Second-order Lagrangians admitting a first-order Hamiltonian formalism

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VI International Meeting on Lorentzian Geometry

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Motivation

The Poincaré-Cartan form of a second-order Lagrangian density on an arbitrary fibred manifold $p\colon E \to N$,

$$\Lambda = Lv, \quad L \in C^\infty(J^2E)$$

is, generally, defined on J^3E.

For certain second-order Lagrangian densities it is known that the Poincaré-Cartan form is projectable onto J^2E.

More surprisingly, there exist second-order Lagrangians for which the associated Poincaré-Cartan form projects not only on J^2E but also on J^1E. Notably, this is the case of the Einstein-Hilbert Lagrange in General Relativity.

GOAL: To characterize the second-order Lagrangians for which the associated Poincaré-Cartan form projects onto J^1E.

Jet bundles

$p\colon E \to N$ arbitrary fibred manifold over a connected n-dimensional manifold N oriented by a volume form $v = dx^1 \wedge \cdots \wedge dx^n$, $\dim E = m+n$.

$p^k\colon J^kE \to N$, k-jet bundle of local sections of p.

Projections $p^k_l\colon J^kE \to J^lE$ for every $k \geq l$.

A fibred system (x^i, y^α) on $U \subseteq E$ induces a coordinate system (x^j, y^α_I) on J^rU given by,

$$y^\alpha_I(j^r_x s) = \frac{\partial^{|I|}(y^\alpha \circ s)}{\partial(x^1)^{i_1}\cdots\partial(x^n)^{i_n}}(x),$$

$I = (i_1, \ldots, i_n) \in \mathbb{N}^n$ integer multi-index of order $|I| = i_1 + \cdots + i_n \leq r$.

We set $y^\alpha_{(jk)} = y^\alpha_{(j)+(k)}$ and $y^\alpha_{(j)} = y^\alpha_j$.

The Legendre form of a second-order Lagrangian density

The Legendre form of $\Lambda = Lv$, $L \in C^\infty(J^2E)$, is the $V^*(p^1)$-valued p^3-horizontal $(n-1)$-form ω_Λ on J^3M locally given by

$$\omega_\Lambda = (-1)^{i-1} v_i \otimes \left(L^{i0}_\alpha dy^\alpha + L^{ij}_\alpha dy^\alpha_j\right),$$

where $v_i = dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$, and

$$L^{ij}_\alpha = \tfrac{1}{2-\delta_{ij}}\frac{\partial L}{\partial y^\alpha_{(ij)}},$$

$$L^{i0}_\alpha = \frac{\partial L}{\partial y^\alpha_i} - \tfrac{1}{2-\delta_{ij}} D_j\left(\frac{\partial L}{\partial y^\alpha_{(ij)}}\right),$$

D_j denoting the total derivative with respect to the coordinate x^j, i.e.,

$$D_j = \frac{\partial}{\partial x^j} + \sum_{|I|=0}^{\infty}\sum_{\alpha=1}^{m} y^\alpha_{I+(j)}\frac{\partial}{\partial y^\alpha_I}.$$

The Poincaré-Cartan form of a second-order Lagrangian density

The Poincaré-Cartan form attached to Λ is the ordinary n-form on J^3E:

$$\Theta_\Lambda = (p^3_2)^*\theta^2 \wedge \omega_\Lambda + \Lambda,$$

where θ^2 is the second-order structure form on J^2E locally given by

$$\theta^2 = \left(dy^\alpha - y^\alpha_i dx^i\right) \otimes \frac{\partial}{\partial y^\alpha} + \left(dy^\alpha_h - y^\alpha_{(hi)} dx^i\right) \otimes \frac{\partial}{\partial y^\alpha_h},$$

and the exterior product of $(p^3_2)^*\theta^2$ and the Legendre form, is taken with respect to the pairing induced by duality, $V(p^1) \times_{J^1E} V^*(p^1) \to \mathbb{R}$.

> The Poincaré-Cartan form of $\Lambda = Lv$, $L \in C^\infty(J^2E)$, projects onto J^2E if and only if the following system holds (cf. [DM], [GM]):
>
> $$\frac{1}{2-\delta_{ib}}\frac{\partial^2 L}{\partial y^\beta_{ac}\partial y^\alpha_{ib}} + \frac{1}{2-\delta_{ia}}\frac{\partial^2 L}{\partial y^\beta_{bc}\partial y^\alpha_{ia}} + \frac{1}{2-\delta_{ic}}\frac{\partial^2 L}{\partial y^\beta_{ab}\partial y^\alpha_{ic}} = 0,$$
>
> $$1 \leq a \leq b \leq c \leq n, \quad \alpha, \beta = 1, \ldots, m.$$

Projecting onto J^1. Local formulation

The projection $p^r_{r-1}\colon J^rE \to J^{r-1}E$ admits an affine bundle structure modelled over the vector bundle

$$W^r = (p^{r-1})^* S^rT^*N \otimes (p^{r-1}_0)^*V(p) \to J^{r-1}E.$$

> **Proposition.** The Poincaré-Cartan form attached to $\Lambda = Lv$, $L \in C^\infty(J^2E)$, projects onto J^1E if and only if L is an affine function with respect to the affine structure of $p^2_1\colon J^2E \to J^1E$ and the following equations hold:
>
> $$\frac{\partial L^{hi}_\alpha}{\partial y^\beta_a} = \frac{\partial L^{ai}_\beta}{\partial y^\alpha_h},$$
>
> with $a, h, i = 1, \ldots, n$, and $\alpha, \beta = 1, \ldots, m$.

Projecting onto J^1. Global formulation

Taking the affine structure of the projection $p^r_{r-1}\colon J^rE \to J^{r-1}E$ into account, we obtain a natural isomorphism of vector bundles,

$$I^r\colon (p^r_{r-1})^*W^r = (p^r)^*S^rT^*N \otimes (p^r_0)^*V(p) \xrightarrow{\cong} V(p^r_{r-1}). \qquad (1)$$

A Lagrangian $L \in C^\infty(J^2E)$ is an affine function with respect to the affine structure of $p^2_1\colon J^2E \to J^1E$ if and only if there exists a—necessarily unique—linear form $w_L\colon W^2 \to \mathbb{R}$ such that,

$$L(\tau + j^2_x s) = w_L(\tau) + L(j^2_x s), \qquad \forall \tau \in S^2T^*_xN \otimes V_{s(x)}(p),\ \forall j^2_x s \in J^2E.$$

By using the natural identification

$$(W^2)^* \cong (p^1)^*S^2TN \otimes (p^1_0)^*V^*(p),$$

the linear form w_L defines a section of the vector bundle $(p^1)^*S^2TN \otimes (p^1_0)^*V^*(p) \to J^1E$. If L is locally given by the formula

$$L = L^{ij}_\alpha y^\alpha_{ij} + L_0, \quad L^{ij}_\alpha, L_0 \in C^\infty(J^1E),$$

then

$$w_L = L^{hi}_\alpha \frac{\partial}{\partial x^h} \odot \frac{\partial}{\partial x^i} \otimes \delta y^\alpha,$$

where $\odot$ denotes symmetric product and $\delta y^\alpha = dy^\alpha|_{V(p)}$.

We consider the section

$$w'_L = \tilde{I}^1 \circ \iota^2 \circ w_L\colon J^1E \to (p^1)^*T^*N \otimes V(p^1_0)$$

obtained by composing the following mappings:

$$J^1E \xrightarrow{w_L} (p^1)^*S^2TN \otimes (p^1_0)^*V^*(p) \xrightarrow{\iota^2} (p^1)^* \otimes^2 TN \otimes (p^1_0)^*V^*(p)$$
$$= (p^1)^*TN \otimes \left[(p^1)^*TN \otimes (p^1_0)^*V^*(p)\right] \xrightarrow{\tilde{I}^1} (p^1)^*TN \otimes V^*(p^1_0),$$

where

- ι^2 is the natural embedding
- $\tilde{I}^1 = 1_{(p^1)^*TN} \otimes ((I^1)^*)^{-1}$ is the isomorphism deduced from (1) for $r = 1$. We obtain $(I^1)^*(\delta y^\alpha_a) = \partial/\partial x^a \otimes \delta y^\alpha$, where $\delta y^\alpha_a = dy^\alpha_a|_{V(p^1_0)}$.

Hence

$$w'_L = 2L^{hi}_\alpha \frac{\partial}{\partial x^i} \otimes \delta y^\alpha_h.$$

We consider the section

$$\mathrm{alt} \circ d_{10}\left(w'_L\right)\colon J^1E \to (p^1)^*TN \otimes \left(\wedge^2 V^*(p^1_0)\right)$$

obtained by composing the following mappings:

$$J^1E \xrightarrow{w'_L} (p^1)^*TN \otimes V^*(p^1_0) \xrightarrow{d_{10}} (p^1)^*TN \otimes \left(\otimes^2 V^*(p^1_0)\right)$$
$$\xrightarrow{\mathrm{alt}} (p^1)^*TN \otimes \left(\wedge^2 V^*(p^1_0)\right),$$

where

- the fibre derivative $d_{10} = d_{J^1E/J^0E}$ of w'_L is given by,

$$d_{10}\left(w'_L\right) = 2\frac{\partial L^{hi}_\alpha}{\partial y^\beta_a}\frac{\partial}{\partial x^i} \otimes \delta y^\beta_a \otimes \delta y^\alpha_h,$$

- $\mathrm{alt}\colon \otimes^2 V^*(p^1_0) \to \wedge^2 V^*(p^1_0)$ is the antisymmetric operator,

$$\mathrm{alt} \circ d_{10}\left(w'_L\right) = \left(\frac{\partial L^{hi}_\alpha}{\partial y^\beta_a} - \frac{\partial L^{ai}_\beta}{\partial y^\alpha_h}\right)\frac{\partial}{\partial x^i} \otimes \delta y^\beta_a \otimes \delta y^\alpha_h,$$

We can conclude:

> **Proposition.** The Poincaré-Cartan form attached to $\Lambda = Lv$, $L \in C^\infty(J^2E)$, projects onto J^1E if and only if L is an affine function with respect to the affine structure of $p^2_1\colon J^2E \to J^1E$ and
>
> $$t^3_L = \mathrm{alt} \circ d_{10}\left(w'_L\right) = \left(\frac{\partial L^{hi}_\alpha}{\partial y^\beta_a} - \frac{\partial L^{ai}_\beta}{\partial y^\alpha_h}\right)\frac{\partial}{\partial x^i} \otimes \delta y^\beta_a \otimes \delta y^\alpha_h,$$
>
> vanishes identically.

Hilbert-Einstein Lagrangian

$p_M\colon M \to N$ bundle of pseudo-Riemannian metrics of a signature (n^+, n^-), $n^+ + n^- = n$. Every coordinate system (x^i) induces a coordinate system (x^i, y_{jk}), where $y_{jk} = y_{kj}$ are defined by,

$$g_x = \sum_{i \leq j} y_{ij}(g_x)(dx^i)_x \otimes (dx^j)_x, \quad \forall g_x \in (p_M)^{-1}(U).$$

The E-H Lagrangian is given by

$$L_{EH} \circ j^2 g = (y^{ij} \circ g)(R^g)^h_{ihj},$$

where R^g is the curvature tensor of the Levi-Civita connection Γ^g of the metric g; hence,

$$(R^{\Gamma^g})^i_{jkl} = \partial(\Gamma^g)^i_{jl}/\partial x^k - \partial(\Gamma^g)^i_{jk}/\partial x^l + (\Gamma^g)^m_{jl}(\Gamma^g)^i_{km} - (\Gamma^g)^m_{jk}(\Gamma^g)^i_{lm}.$$

The local expression for L_{EH} is readily seen to be

$$L_{EH} = \tfrac{1}{2}y^{ij}y^{hd}\left(y_{dj,hi} - y_{ij,dh} - y_{dh,ij} + y_{hi,dj}\right) + L_0,$$

where

$$\begin{aligned} L_0 = \tfrac{1}{2}y^{ij}\Big\{ & y^{hm}y_{mr,j}y^{rd}\left(y_{id,h} + y_{hd,i} - y_{ih,d}\right) \\ & - y^{hm}y_{mr,h}y^{rd}\left(y_{id,j} + y_{jd,i} - y_{ij,d}\right) \\ & + \tfrac{1}{2}y^{hr}y^{md}\left(y_{id,j} + y_{jd,i} - y_{ij,d}\right)\left(y_{hr,m} + y_{mr,h} - y_{hm,r}\right) \\ & - \tfrac{1}{2}y^{hr}y^{md}\left(y_{id,h} + y_{hd,i} - y_{ih,d}\right)\left(y_{jr,m} + y_{mr,j} - y_{jm,r}\right)\Big\}. \end{aligned}$$

Hence L_{EH} is an affine function and its Poincaré-Cartan form projects onto J^1M if and only if:

$$0 = 2\frac{\partial\left(L_{EH}\right)^{hi}_{rs}}{\partial y_{ht,a}} - \frac{\partial\left(L_{EH}\right)^{ai}_{ht}}{\partial y_{rs,h}} - \frac{\partial\left(L_{EH}\right)^{ah}_{ht}}{\partial y_{rs,i}},$$

where

$$\left(L_{EH}\right)^{ij}_{rs} = \tfrac{1}{2-\delta_{ij}}\frac{\partial L_{EH}}{\partial y^{rs}_{(ij)}},$$

and the result follows immediately as $(L_{EH})^{ij}_{rs}$ does not depend on the variables $y_{ij,k}$.

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