begin{array}{cc|ccc} \kappa & -b & & \mathbf{0} & \\ b & \kappa & & & \\ \hline & & \kappa_3 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \kappa_n \end{array}\right], b \neq 0; \quad \text{III. } S \approx \left[\begin{array}{cc|ccc} \kappa & 0 & & \mathbf{0} & \\ 1 & \kappa & & & \\ \hline & & \kappa_3 & & \\ & & & \ddots & \\ \mathbf{0} & & & & \kappa_n \end{array}\right]; \quad \text{IV. } S \approx \left[\begin{array}{ccc|ccc} \kappa & 0 & 0 & & \mathbf{0} & \\ 0 & \kappa & 1 & & & \\ -1 & 0 & \kappa & & & \\ \hline & & & \kappa_4 & & \\ & & & & \ddots & \\ \mathbf{0} & & & & & \kappa_n \end{array}\right].$$

A hypersurface M in a pseudo-Riemannian space form is said to be *isoparametric* if the minimal polynomial of its shape operator S is constant (both coefficients and degree).

The characteristic polynomial $Q_S(t)$ of the shape operator S is given by

$$Q_S(t) = \det(tI - S) = \sum_{k=0}^{n} a_k t^{n-k}, \quad \text{with } a_0 = 1, \quad a_k = -\frac{1}{k}\sum_{j=1}^{k} a_{k-j}\operatorname{tr}(S^j).$$

The *k-th mean curvature* of M is defined by $\binom{n}{k} H_k = (-\varepsilon)^k a_k$, and the *k-th Newton transformation* of M is the operator $P_k : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$ defined by

$$P_k = \sum_{j=0}^{k} a_{k-j} S^j, \quad \text{or inductively by} \quad P_0 = I \text{ and } P_k = a_k I + S \circ P_{k-1}.$$

Associated to each Newton transformation P_k we can define the second-order linear differential operator $L_k : \mathcal{C}^\infty(M) \longrightarrow \mathcal{C}^\infty(M)$ given by

$$L_k(f) := \operatorname{div}(P_k(\nabla f)) = \operatorname{tr}(P_k \circ \nabla^2 f).$$

2 Examples

First, we need to compute $L_k\psi$ and $L_k N$, and to do that we are going to compute the operator L_k acting on the coordinate functions of the immersion ψ and the Gauss map N, that is, the functions $\langle \psi, a\rangle$ and $\langle N, a\rangle$ respectively, where $a \in \mathbb{L}^{n+1}$ is an arbitrary fixed vector. A direct computation shows that

$$L_k\psi = c_k H_{k+1} N - cc_k H_k \psi, \tag{1}$$
$$L_k N = -\varepsilon C_k \nabla H_{k+1} - \varepsilon C_k\big(nH_1 H_{k+1} - (n-k-1)H_{k+2}\big)N + \varepsilon cc_k H_{k+1}\psi, \tag{2}$$

where c_k and C_k are constant. On the other hand, in the case where A is self-adjoint we deduce that

$$\nabla\langle b, \psi\rangle = b^T = c_k \nabla H_k. \tag{3}$$

The following auxiliar result is the key point in the proof of the main theorems of classification.

Lemma 1 If $\psi : M \longrightarrow \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ satisfies the condition $L_k\psi = A\psi + b$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some constant vector $b \in \mathbb{R}_q^{n+2}$, then H_k is constant if and only if H_{k+1} is constant.

The goal of this section is to show some examples of hypersurfaces in the Lorentzian space form $\mathbb{M}_c^{n+1}$ satisfying the condition $L_k\psi = A\psi + b$.

Example 1 *Every hypersurface with $H_{k+1} \equiv 0$ and constant k-th mean curvature H_k trivially satisfies $L_k\psi = A\psi + b$, with $A = -cc_k H_k I_{n+2} \in \mathbb{R}^{(n+2)\times(n+2)}$ and $b = 0$.*

Example 2 (Totally umbilical hypersurfaces)
As is well known, totally umbilical hypersurfaces in $\mathbb{M}_c^{n+1}$ are obtained as the intersection of $\mathbb{M}_c^{n+1}$ with a hyperplane of $\mathbb{R}_q^{n+2}$, and the causal character of the hyperplane determines the type of the hypersurface. It is not difficult to see that for every $\tau \in \mathbb{R}$ with $\langle a, a\rangle - c\tau^2 \neq 0$, the set

$$M_\tau = \{x \in \mathbb{M}_c^{n+1} : \langle a, x\rangle = \tau\}, \qquad \langle a, a\rangle \in \{-1, 0, 1\},$$

is a totally umbilical hypersurface in $\mathbb{M}_c^{n+1}$, and that M_τ satisfies the equation $L_k\psi = A\psi + b$ with

$$A = -\frac{c_k(\varepsilon c\tau)^k\big(\varepsilon\tau^2 + c|\langle a, a\rangle - c\tau^2|\big)}{|\langle a, a\rangle - c\tau^2|^{(k+2)/2}} I_{n+2} \quad \text{and} \quad b = \frac{c_k(\varepsilon c\tau)^{k+1}}{|\langle a, a\rangle - c\tau^2|^{(k+2)/2}} a.$$

In particular, $b = 0$ only when $\tau = 0$, and then M_0 is a totally geodesic hypersurface in $\mathbb{M}_c^{n+1}$. Now we will see the different possibilities.

$\langle a, a\rangle$	For $c = 1$, $M_\tau \subset \mathbb{M}_c^{n+1} = \mathbb{S}_1^{n+1}$	For $c = -1$, $M_\tau \subset \mathbb{M}_c^{n+1} = \mathbb{H}_1^{n+1}$
-1	$M_\tau \equiv \mathbb{S}^n(\sqrt{\tau^2+1})$	$M_\tau \equiv \mathbb{S}_1^n(\sqrt{\tau^2-1})$ ó $M_\tau \equiv \mathbb{H}^n(\sqrt{1-\tau^2})$
0	$M_\tau \equiv \mathbb{R}^n$	$M_\tau \equiv \mathbb{R}_1^n$
1	$M_\tau \equiv \mathbb{H}^n(-\sqrt{\tau^2-1})$ ó $M_\tau \equiv \mathbb{S}_1^n(\sqrt{1-\tau^2})$	$M_\tau \equiv \mathbb{H}_1^n(-\sqrt{\tau^2+1})$

Example 3 (Product of two totally umbilical submanifolds)
Let $f : \mathbb{M}_c^{n+1} \to \mathbb{R}$ be the differentiable function defined by $f(x) = \langle Dx, x\rangle$, where D is the matrix $D = \operatorname{diag}[\delta_1, 1, \delta_1, \ldots, \delta_1, \delta_2, \ldots, \delta_2]$ with $\delta_1, \delta_2 \in \{0, 1\}$ and $\delta_1 + \delta_2 = 1$. Then, for every $r > 0$ and $\rho = \pm 1$ with $r^2 - c\rho \neq 0$, the level set $M^n = f^{-1}(\rho r^2)$ is a hypersurface in $\mathbb{M}_c^{n+1}$, provided that $(\delta_1, \delta_2, \rho, c) \notin \{(0, 1, -1, 1), (1, 0, 1, -1)\}$. It is not difficult to see that M^n satisfies the condition $L_k\psi = A\psi + b$, with $b = 0$ and $A = \operatorname{diag}[\lambda, \theta, \lambda, \ldots, \lambda, \mu, \ldots \mu]$ where

$$\lambda = \frac{cc_k H_{k+1}(\delta_1 - \rho cr^2)}{r\sqrt{|\rho - cr^2|}} - cc_k H_k, \theta = \frac{cc_k H_{k+1}(1 - \rho cr^2)}{r\sqrt{|\rho - cr^2|}} - cc_k H_k, \mu = \frac{cc_k H_{k+1}(\delta_2 - \rho cr^2)}{r\sqrt{|\rho - cr^2|}} - cc_k H_k.$$

The following two tables show the different hypersurfaces in $\mathbb{M}_c^{n+1}$.

$c = 1$: Standard products in $\mathbb{S}_1^{n+1}$

δ_1	δ_2	ρ	Hypersurface
1	0	1	$\mathbb{S}_1^m(r) \times \mathbb{S}^{n-m}(\sqrt{1-r^2})$
1	0	-1	$\mathbb{H}^m(-r) \times \mathbb{S}^{n-m}(\sqrt{1+r^2})$
0	1	1	$\mathbb{S}_1^m(\sqrt{1-r^2}) \times \mathbb{S}^{n-m}(r)$ $\mathbb{H}_1^m(-\sqrt{r^2-1}) \times \mathbb{S}^{n-m}(r)$

$c = -1$: Standard products in $\mathbb{H}_1^{n+1}$

δ_1	δ_2	ρ	Hypersurface
1	0	-1	$\mathbb{H}_1^m(-r) \times \mathbb{S}^{n-m}(\sqrt{r^2-1})$
0	1	1	$\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}_1^{n-m}(r)$
0	1	-1	$\mathbb{S}^m(\sqrt{r^2-1}) \times \mathbb{H}^{n-m}(-r)$ $\mathbb{H}^m(-\sqrt{1-r^2}) \times \mathbb{H}^{n-m}(-r)$

Example 4 (A quadratic hypersurface with non-diagonalizable shape operator)
Let R be a self-adjoint endomorphism of $\mathbb{R}_q^{n+2}$, that is, $\langle Rx, y\rangle = \langle x, Ry\rangle$, for all $x, y \in \mathbb{R}_q^{n+2}$. Let $f : \mathbb{M}_c^{n+1} \to \mathbb{R}$ be a quadric function defined by $f(x) = \langle Rx, x\rangle$, and assume that the minimal polynomial of R is given by $\mu_R(t) = t^2 + at + b$, $a, b \in \mathbb{R}$, with $a^2 - 4b \le 0$. Then, for every $d \in \mathbb{R}$ with $\mu_R(cd) \neq 0$, the level set $M = f^{-1}(d)$ is a Lorentzian hypersurface in $\mathbb{M}_c^{n+1}$. It is not difficult to show that the shape operator S and its minimal polynomial are given, respectively, by

$$SX = -\frac{1}{|\mu_R(cd)|^{1/2}}(RX - cdX), \quad \text{and} \quad \mu_S(t) = t^2 - \frac{a + 2cd}{|\mu_R(cd)|^{1/2}} t + \frac{b + d^2 + acd}{|\mu_R(cd)|}.$$

Therefore, every k-th mean curvature is constant. On the other hand, since the discriminant of $\mu_S(t)$ is not positive, the shape operator is non-diagonalizable.

Finally, from (1) we obtain that $L_k\psi = A\psi$, where A is the matrix given by

$$A = \frac{c_k H_{k+1}}{|\mu_R(cd)|^{1/2}} R - \Big(\frac{c_k H_{k+1} cd}{|\mu_R(cd)|^{1/2}} + cc_k H_k\Big) I.$$

3 Classification

Theorem 2

Let $\psi : M^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ be an orientable hypersurface immersed into the space form $\mathbb{M}_c^{n+1}$. Then the immersion satisfies the condition $L_k\psi = A\psi$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$, if and only if it is one of the following hypersurfaces:

- *a hypersurface having zero $(k+1)$-th mean curvature and constant k-th mean curvature;*
- *an open piece of a standard pseudo-Riemannian product in $\mathbb{S}_1^{n+1}$: $\mathbb{S}_1^m(r) \times \mathbb{S}^{n-m}(\sqrt{1-r^2})$, $\mathbb{H}_1^m(-r) \times \mathbb{S}^{n-m}(\sqrt{1+r^2})$, $\mathbb{H}^m(-\sqrt{r^2-1}) \times \mathbb{S}^{n-m}(r)$;*
- *an open piece of a standard pseudo-Riemannian product in $\mathbb{H}_1^{n+1}$: $\mathbb{H}_1^m(-r) \times \mathbb{S}^{n-m}(\sqrt{r^2-1})$, $\mathbb{H}^m(-\sqrt{1+r^2}) \times \mathbb{S}_1^{n-m}(r)$, $\mathbb{H}^m(-\sqrt{1-r^2}) \times \mathbb{H}^{n-m}(-r)$;*
- *an open piece of a quadratic hypersurface $\{x \in \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2} | \langle Rx, x\rangle = d\}$, where R is a self-adjoint constant matrix whose minimal polynomial is $t^2 + at + b$ with $a^2 + 4b \le 0$.*

Proof. We have already checked in section 2 that each one of the hypersurfaces mentioned in Theorem 2 does satisfy the condition $L_k\psi = A\psi$. Conversely, let us assume that $\psi : M \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ satisfies the condition $L_k\psi = A\psi$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$. Since $b = 0$, from (3) we get that H_k is constant on M, and from Lemma 1 we know that H_{k+1} is constant on M.

Let us assume that $H_{k+1} \neq 0$ (otherwise, there is nothing to prove). By using (1) and (2), we find that

$$A\psi = c_k H_{k+1} N - cc_k H_k \psi, \tag{4}$$
$$AX = -c_k H_{k+1} SX - cc_k H_k X, \tag{5}$$
$$AN = \alpha N + \varepsilon cc_k H_{k+1}\psi, \tag{6}$$

with $\alpha = -\varepsilon C_k(nH_1 H_{k+1} - (n-k-1)H_{k+2})$. Taking covariant derivative in (6), and using (5), we find

$$\overline{\nabla}_X(AN) = \langle\nabla\alpha, X\rangle N - \alpha SX + \varepsilon cc_k H_{k+1} X,$$
$$\overline{\nabla}_X(AN) = -A(\nabla_X^0 N) = -A(SX) = c_k H_{k+1} S^2 X + cc_k SX.$$

From the last two equations we deduce that α is constant on M, and that the operator S satisfies the equation

$$S^2 + \lambda S - \varepsilon cI = 0, \quad \text{with} \quad \lambda = \frac{\alpha}{c_k H_{k+1}} + \frac{cH_k}{H_{k+1}} = \text{constant}. \tag{7}$$

As a consequence, M is an isoparametric hypersurface in $\mathbb{M}_c^{n+1}$ and the minimal polynomial of its shape operator S is of degree at most two. We claim that M is not totally umbilical. Otherwise, from Example 2 we get that it should be totally geodesic, but this is a contradiction, since we are supposing that H_{k+1} is a non-zero constant. Thus, the minimal polynomial of S is exactly of degree two. If S is diagonalizable, then M has exactly two distinct constant principal curvatures, and then it is an open piece of a standard pseudo-Riemannian product (Example 3) [3,4].

Suppose now that S is not diagonalizable, so that the minimal polynomial of S is given by $\mu_S(t) = t^2 + \lambda t - \varepsilon c$, with discriminant $d_S = \lambda^2 + 4\varepsilon c \le 0$. From equations (4)-(7) we easily deduce that the minimal polynomial of A is given by $\mu_A(t) = t^2 + a_1 t + a_0$, where $a_1 = 2cc_k H_k - \lambda c_k H_{k+1}$ and $a_0 = c_k^2 H_k^2 - \lambda cc_k^2 H_k H_{k+1} - \varepsilon cc_k^2 H_{k+1}^2$ are constant. Since the discriminant d_A of $\mu_A(t)$ is given by $d_A = c_k^2 H_{k+1}^2 d_S$, then A also is not diagonalizable. On the other hand, $\langle A\psi, \psi\rangle = c_k H_k$ is constant and $\mu_A(-cc_k H_k) \neq 0$, then M is an open piece of a quadratic hypersurface as in Example 4. This concludes the proof of Theorem 2. ■

Theorem 3

Let $\psi : M^n \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ be an orientable hypersurface immersed into the space form $\mathbb{M}_c^{n+1}$. Assume that H_k is constant. Then the immersion satisfies the condition $L_k\psi = A\psi + b$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+1)\times(n+1)}$and some non-zero constant vector $b \in \mathbb{R}_q^{n+2}$, if and only if

- *$c = 1$ and it is an open piece of a totally umbilical hypersurface in $\mathbb{S}_1^{n+1} \subset \mathbb{R}_1^{n+2}$: $\mathbb{S}^n(r)$, $r > 1$; $\mathbb{H}^n(-r)$,$r > 0$; $\mathbb{S}_1^n(r)$, $0 < r < 1$; $\mathbb{R}^n$.*
- *$c = -1$ and it is an open piece of a totally umbilical hypersurface in $\mathbb{H}_1^{n+1} \subset \mathbb{R}_2^{n+2}$: $\mathbb{H}_1^n(-r)$, $r > 1$; $\mathbb{H}^n(-r)$,$0 < r < 1$; $\mathbb{S}_1^n(r)$, $r > 0$; $\mathbb{R}_1^n$.*

Proof. We have already checked in section 2 that each one of the hypersurfaces mentioned in Theorem 3 does satisfy the condition $L_k\psi = A\psi + b$. Conversely, let us assume that $\psi : M \to \mathbb{M}_c^{n+1} \subset \mathbb{R}_q^{n+2}$ satisfies the condition $L_k\psi = A\psi + b$, for some self-adjoint constant matrix $A \in \mathbb{R}^{(n+2)\times(n+2)}$ and some non-zero constant vector b. Since H_{k+1} is assumed to be constant on M, from Lemma 1 we know that H_{k+1} is constant on M. The case $H_{k+1} = 0$ cannot occur, because in that case we have $b = 0$ (see Example 1).

Let us assume that $H_{k+1} \neq 0$. From (3), we obtain that $b^\top = 0$ and that the function $\langle b, \psi\rangle$ is constant on M. Now we use $\langle AN, \psi\rangle = \langle N, A\psi\rangle$ to deduce that

$$\langle b, N\rangle = \frac{cH_k}{H_{k+1}}\langle b, \psi\rangle = \text{constant}.$$

Since $b = \varepsilon\langle b, N\rangle N + c\langle b, \psi\rangle\psi$, taking covariant derivate in this equation we have

$$-\varepsilon\langle b, N\rangle SX + c\langle b, \psi\rangle X = 0,$$

for any tangent vector field X. If $\langle b, N\rangle \neq 0$, then M is totally umbilical (but not totally geodesic). Otherwise, $b = c\langle b, \psi\rangle\psi$ and then $\langle b, N\rangle = 0$, but this implies $b = 0$. That concludes the proof of Theorem 3. ■

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