Decoupling and Exact Solutions of Einstein Equations in Almost Kähler Variables

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Poster Session

VI International Meeting on Lorentzian Geometry
Granada, Spain

September 6–9, 2011

Summary

The Einstein equations written in certain nonholonomic variables with 2+2 splitting have a decoupling/splitting property which allows us to construct exact solutions in very general forms following methods elaborated in Refs. [1]. The "general" solutions are parametrized by generic off-diagonal metrics determined by corresponding classes of generating and integration functions depending on all spacetime coordinates.

We show how nonholonomic almost Kähler variables can be associated to off–diagonal Einstein manifolds. Such variables are important for performing deformation and A-brane quantization of Einstein gravity, formal "anisotropic" generalizations and elaborating noncommutative generalizations [2] and computing quantum corrections to solutions.

We study geometric criteria when generic off-diagonal solutions define Lorenz manifolds and satisfy the Cauchy problem. There are discussed extensions of the method for constructing exact solutions in modified theories of gravity and Ricci flows [3].

References

[1] Off–diagonal Solutions in Gravity

1. S. Vacaru, Decoupling of EYMH equations, off-diagonal solutions, and black ellipsoids and solitons, arXiv: 1108.2022
2. - // -, On general solutions in Einstein gravity, Int. J. Geom. Meth. Mod. Phys. 8 (2011) 9-21; arXiv: 1106.4660
3. - // -, On general solutions in Einstein and high dimensional gravity, Int. J. Theor. Phys. 49 (2010) 884-913; arXiv: 0909.3949v4
4. - // -, Curve flows and solitonic hierarchies generated by Einstein metrics, Acta Applicandae Mathematicae 110 (2010) 73-107; arXiv: 0810.0707

[2] Deformation, A–brane and Two Connections Quantization of Gravity

1. S. Vacaru, Einstein gravity as a nonholonomic almost Kaehler geometry, Lagrange-Finsler variables, and deformation quantization, J. Geom. Phys. 60 (2010) 1289-1305; arXiv: 0709.3609
2. - // -, Two-connection remormalization and nonholonomic gauge models of Einstein gravity, Int. J. Geom. Meth. Mod. Phys. 7 (2010) 713-744; arXiv:0902.0961
3. - // -, Branes and quantization for an A-model complexification of Einstein gravity in almost Kaehler variables, Int. J. Geom. Meth. Mod. Phys. 6 (2009) 873-909; arXiv: 0810.4692

[3] Modified Theories of Gravity and Ricci Flows

1. - // -, Finsler black holes induced by noncommutative anholonomic distributions in Einstein gravity, Class. Quant. Grav. 27 (2010) 105003; arXiv: 0907.4278
2. - // -, Critical remarks on Finsler modifications of gravity and cosmology by Zhe Chang and Xin Li, Phys. Lett. B 690 (2010) 224-228; arXiv: 1003.0044
3. - // -, Spectral functionals, nonholonomic Dirac operators, and noncommutative Ricci flows, J. Math. Phys. 50 (2009) 073503; arXiv: 0806.3814

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Decoupling Property of Einstein eqs

Nonholonomic 2+2 splitting in GR

Goal: Find such nonholonomic frames of reference $e_\alpha = e_\alpha^{\alpha'} \partial_{\alpha'}$ when the Einstein eqs decouple in a general form and then integrate in very general forms, with generic off–diagonal metrics depending on all coordinates.

Non–integrable (nonholonomic) 2+2 spacetime splitting in GR $(V, \mathbf{g})$,

4–d pseudo–Riemannian V, $\mathbf{g} = g_{\alpha\beta}$ solution of $E_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \Upsilon_{\alpha\beta}$.

Conventional $2+2$ splitting: indices $\alpha, \beta, ... = (i, a), (j, b), ...$ for $i, j, k, ... = 1, 2$; $a, b, c, .. = 3, 4$; coordinates $u^\alpha = (x^i, y^a) = (x^1, x^2, y^3, y^4)$, or $u = (x, y)$, partial derivatives $\partial_\alpha := \partial / \partial u^\alpha$; $\partial_\alpha = (\partial_i, \partial_a)$

N–adapted frames/ bases: $\mathbf{N} : TV = hTV \oplus vTV$; $\mathbf{N} = N_i^a(x, y) \partial_a \otimes dx^i$

Structure of nonholonomic frames, $[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = \mathbf{w}^{\gamma}{}_{\alpha\beta}(u) \mathbf{e}_\gamma$,

$$\mathbf{e}_\alpha := (\mathbf{e}_i = \partial_i - N_i^a \partial_a, e_b = \partial_b),$$
$$\mathbf{e}^\beta := (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i).$$

Decoupling Property of Einstein eqs

N–adapted metrics and connections

Via frame transforms $\forall \mathbf{g}$ in GR can be represented in two equivalent forms:

1) With respect to coordinate bases: $\mathbf{g} = \underline{g}_{\alpha\beta}(u)\, du^{\alpha} \otimes du^{\beta}$

$$\text{for } \underline{g}_{\alpha\beta} = \begin{bmatrix} g_{ij} + N_i^a N_j^b g_{ab} & N_j^e g_{ae} \\ N_i^e\, g_{be} & g_{ab} \end{bmatrix}, \text{where } N_i^a \neq A_{bi}^a(x) y^b;$$

we do not consider Kaluza–Klein gravity.

2) N–adapted, $\mathbf{g} = \mathbf{g}_{\alpha\beta}(u)\mathbf{e}^{\alpha} \otimes \mathbf{e}^{\beta} = g_{ij}(x,y)\, e^i \otimes e^j + g_{ab}(x,y)\mathbf{e}^a \otimes \mathbf{e}^b$

Levi–Civita connection: $\nabla = \widehat{\mathbf{D}} + \widehat{\mathbf{Z}}[\widehat{\mathbf{T}}]$, $\forall$ values from $\mathbf{g}$, $\nabla \mathbf{g} = 0$, zero ∇–torsion;

"auxiliary" connection $\widehat{\mathbf{D}} = \widehat{\Gamma}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$: 1) $\widehat{\mathbf{D}}\mathbf{g} = 0$, 2) $\widehat{T}^i{}_{jk} = 0$, $\widehat{T}^a{}_{bc} = 0$.

$\widehat{L}^i_{jk} = \frac{1}{2} g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk})$, $\widehat{L}^a_{bk} = e_b(N^a_k) + \frac{1}{2} g^{ac}(e_k g_{bc} - g_{dc}\, e_b N^d_k - g_{db}\, e_c N^d_k)$,

$\widehat{C}^i_{jc} = \frac{1}{2} g^{ik} e_c g_{jk}$, $\widehat{C}^a_{bc} = \frac{1}{2} g^{ad}(e_c g_{bd} + e_c g_{cd} - e_d g_{bc})$

Torsion $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta}$: $\widehat{T}^i{}_{ja} = \widehat{C}^i_{jb}$, $\widehat{T}^a_{ji} = -\Omega^a{}_{ji}$, $\widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N^c_j)$.

If we chose N_i^a that $\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta} = 0$, we get $\widehat{\mathbf{Z}}[\widehat{\mathbf{T}}] = 0$ and $\Gamma^{\gamma}{}_{\alpha\beta} = \widehat{\Gamma}^{\gamma}{}_{\alpha\beta}$, even $\nabla \neq \widehat{\mathbf{D}}$

Decoupling Property of Einstein eqs

Einstein eqs in N–adapted form

Lemma: With respect to N–adapted frames with 2+2 splitting,

the Einstein eqs can be written in terms of $\widehat{\mathbf{D}}$,

$$\text{N–adapted eqs:} \quad \widehat{\mathbf{R}}_{\ \beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\ {}^{s}R = \Upsilon_{\beta\delta},$$

$$\text{LC–conditions:} \quad \widehat{L}^{c}_{aj} = e_a(N^c_j),\ \widehat{C}^{i}_{jb} = 0,\ \Omega^{a}_{\ ji} = 0,$$

$\widehat{\mathbf{R}}_{\ \beta\delta}$ for $\widehat{\Gamma}^{\gamma}_{\ \alpha\beta}$, ${}^{s}R = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\ \beta\delta}$ and $\Upsilon_{\beta\delta} \rightarrow \varkappa T_{\beta\delta}$ for $\widehat{\mathbf{D}} \rightarrow \nabla$.

In the simplest form, the decoupling property can be proved for metrics, $\mathbf{g}_{\alpha'\beta'}$, when ${}^{K}\mathbf{g}_{\alpha\beta} = e^{\alpha'}_{\alpha} e^{\beta'}_{\beta}\mathbf{g}_{\alpha'\beta'}$, $(u^{\alpha} = (x^k, v, y^4)$, ansatz with Killing symmetry $\partial/\partial y^4$,

for $g_{\alpha\beta} = diag[g_i(x^k), h_a(x^k, v)], y^3 := v$ and $N^3_i = w_i(x^k, v),\ N^4_i = n_i(x^k, v)$,

$$\begin{aligned} {}^{K}\mathbf{g} &= g_i(x^k)dx^i \otimes dx^i + h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a, \\ \mathbf{e}^3 &= dt + w_i(x^k, v)dx^i,\ \mathbf{e}^4 = dy^4 + n_i(x^k, v)dx^i. \end{aligned}$$

Decoupling Property of Einstein eqs

Theorem 1 (Decoupling of N–adapted Einstein eqs with 2+2 splitting)

Theorem 1 (Decoupling): The nontrivial comps of Einstein eqs for ansatz ${}^{K}\mathbf{g}$,

source $\Upsilon^{\alpha}_{\beta} = diag[\Upsilon^1_1 = \Upsilon^2_2 = \Upsilon_2(x^k, v), \Upsilon^3_3 = \Upsilon^4_4 = \Upsilon_4(x^k)]$
(with $a^{\bullet} = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial v$), parameters ${}^{A}\theta$, are

$$-\widehat{R}^1_1 = -\widehat{R}^2_2 = \frac{1}{2g_1 g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = \Upsilon_4(x^k, {}^{A}\theta)$$

$$-\widehat{R}^3_3 = -\widehat{R}^4_4 = \frac{1}{2h_3 h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] = \Upsilon_2(x^k, v, {}^{B}\theta),$$

$$\widehat{R}_{3k} = \frac{w_k}{2h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] + \frac{h_4^*}{4h_4}(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}) - \frac{\partial_k h_4^*}{2h_4} = 0,$$

$$\widehat{R}_{4k} = \frac{h_4}{2h_3} n_k^{**} + (\frac{h_4}{h_3} h_3^* - \frac{3}{2} h_4^*)\frac{n_k^*}{2h_3} = 0,$$

LC conditions: $w_i^* = \mathbf{e}_i \ln|h_4|, \mathbf{e}_k w_i = \mathbf{e}_i w_k,\ n_i^* = 0,\ \partial_i n_k = \partial_k n_i.$

Remark: We can not "see" such a decoupling/separation of eqs if we use directly the LC in non–N–adapted frames.

Integration of N–adapted Einstein eqs

Theorem 2 (Integral Varieties)

Theorem 2 (constructing off–diagonal "one–Killing" solutions)

For $h^*_{3,4} \neq 0, \Upsilon_{2,4} \neq 0$, the gravitational field eqs for ansatz ${}^K\mathbf{g}$ and $\widehat{\mathbf{D}}$ are equivalent to

$$\begin{aligned} \varepsilon_1 \ddot{\psi} + \varepsilon_2 \psi'' &= 2\Upsilon_4(x^k, {}^A\theta) \\ h_4^* &= 2h_3h_4\Upsilon_2(x^i, v, {}^B\theta)/\phi^* \\ \beta w_i + \alpha_i &= 0, n_i^{**} + \gamma n_i^* = 0, \end{aligned}$$

for $\varepsilon_i = \pm 1, \alpha_i = h_4^* \partial_i \phi, \beta = h_4^* \ \phi^*, \phi = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|, \gamma = \left(\ln \frac{|h_4|^{3/2}}{|h_3|} \right)^*$, parameters ${}^A\theta$,

solution: $g_i = \varepsilon_i e^{\psi(x^k, {}^C\theta)}, \ h_4 = {}_0h_4(x^k, {}^K\theta) \pm \ 2 \int \frac{(\exp[2 \ \phi(x^k, v, {}^D\theta)])^*}{\Upsilon_2(x^i, v, {}^B\theta)} dv,$

$h_3 = \pm \ \frac{1}{4} \left[\sqrt{|h_4^*|} \right]^2 \exp[-2 \ \phi(x^k, v, {}^D\theta)],$

$w_i = -\partial_i \phi / \phi^*, \ n_k = \ {}_1n_k(x^i, {}^J\theta) + \ {}_2n_k(x^i, {}^F\theta) \int [h_3/(\sqrt{|h_4|})^3] dv,$

LC conditions: $w_i^* = \mathbf{e}_i \ln |h_4|, \mathbf{e}_k w_i = \mathbf{e}_i w_k, \ n_i^* = 0, \ \partial_i n_k = \partial_k n_i \rightarrow \ {}_2n_i = 0.$

Integration of N–adapted Einstein eqs

Non–Killing solutions with half–conformal symmetry and N–deformations

Dependence on $y = y^4$ via "vertical" conf. transfs $\omega^2(x^j, v, y), \partial a/\partial y := a^\circ$,

$\omega^2 = 1$ results in solutions with Killing symmetry,

$$\mathbf{g} = g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j, v, y)h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a,$$
$$\mathbf{e}^3 = dy^3 + w_i(x^k, v)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k, v)dx^i,$$
$$\mathbf{e}_k\omega = \partial_k\omega + w_k\omega^* + n_k\omega^\circ = 0.$$

N–deformations & exact solutions with 'gravitat. polarizations' $\eta_\alpha,\ \eta_i^a$,

N–deforms, $^\star\mathbf{g} = [\ ^\star g_i,\ ^\star h_a,\ ^\star N_k^a] \rightarrow\ ^\eta\mathbf{g} = [\ g_i, h_a, N_k^a]$, of fund. geom. objs,

$$^\eta\mathbf{g} = \eta_i(x^k, v)\ ^\star g_i(x^k, v)dx^i \otimes dx^i + \eta_a(x^k, v)\ ^\star h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a,$$

$$\mathbf{e}^3 = dv + \eta_i^3(x^k, v)\ ^\star w_i(x^k, v)dx^i,\ \mathbf{e}^4 = dy^4 + \eta_i^4(x^k, v)\ ^\star n_i(x^k, v)dx^i.$$

$^\star\mathbf{g}$, given metric/solution in GR,$\rightarrow\ ^\eta\mathbf{g}$, being a "parametric/noncommutative/stochastic ..." exact solution in GR or EFG.

Integration of N–adapted Einstein eqs

On "general" classes of off–diagonal solutions

- a) "∀" solution of Einstein eqs, $g_{\alpha'\beta'}$, via $e_\alpha = e_\alpha^{\alpha'}(x^i, y^a)e_{\alpha'}$, $\mathbf{g}_{\alpha\beta} = e_\alpha^{\alpha'} e_\beta^{\beta'} g_{\alpha'\beta'}, \rightarrow$

$$\mathbf{g}_{\alpha\beta} = \left| \begin{array}{cccc} g_1 + \omega^2(w_1{}^2 h_3 + \omega^2(n_1{}^2 h_4) & \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & \omega^2\ w_1 h_3 & \omega^2\ n_1 h_4 \\ \omega^2(w_1 w_2 h_3 + n_1 n_2 h_4) & g_2 + \omega^2(w_2{}^2 h_3 + n_2^2 h_4) & \omega^2\ w_2 h_3 & \omega^2\ n_2 h_4 \\ \omega^2\ w_1 h_3 & \omega^2\ w_2 h_3 & h_3 & 0 \\ \omega^2\ n_1 h_4 & \omega^2\ n_2 h_4 & 0 & h_4 \end{array} \right|$$

b) "Very general" ansatz

$$\mathbf{g} = g_i dx^i \otimes dx^i + \omega^2 h_a \underline{h}_a \mathbf{e}^a \otimes \mathbf{e}^a,$$

$$\mathbf{e}^3 = dy^3 + (w_i + \underline{w}_i)\, dx^i,\ \mathbf{e}^4 = dy^4 + (n_i + \underline{n}_i)\, dx^i,$$

$$g_i = g_i(x^k), g_a = \omega^2(x^i, y^c)\ h_a(x^k, y^3)\underline{h}_a(x^k, y^4),\ \text{not summation on "a"},$$

$$N_i^3 = w_i(x^k, y^3) + \underline{w}_i(x^k, y^4), N_i^4 = n_i(x^k, y^3) + \underline{n}_i(x^k, y^4),$$

are functions of necessary smooth class generating solutions of Einstein eqs.

- Concept of general solutions for systems of nonlinear partial differential eqs? Topology, symmetries etc. Arbitrariness, uniqueness, sources? Complex/supersymmetric/ nonholonomic / quantum distributions – applications to modern gravity and physics. Higher dimensions - "shell by shell". Almost Kähler structures etc, generalized (algebroid etc) symmetries. Nontrivial topology etc. Exact solutions in astrophysics, cosmology: black ellipsoids/toruses, wormholes, solitons, Dirac waves, pp–waves etc

Cauchy Problem & Almost Kähler Variables

Cauchy problem and decoupling property

Fundamental result (Choquet–Bruhat, 1952): $\exists$ a set of hyperbolic equations underlying the Einstein eqs with cosmological constant. **Goal:** to study the evolution (Cauchy) problem in N–adapted form and preserving the decoupling property.

Definition (N–adapted wave coordinates): A set $\{\widehat{u}^{\mu} = (\widehat{x}^{i}, \widehat{y}^{a})\}$

is canonically N–harmonic, i.e. it both harmonic and adapted to a splitting **N** if $\widehat{\square}\widehat{u}^{\mu} = 0$, where d'Alambert operator $\widehat{\square} := \widehat{\mathbf{D}}_{\alpha}\ \widehat{\mathbf{D}}^{\alpha}$ acts on $f(x,y)$,

$$\begin{aligned} \widehat{\square} f &:= (\sqrt{|\mathbf{g}_{\alpha\beta}|})^{-1}\mathbf{e}_{\mu}\left(\sqrt{|\mathbf{g}_{\alpha\beta}|}\mathbf{g}^{\mu\nu}\mathbf{e}_{\nu} f\right) \\ &= (\sqrt{|g_{kl}|})^{-1}\mathbf{e}_{i}\left(\sqrt{|g_{kl}|}g^{ij}\mathbf{e}_{j} f\right) + (\sqrt{|g_{cd}|})^{-1}\partial_{a}\left(\sqrt{|g_{kl}|}g^{ab}e_{b} f\right). \end{aligned}$$

Lemma: In canonical N–harmonic coordinates, the Einstein eqs with Λ can be

written $\widehat{\mathbf{E}}^{\alpha\beta} = \widehat{\square}\mathbf{g}^{\alpha\beta} - \mathbf{g}^{\tau\theta}\left[(\mathbf{g}^{\alpha\mu}\widehat{\Gamma}^{\beta}{}_{\mu\nu} + \mathbf{g}^{\alpha\mu}\widehat{\Gamma}^{\beta}{}_{\mu\nu})\widehat{\Gamma}^{\nu}{}_{\tau\theta} + 2\mathbf{g}^{\gamma\mu}\widehat{\Gamma}^{\alpha}{}_{\mu\theta}\widehat{\Gamma}^{\beta}{}_{\tau\gamma}\right] - 2\Lambda\mathbf{g}^{\alpha\beta} = 0;$ such PDE for $\mathbf{g}^{\alpha\beta}$ form a system of 2d order quasi–linear N–adapted wave–type eqs.

Cauchy Problem & Almost Kähler Variables

Theorem on N–adapted solutions

We can apply the standard theory of hyperbolic PDE.
Let H^k_{loc} are the Sobolev spaces of functions which are in $L^2(K)$ for any compact set K when their distributional derivatives are considered up to an integer order k also in $L^2(K)$.
We use N–adapted wave coordinates with additional formal $3+1$ splitting, $\widehat{u}^{\mu} = ({}^t\widehat{u}, \widehat{u}^{\bar{i}})$.
Standard results from the theory of PDE give rise to

Theorem: The field eqs for Einstein manifolds have a unique solution $\mathbf{g}^{\alpha\beta}$ defined by N–adapted PDE stated on an open neighborhood $\mathcal{U} \subset \mathbb{R} \times \mathbb{R}^3$ of $\mathcal{O} \subset \{0\} \times \mathbb{R}^3$ with any initial data $\mathbf{g}^{\alpha\beta}(0, \widehat{u}^{\bar{i}}) \in H^{k+1}_{loc}$ and $\frac{\partial \mathbf{g}^{\alpha\beta}(0,\widehat{u}^{\bar{i}})}{\partial({}^t\widehat{u})} \in H^{k+1}_{loc}, k > 3/2$. The set $\mathcal{U}$ can be chosen in a form that $(\mathcal{U}, \mathbf{g}^{\alpha\beta})$ is globally hyperbolic with Cauchy surface $\mathcal{O}$.

On initial data sets and global nonholonomic evolution, see arXiv: 1108.2022

Cauchy Problem & Almost Kähler Variables

Generating analogous Lagrange structures

Generating analogous Lagrange structures

Definition: $\mathcal{L}(x,y)$ is a regular generating function if $\tilde{g}_{ab} = \frac{1}{2}\frac{\partial^2 \mathcal{L}}{\partial y^a \partial y^b}$, $det|\tilde{g}_{ab}| \neq 0$, on an open region $\mathcal{U}$ of a Lorentz manifold **V**.

Theorem: $\exists$ 3 fundamental geom. structure induced by $\mathcal{L}$ on $\mathcal{U}$:

1. **Canonical** $\tilde{N}^i_j(x,y) := \frac{\partial \tilde{G}^i}{\partial y^j}$, $\tilde{G}^i = \frac{1}{4}\tilde{g}^{ij}(\frac{\partial^2 \mathcal{L}}{\partial y^i \partial x^k} y^k - \frac{\partial \mathcal{L}}{\partial x^i})$, for $x^i(\tau), y^i = \frac{dx^i}{d\tau}$, $\frac{d^2 x^i}{d\tau^2} + 2\tilde{G}^i(x^k, \frac{dx^j}{d\tau}) = 0$ equivalent to E–L eqs $\frac{d}{d\tau}(\frac{\partial \mathcal{L}}{\partial y^i}) - \frac{\partial \mathcal{L}}{\partial x^i} = 0$
2. a total metric $\tilde{\mathbf{g}} = \tilde{g}_{ij}(u)\, e^i \otimes e^j + \tilde{g}_{ab}(u)\tilde{\mathbf{e}}^a \otimes \tilde{\mathbf{e}}^b$, $e^i = dx^i$, $\tilde{\mathbf{e}}^b = dy^b + \tilde{N}^b_j(u) dx^j$
3. Cartan connection ${}^c\tilde{\mathbf{D}}$ completely determined by $\mathcal{L}$, and/or $\tilde{\mathbf{g}}$, following conditions ${}^c\tilde{\mathbf{D}}\tilde{\mathbf{g}} = 0$; ${}^c\tilde{\mathbf{T}}[\tilde{\mathbf{g}}] \neq 0$, but $h\, {}^c\tilde{\mathbf{T}} = 0$ and $v\, {}^c\tilde{\mathbf{T}} = 0$, when
   ${}^c\widetilde{L}^i_{jk} = \frac{1}{2}\tilde{g}^{ir}(\tilde{\mathbf{e}}_k \tilde{g}_{jr} + \tilde{\mathbf{e}}_j \tilde{g}_{kr} - \tilde{\mathbf{e}}_r \tilde{g}_{jk})$, ${}^c\widetilde{L}^a_{bk} \rightarrow {}^c\widetilde{L}^i_{jk}$,
   ${}^c\widetilde{C}^i_{jc} \rightarrow {}^c\widetilde{C}^a_{bc}$, ${}^c\widetilde{C}^a_{bc} = \frac{1}{2}\tilde{g}^{ad}(e_c \tilde{g}_{bd} + e_c \tilde{g}_{cd} - e_d \tilde{g}_{bc})$
   ${}^c\tilde{\mathbf{D}} = \{ {}^c\widetilde{\Gamma}^\alpha_{\beta\gamma} = ({}^c\widetilde{L}^i_{jk}, {}^c\widetilde{C}^a_{bc})\}$, distortion determined by $\tilde{g}$, ${}^c\tilde{\mathbf{D}} = \nabla + {}^c\tilde{\mathbf{Z}}[{}^c\tilde{\mathbf{T}}(\tilde{g})]$

Cauchy Problem & Almost Kähler Variables

Almost Kähler models/variables, 2+2 splitting

Lemma: $\mathcal{L}(x,y)$ induces canonical

- almost complex structure: $\tilde{\mathbf{J}}(\tilde{\mathbf{e}}_i) = -e_i$ and $\tilde{\mathbf{J}}(e_i) = \tilde{\mathbf{e}}_i$
- 1–form $\tilde{\omega} = \frac{1}{2}\frac{\partial \mathcal{L}}{\partial y^i} e^i$
- 2–form $\tilde{\theta}(\mathbf{X},\mathbf{Y}) := \tilde{\mathbf{g}}(\mathbf{J}\mathbf{X},\mathbf{Y}) = \tilde{\mathbf{g}}_{ij}(x,y)\tilde{\mathbf{e}}^i \wedge e^i$

Theorem: A canonical almost Kähler structure on $\mathcal{U}$ is defined by ${}^{\theta}\widetilde{\mathbf{D}} = {}^{c}\widetilde{\mathbf{D}}$, ${}^{\theta}\widetilde{\mathbf{D}}_{\mathbf{X}}\tilde{\theta} = 0$ and ${}^{\theta}\widetilde{\mathbf{D}}_{\mathbf{X}}\tilde{\mathbf{J}} = 0$

GR in almost Kähler variables

Via frame transforms $\tilde{\mathbf{e}}_{\alpha} = \tilde{e}_{\alpha}^{\ \alpha'}(x^i, y^a)\mathbf{e}_{\alpha'}$, $\tilde{\mathbf{g}}_{\alpha\beta} = \tilde{e}_{\alpha}^{\ \alpha'}\tilde{e}_{\beta}^{\ \beta'}\mathbf{g}_{\alpha'\beta'}$, and $\nabla = {}^{c}\tilde{\mathbf{D}} - {}^{c}\tilde{\mathbf{Z}}$ distortions any fundamental geometric/physical object, physical equations and solutions of Einstein eqs, $\mathbf{g}_{\alpha'\beta'}$, can be encoded equivalently into a nonholonomic almost Kähler geometry. This allows us to perform deformation and A–brane quantization of gravity theories and Ricci flows [2,3].

Summary, Conclusions & Perspectives

- **Main Results:**
  - With respect to N–adapted frames with 2+2 splitting, it is possible a general decoupling of Einstein eqs and integration of equivalent PDE in very general forms.
  - The generated metrics are generic off–diagonal, depend on all coordinates, being determined by certain classes of generating and integration functions, constants and sets of parameters.
- Physically important theories (Einstein, Hořava–Lifshitz, Einstein–Finser etc) re–formulated as almost Kähler geometries.
- Application and generalizations of methods of deformation / A-brane, two connection and/or geometric quantization and certain ideas on "anisotropic renormalization of QG theories.
- Generalizations to (non) commutative/ quantum / nonholonomic Ricci flow theory.