A Lorentz metric on the manifold of positive definite (2×2)-matrices and foliations by ellipses

Marcos Salvai
FaMAF-CIEM, Córdoba, Argentina
salvai@famaf.unc.edu.ar

Let $\mathcal{E}$ be the set of all ellipses in the plane centered at zero (with axes not necessarily parallel to the coordinate axes), which may be canonically identified with the manifold $\mathcal{S}_+$ of all positive definite (2×2)-matrices.

The group $G = Gl_2^+(\mathbb{R})$ acts smoothly on $\mathcal{S}_+$ by $g \cdot A = gAg^T$. Consider on G the bi-invariant metric of signature $(2,2)$ given at the identity by the opposite of the canonical inner product of the split quaternions, that is, such that $\langle X, X \rangle = -\det X$ for all $X \in M_2(\mathbb{R})$, and endow $\mathcal{E} \cong \mathcal{S}_+$ with the Lorentz metric pushed down from G via the canonical projection $G \to \mathcal{S}_+$.

We use this Lorentz metric on $\mathcal{E}$ to describe all foliations of (open sets of) the pointed plane $\mathbb{R}^2 - \{0\}$ by ellipses. More precisely, we prove that a smooth curve γ in $\mathcal{E}$ determines a foliation of an open set of the pointed plane if and only if γ is time-like. Moreover, if the curve γ in $\mathcal{E}$ is defined on the whole real line, $\langle \dot{\gamma}, \dot{\gamma} \rangle = -1$, and $\langle \dot{\gamma}, X \circ \gamma \rangle$ is a bounded function, then the corresponding foliation covers the whole pointed plane (here X denotes the unique unit future directed G-invariant vector field X on $\mathcal{E}$). We also provide examples of causal curves of pairwise disjoint ellipses which do not determine a foliation.

The general setting is the characterization of the foliations of a smooth manifold by submanifolds congruent to a given one by the action of a group H, in terms of the H-invariant geometry of this set of submanifolds: Let N be a smooth manifold acted on smoothly by a group H, let M be a submanifold of N and E the set of submanifolds of N congruent to M via G. The problem consists in describing geometrically which subsets F of E determine foliations of (open subsets of) N. The paradigm is the paper [1], where fibrations of S^3 by great circles are characterized in this way. See also [2] (a partial generalization of [1]) and [3], with the global foliations of $\mathbb{R}^3$ by lines, which includes a pseudo-Riemannian reformulation of the main result of [1]. In our case, N is the pointed plane, M is the circle, $H = G$, $E = \mathcal{E}$ and F the set of trajectories of time-like curves.

[1] H Gluck, F Warner, *Great circle fibrations of the three-sphere*, Duke Math. J. 50 (1983), 107-132.

[2] M. Salvai, *Affine maximal torus fibrations of a compact Lie group*, International J. Math. 13 (3) (2002) 217-226.

[3] M. Salvai, *Global smooth fibrations of* $\mathbb{R}^3$ *by oriented lines*, Bull. London Math. Soc. 41 (2009) 155-163.