A Lorentz metric on the manifold of positive definite (2 x 2)-matrices and foliations by ellipses

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1. Preliminaries

A subset E of $\mathbb{R}^2$ is an ellipse (centered at zero) if there exist an orthonormal basis u, v of $\mathbb{R}^2$ and positive numbers a, b such that

$$E = \left\{ xu + yv \mid (x/a)^2 + (y/b)^2 = 1 \right\}.$$

Let $\mathcal{E}$ be the set of all ellipses in the plane centered at zero and let $\mathcal{S}_+$ be the manifold of all positive definite (2×2)-matrices.
The group $G = Gl_2^+ (\mathbb{R})$ acts smoothly on $\mathcal{S}_+$ by $g \cdot A = gAg^T$.
Among the several ways of identifying $\mathcal{E}$ with $\mathcal{S}_+$ we choose:

$$F : \mathcal{S}_+ \to \mathcal{E},\ F(A) = E_A, \qquad \text{where}$$

$$E_A = \left\{ w \in \mathbb{R}^2 \mid \left\langle A^{-1} w, w \right\rangle = 1 \right\} = \left\{ A^{1/2} z \mid |z| = 1 \right\},$$

since it is equivariant with respect to the canonical actions of the group G on $\mathcal{S}_+$ and $\mathcal{E}$.

Endow $\mathcal{E} \cong \mathcal{S}_+$ with the G-invariant ***Lorentz metric*** of signature $(+, -, -)$ such that for $X \in T_I \mathcal{S}_+ = \left\{ X \in \mathbb{R}^{2\times 2} \mid X^T = X \right\}$,

$$\|X\| := \langle X, X \rangle = \det X.$$

Consider on G the bi-invariant metric of signature $(2, 2)$ given at the identity by the canonical inner product of the split quaternions, that is, such that

$$\|(A, X)\| = \det \left(A^{-1} X \right)$$

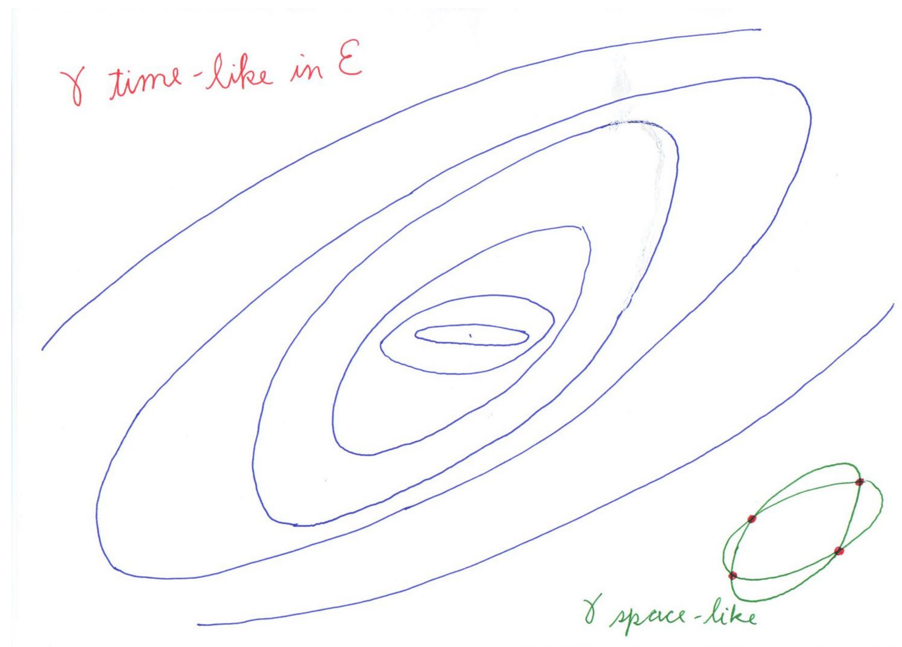
for $A \in G$ and $X \in T_A G = \mathbb{R}^{2\times 2}$. Via the canonical projection $G \to \mathcal{S}_+$, this metric on G pushes down to the Lorentz metric above on $\mathcal{E}$. Hence, the latter is a G-normal metric.

A curve γ in $\mathcal{E}$ is time-like if $\langle \gamma', \gamma' \rangle > 0$.
If $X \in T_I \mathcal{S}_+$ with $\|X\| > 0$, then X is said to be future-pointing if X_{11} and X_{22} are both positive. This induces a G-invariant ***temporal orientation*** on $\mathcal{S}_+$.

2. Foliations by ellipses

We use the Lorentz metric on $\mathcal{E}$ to describe all foliations of (open sets of) the pointed plane $\mathbb{R}^2 - \{0\}$ by ellipses.

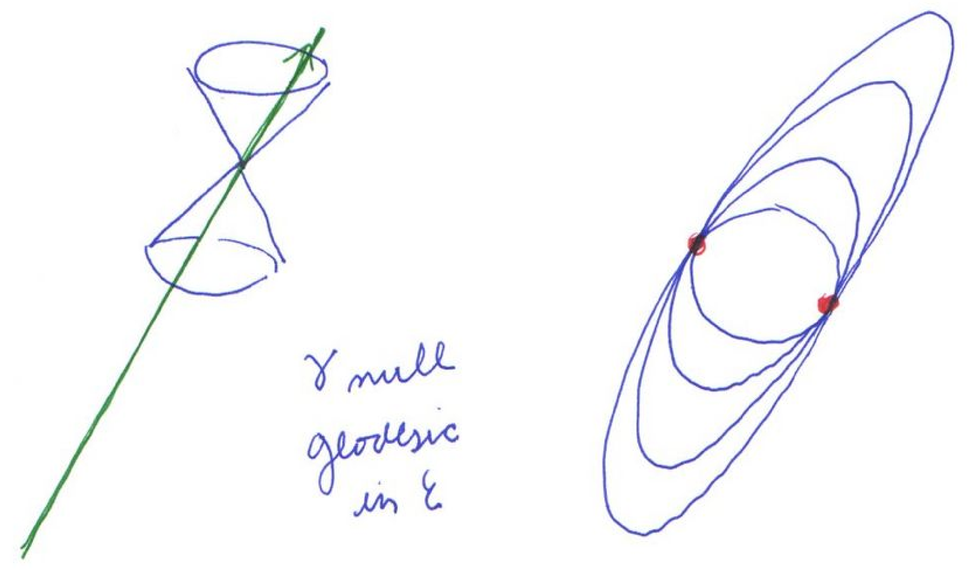
Theorem 1. *A smooth curve γ in $\mathcal{E}$ determines a foliation of an open set of the pointed plane if and only if γ is time-like. The ellipses are nested in increasing order if γ is future-pointing.*

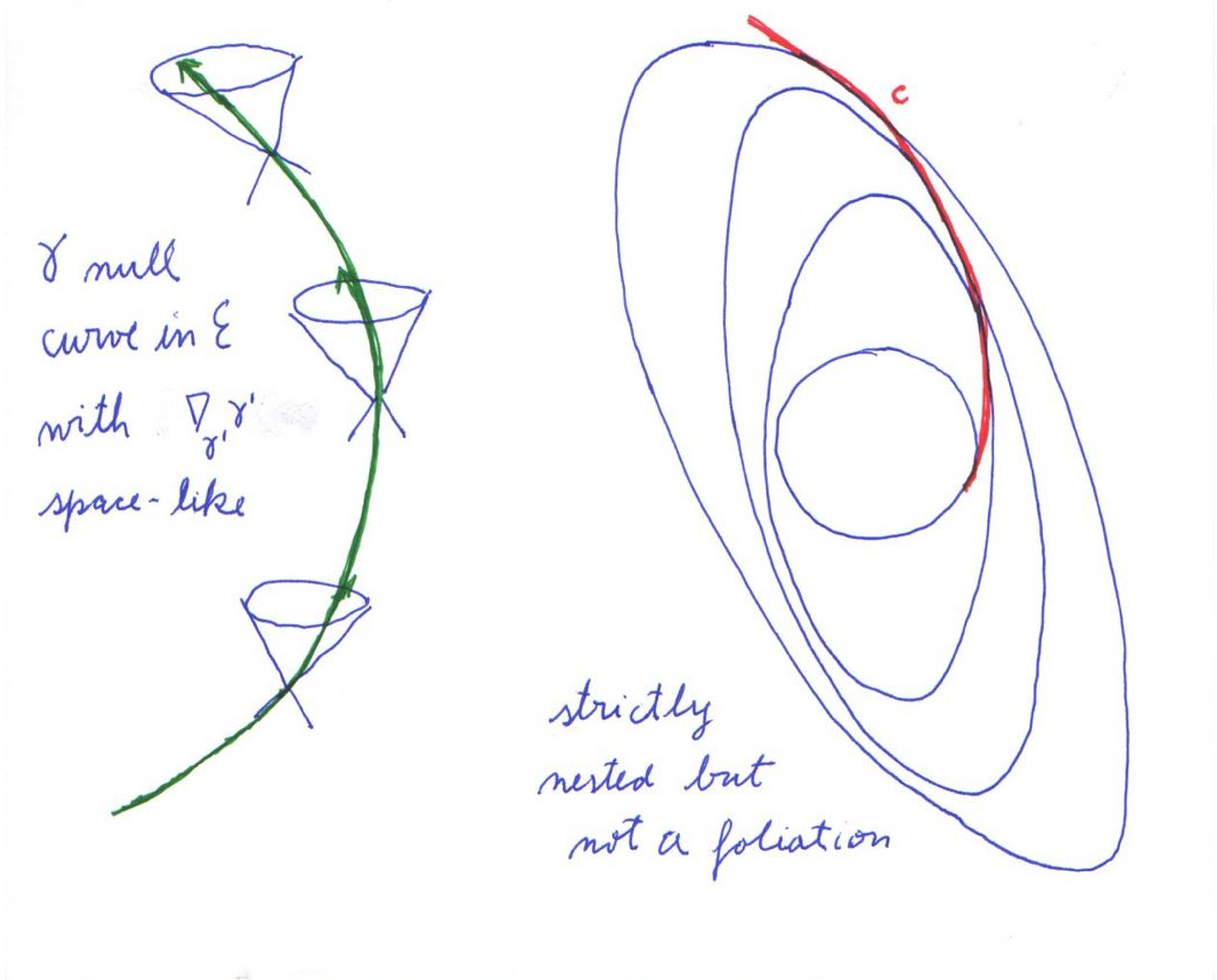


Let V denote the unique unit future-pointing G-invariant vector field X on $\mathcal{E}$, that is, $V_B = (B, B)$. We say that a curve in $\mathcal{E}$ is nondegenerate if no element in its image is a circle.

Theorem 2. *Let γ be a nondegenerate unit speed curve in $\mathcal{E}$ defined on the whole real line such that $\langle \dot\gamma, V \circ \gamma \rangle$ is a bounded function. Then the corresponding foliation by ellipses covers the whole pointed plane.*

Theorem 3. *Let $\gamma : (a, b) \to \mathcal{S}_+$ be a regular null curve with space-like acceleration, that is, $\|\nabla_{\gamma'} \gamma'\| < 0$, and let E_t be the ellipse associated with γ_t, as above. Then the ellipses E_t are strictly nested ($E_t \cap E_s = \emptyset$ if $s \neq t$) and there exists a regular curve $c : (a, b) \to \mathbb{R}^2$ with $c(t) \in E_t$ and $c'(t) \in T_{c(t)} E_t$ for all t. In particular, γ does not determine a foliation of an open set of the plane.*





Proposition 4. *$\mathcal{E}$ is isometric to the warped product $\mathbb{R} \times_- H$, where H is the hyperbolic plane.*

3. General setting: Foliations by congruent submanifolds

The general setting is the characterization of the foliations of a smooth manifold by submanifolds congruent to a given one by the action of a group H, in terms of the H-invariant geometry of this set of submanifolds:
Let N be a smooth manifold acted on smoothly by a group H, let M be a submanifold of N and E the set of submanifolds of N congruent to M via H. The problem consists in describing geometrically which subsets F of E determine foliations of (open subsets of) N.
The paradigm is the paper [1], where fibrations of S^3 by great circles are characterized in this way. See also [2] and [3].
In our case, N is the pointed plane, M is the circle, $H = G$, $E = \mathcal{E}$ and F is the set of trajectories of time-like curves.

[1] H. Gluck, F. Warner, *Great circle fibrations of the three-sphere*, Duke Math. J. 50 (1983), 107-132.
[2] M. Salvai, *Affine maximal torus fibrations of a compact Lie group*, International J. Math. 13 (3) (2002) 217-226.
[3] M. Salvai, *Global smooth fibrations of $\mathbb{R}^3$ by oriented lines*, Bull. London Math. Soc. 41 (2009) 155-163.