Clifford cohomology on hermitian manifolds

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Abstract

In [5] Michelsohn elaborates a detailed analysis of the Clifford cohomology on Kähler manifolds. For this she considers the bundle $Cl_{\mathbb{C}}(M) = Cl(M) \oplus \mathbb{C}$ and a triple of parallel operators $\mathfrak{L}$, $\overline{\mathfrak{L}}$ and $\mathfrak{H}$ defined on it and which carry an intrinsic $\mathfrak{sl}(2)$-structure of $Cl_{\mathbb{C}}(M)$. This, together with J, yields a decomposition

$$Cl_{\mathbb{C}}(M) \equiv \oplus_{|p+q| \leq n} Cl^{p,q}(M).$$

Taking the hermitian Dirac operators $\mathfrak{D}$ and $\overline{\mathfrak{D}}$ associated to the Levi-Civita connection she obtains that $\mathfrak{D}^2 = \overline{\mathfrak{D}}^2 = 0$ and $\mathfrak{D} + \overline{\mathfrak{D}} = 1/2D$, where D is the corresponding Dirac operator, and $\overline{\mathfrak{D}}$ is the formal adjoint of $\mathfrak{D}$.
In [1] the authors define a formally holomorphic connection over those hermitian manifolds which satisfy the third curvature condition. The expresion for this connection is

$$\nabla_X = \nabla^{L.C.}_X - \frac{1}{2} J(\nabla^{L.C.}_X J) \tag{1}$$

where $\nabla^{L.C.}$ represents the Levi-Civita connection and $X \in T_{\mathbb{C}}M$.
In this contribution we use the algebraic theory of the Clifford algebra $Cl_{\mathbb{C}}(M)$ developed by Michelsohn and this formally holomorphic connection to obtain similar operators to $\mathfrak{D}$ and $\overline{\mathfrak{D}}$, $\mathfrak{D}^{\nabla}$ and $\overline{\mathfrak{D}}^{\nabla}$, defined on certain hermitian non Kähler manifolds and which satisfy similar properties, $(\mathfrak{D}^{\nabla})^2 = (\overline{\mathfrak{D}}^{\nabla})^2 = 0$ and such that $\overline{\mathfrak{D}}^{\nabla}$ is the formal adjoint of $\mathfrak{D}^{\nabla}$.

1. Preliminaries

An *almost hermitian manifold* $(M^{2n}, J, \langle\cdot,\cdot\rangle)$ is a real manifold of dimension $2n$ endowed with an almost complex structure J and with a metric $\langle\cdot,\cdot\rangle$ compatible with J, that is $\langle JX, JY\rangle = \langle X, Y\rangle; \forall X, Y \in TM$.
A *hermitian* manifold is an almost hermitian manifold $(M^{2n}, J, \langle\cdot,\cdot\rangle)$ such that

$$(\nabla^{L.C.}_X J)Y = (\nabla^{L.C.}_{JX} J)JY$$

for any vector fields $X, Y \in TM$ and where $\nabla^{L.C.}$ denotes the Levi-Civita connection associated to the metric $\langle\cdot,\cdot\rangle$.
It is a well known fact, see for example [1], that on any hermitian manifold the torsion, T^{∇}, of the connection given by the expression (1) satisfies $T^{\nabla}(X,Y) - T^{\nabla}(JX, JY) = 0; \forall X, Y \in TM$. Furthermore, if the hermitian manifold M satisfies the third curvature condition, then the connection ∇ is formally holomorphic.
Lets $\{e_1, Je_1, \ldots, e_n, Je_n\}$ be an associated J-basis, the complexified of the vector space TM, $T_{\mathbb{C}}M$, is generated by the elements $\epsilon_k = 1/2(e_k - iJe_k)$ and $\overline{\epsilon}_k = 1/2(e_k + iJe_k)$ for $k = 1, \ldots, n$. The extension of the almost complex structure to $V_{\mathbb{C}}$ in the natural way induces the decomposition $T_{\mathbb{C}}M = T^{(1,0)}M \otimes T^{(0,1)}M$, where the vectors ϵ_k and $\overline{\epsilon}_k$ are basis for $T^{(1,0)}M$ and $T^{(0,1)}M$ respectively.
The Clifford algebra $Cl(M)$ associated to the tangent fiber bundle is defined as the quotient T/I, where $T = \sum_{r=0}^{\infty} \otimes^r TM$ is the *tensor algebra* and I is the two-side ideal generated by all elements of the form $v \otimes v + \|v\| \cdot 1$, $v \in V$. So, the Clifford algebra $ClM)$ is the unitary associative algebra equipped with a canonical embedding $TM \subset Cl(M)$, and it is characterized by the universal property that any linear map $\phi : M \to A$ into an associative algebra, A, with unit, such that $\phi(v) \cdot \phi(v) = -\|v\| \cdot 1$ for all v, extends to a unique algebra homomorphism $\hat{\phi} : Cl(M) \to A$. It´s not difficult to prove that the Clifford algebra $Cl_{\mathbb{C}}(M)$ is generated by the elements of the form $\epsilon_I \overline{\epsilon}_J = \epsilon_{i_1} \cdot \ldots \cdot \epsilon_{i_r} \cdot \overline{\epsilon}_{j_1} \cdot \ldots \cdot \overline{\epsilon}_{j_s}$, where I and J are increasing elements of the set $\{i, \ldots, n\}$ not necessarily disjoints, and which satisfy the relations $\epsilon_k \overline{\epsilon}_j + \overline{\epsilon}_j \epsilon_k = -\delta_{ij}$ and $\epsilon_k \epsilon_k = \overline{\epsilon}_k \overline{\epsilon}_k = 0$.
The almost complex structure can also be extended to $Cl_{\mathbb{C}}(M)$ by setting $\mathfrak{J}(w_1 \cdot \ldots \cdot w_k) = \frac{1}{i} \sum_{i=1}^{k} w_1 \cdot \ldots \cdot J(w_i) \cdot \ldots \cdot w_k$ which satisfy $\mathfrak{J}(\epsilon_I \overline{\epsilon}_J) = (|I| - |J|)\epsilon_I \overline{\epsilon}_J$ and define the decomposition

$$Cl_{\mathbb{C}}(M) = \bigoplus_{p=-n}^{n} Cl^p$$

where $Cl^p = \{\varphi \in Cl_{\mathbb{C}}(M) : \mathfrak{J}\varphi = p\varphi\}$.
There exist three intrinsically linear maps $\mathfrak{L}, \overline{\mathfrak{L}}, \mathfrak{H} : Cl_{\mathbb{C}}(M) \longrightarrow Cl_{\mathbb{C}}(M)$ defined as follows:

$$\mathfrak{L}(\varphi) = -\sum_{i=1}^{n} \epsilon_i \varphi \overline{\epsilon}_i, \quad \overline{\mathfrak{L}}(\varphi) = -\sum_{i=1}^{n} \overline{\epsilon}_i \varphi \epsilon_i, \quad \mathfrak{H}(\varphi) = [\mathfrak{L}, \overline{\mathfrak{L}}](\varphi)$$

for $\varphi \in Cl_{\mathbb{C}}(M)$.
It is not difficult to prove that the operators $\mathfrak{L}$, $\overline{\mathfrak{L}}$, $\mathfrak{H}$ verify the following relations: $[\mathfrak{L}, \overline{\mathfrak{L}}] = \mathfrak{H}$, $[\mathfrak{H}, \mathfrak{L}] = 2\mathfrak{L}$, $[\mathfrak{H}, \overline{\mathfrak{L}}] = -2\overline{\mathfrak{L}}$ and hence they define a representation of $\mathfrak{sl}(2)$ on $Cl_{\mathbb{C}}(M)$.
Each of the operators $\mathfrak{L}$, $\overline{\mathfrak{L}}$, $\mathfrak{H}$ commutes with $\mathfrak{J}$, therefore it is posible to define the subspaces

$$Cl^{p,q} = \{\varphi \in Cl_{\mathbb{C}}(M) : \mathfrak{h}\varphi = q\varphi \text{ and } \mathfrak{J}\varphi = p\varphi\}$$

and obtain the decomposition

$$Cl_{\mathbb{C}}(M) = \otimes_{p,q} Cl^{p,q}$$

2. Complex Dirac operators

It is a well known result that any unitary connection on $T_{\mathbb{C}}M$ extends canonically to the bundle $Cl_{\mathbb{C}}(M)$ as a derivation, i. e., such that

$$\nabla(\varphi \cdot \psi) = \nabla\varphi \cdot \psi + \varphi \cdot \nabla\psi$$

for all $\varphi, \psi \in \Gamma(Cl_{\mathbb{C}}(M))$. Each of the operators $\mathfrak{J}$, $\mathfrak{H}$, $\mathfrak{L}$ and $\overline{\mathfrak{L}}$ is parallel in this connection. In particular, the subspaces $Cl^{p,q}(M)$ are preserved under covariant differentiation.
We introduce now two differential operators $\mathfrak{D}^{S^{\nabla}}, \overline{\mathfrak{D}}^{S^{\nabla}} : \Gamma(Cl_{\mathbb{C}}(M)) \to \Gamma(Cl_{\mathbb{C}}(M))$ by the formulas

$$\mathfrak{D}^{S^{\nabla}} = \sum_{j=1}^{n} \epsilon_j \cdot \nabla_{\overline{\epsilon}_j} + \frac{1}{2} \sum_{i=1}^{n} \epsilon_{S(e_i)e_i}$$

$$\overline{\mathfrak{D}}^{S^{\nabla}} = \sum_{j=1}^{n} \overline{\epsilon}_j \cdot \nabla_{\epsilon_j} + \frac{1}{2} \sum_{i=1}^{n} \overline{\epsilon}_{S(e_i)e_i}$$

where $\epsilon_{S(e_i)e_i} = 1/2(S(e_i)e_i - iJS(e_i)e_i)$.

2.1. Clifford cohomology

Theorem. If the divergence associated to the connection ∇ coincide with the divergence associated to the Levi-Civita connection then the operators $\mathfrak{D}^{S^{\nabla}}$ and $\overline{\mathfrak{D}}^{S^{\nabla}}$ are formally adjoint.

Demostration:
At each $p \in M$ it is possible to choose local frames $\epsilon_1, \ldots, \epsilon_n, \overline{\epsilon}_1, \ldots, \overline{\epsilon}_n$ such that $(\nabla\epsilon_k)_p = (\nabla\overline{\epsilon}_k)_p = 0$ for every $k \in \{1, \ldots, n\}$. For φ, $\psi \in \Gamma(Cl_{\mathbb{C}}(M))$ we define the complex vector U given by $\langle V, U\rangle = 1/4((V - iJV) \cdot \varphi, \psi)$ for any tangent vectors V, where $(\cdot,\cdot)$ is the hermitian inner product defined in $Cl_{\mathbb{C}}(M)$ associated to the scalar product $\langle\cdot,\cdot\rangle$ in the usual way and where "$\cdot$" denotes the Clifford product. Then at the point p we have

$$div^{\nabla} U = div^{\nabla^{L.C.}} + \sum_{i=1}^{n} (\varphi, S(\overline{\epsilon}_i)\epsilon_i) = \sum_{i=1}^{n} \overline{\epsilon}_i(\epsilon_i \cdot \varphi, \psi).$$

We consider know the auxiliar operator

$$\mathfrak{D}^{\nabla} = \sum_{i=1}^{n} \epsilon_i \cdot \nabla_{\overline{\epsilon}_i}$$

which at the point p satisfy

$$(\mathfrak{D}^{\nabla}\varphi, \psi) = div^{\nabla} U + \sum_{i=1}^{n} (\varphi, \overline{\epsilon}_i \cdot \nabla_{\epsilon_i}\psi)$$

hence

$$(\mathfrak{D}^{S^{\nabla}}\varphi, \psi) = (\varphi, \overline{\mathfrak{D}}^{S^{\nabla}}\psi) + div^{\nabla^{L.C.}} U$$

and finally

$$\int_M (\mathfrak{D}^{S^{\nabla}}\varphi, \psi) = \int_M (\varphi, \overline{\mathfrak{D}}^{S^{\nabla}}\psi) + \int_M div^{\nabla^{L.C.}} U = \int_M (\varphi, \overline{\mathfrak{D}}^{S^{\nabla}}\psi)$$

□

Theorem. The operators $\mathfrak{D}^{\nabla} = \sum_{i=1}^{n} \epsilon_i \cdot \nabla_{\overline{\epsilon}_i}$ and $\overline{\mathfrak{D}}^{\nabla} = \sum_{i=1}^{n} \overline{\epsilon}_i \cdot \nabla_{\epsilon_i}$ satisfy the following equalities

$$(\mathfrak{D}^{\nabla})^2 = 0 = (\overline{\mathfrak{D}}^{\nabla})^2$$

Demostration:
As above we consider at each point $p \in M$ local frames $\epsilon_1, \ldots, \epsilon_n, \overline{\epsilon}_1, \ldots, \overline{\epsilon}_n$ such that $(\nabla\epsilon_k)_p = (\nabla\overline{\epsilon}_k)_p = 0$ for every $k \in \{1, \ldots, n\}$. For φ, $\psi \in \Gamma(Cl_{\mathbb{C}}(M))$ we have

$$(\mathfrak{D}^{\nabla})^2 = \sum_{j<k}^{n} \epsilon_j \cdot \epsilon_k \cdot R^{\nabla}(\overline{\epsilon}_j, \overline{\epsilon}_k) + \sum_{j<k}^{n} \epsilon_j \cdot \epsilon_k \cdot \nabla_{T^{\nabla}(\overline{\epsilon}_j, \overline{\epsilon}_k)} = 0$$

$$(\overline{\mathfrak{D}}^{\nabla})^2 = \sum_{j<k}^{n} \overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot R^{\nabla}(\epsilon_j, \epsilon_k) + \sum_{j<k}^{n} \overline{\epsilon}_j \cdot \overline{\epsilon}_k \cdot \nabla_{T^{\nabla}(\epsilon_j, \epsilon_k)} = 0$$

□

Furthermore, the complexes

$$\ldots \xrightarrow{\mathfrak{D}^{\nabla}} \Gamma(Cl^{p-1,q-1}) \xrightarrow{\mathfrak{D}^{\nabla}} \Gamma(Cl^{p,q}) \xrightarrow{\mathfrak{D}^{\nabla}} \Gamma(Cl^{p+1,q+1}) \xrightarrow{\mathfrak{D}^{\nabla}} \ldots$$

$$\ldots \xleftarrow{\overline{\mathfrak{D}}^{\nabla}} \Gamma(Cl^{p-1,q-1}) \xleftarrow{\overline{\mathfrak{D}}^{\nabla}} \Gamma(Cl^{p,q}) \xleftarrow{\overline{\mathfrak{D}}^{\nabla}} \Gamma(Cl^{p+1,q+1}) \xleftarrow{\overline{\mathfrak{D}}^{\nabla}} \ldots$$

are elliptic. To see this let $\phi = (\lambda_j e_j + \mu_j J e_j)^{\flat} \in T^*M$ be a real 1-form in M, then the principal symbols of $\mathfrak{D}^{\nabla}$ and $\overline{\mathfrak{D}}^{\nabla}$ are given by $\sigma(\mathfrak{D}^{\nabla}, \phi) = \xi\cdot$ and $\sigma(\overline{\mathfrak{D}}^{\nabla}, \phi) = \overline{\xi}\cdot$ respectively, where $\xi = 1/4(\phi - iJ\phi)$.
Hence, for the operator $\mathfrak{D}^{\nabla}$ there are defined finite dimensional Clifford cohomology groups

$$\mathfrak{H}^{p,q}(M) = (Ker\,\mathfrak{D}^{\nabla} / Im\,\mathfrak{D}^{\nabla}) \cap \Gamma(Cl^{p,q})$$

which are isomorphic to the groups

$$H^{p,q}(M) = Ker(\Delta) \cap \Gamma(Cl^{p,q})$$

where Δ is the Laplacian $\Delta = \mathfrak{D}^{\nabla}\overline{\mathfrak{D}}^{\nabla} + \mathfrak{D}^{\nabla}\overline{\mathfrak{D}}^{\nabla}$. The argument for the operator $\overline{\mathfrak{D}}^{\nabla}$ is the same.

3. Examples

In [2] the author proves that on the homogeneous natural reductive space $M = U(3)/(U(1) \times U(1) \times U(1))$ there exist three different hermitian structures, J_1, J_2 and J_3, defined in the following way: Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of $U(3)$ and $U(1) \times U(1) \times U(1)$ respectively, then we have the reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$, here $\mathfrak{m}$ is identified with the tangent space of M at a point o. Let denote by D_{ij} the (3×3)-matrix consisting of a single 1 in the i-th row and j-th column, and zeros in the rest, $E_{ij} = 1/\sqrt{2}(D_{ij} - D_{ji})$ and $F_{i,j} = i/\sqrt{2}(D_{ij} + D_{ji})$. The matrices $e_1 = E_{12}$, $e_2 = F_{12}$, $e_3 = E_{13}$, $e_4 = F_{13}$, $e_5 = E_{23}$, $e_6 = F_{23}$ generate a basis of $\mathfrak{m}$. So, the almost complex structures J_1, J_2 and J_3 are given by:

$$\begin{array}{lll} J_1(e_1) = e_2, & J_1(e_3) = e_4, & J_1(e_5) = e_6 \\ J_2(e_1) = e_2, & J_2(e_3) = e_4, & J_2(e_5) = -e_6 \\ J_3(e_1) = -e_2, & J_3(e_3) = e_4, & J_3(e_5) = e_6 \end{array}$$

It is prove too that this manifold endowed with any of these hermitian structures satisfy de third curvature condition. Furthermore, it is not difficult to see that in any of this cases the operators $\mathfrak{D}^{S^{\nabla}}$ and $\overline{\mathfrak{D}}^{S^{\nabla}}$ coincide with the operators $\mathfrak{D}^{\nabla}$ and $\overline{\mathfrak{D}}^{\nabla}$ respectively.

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