NON-NULL HELICOIDAL SURFACES AS NON-NULL BONNET SURFACES

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Abstract

In this study, we obtain the equivalent of the Codazzi-Mainardi equations for spacelike and timelike surfaces in three dimensional Lorentz space $\mathbb{R}^3_1$. Also, we find the necessary and sufficient conditions for spacelike and timelike helicoidal surfaces with non-null axis of being Bonnet surfaces.

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1 Introduction

Surfaces which admit a one-parameter family of isometric deformations preserving the mean curvature called Bonnet surfaces. In 1867, Bonnet proved that any surfaces with constant mean curvature in $\mathbb{R}^3$(which is not totally umbilical) is a Bonnet surface (see [10]). Lawson extended Bonnet's results to any surface with constant mean curvature in Riemannian 3-manifold of constant curvature c, $\mathbb{R}^3(c)$, [5]. Also, it is proved that any Bonnet surfaces of non-constant mean curvature depend on six arbitrary constants. The similar problems for surfaces in space form $\mathbb{R}^3(c)$ and for spacelike surfaces in indefinite space form $\mathbb{R}^3(c)$ were studied by Chen and Li in [14]. A geometric characterization of helicoidal surfaces of constant mean curvature, the helicoidal surfaces as Bonnet surfaces and the tangent developable surfaces as Bonnet surfaces were investigated by Roussos in [6], [7] and [8], respectively. More recently, timelike surfaces in Lorentzian space forms which admit a one-parameter family of isometric deformations preserving the mean curvature were studied by Fujioka and Inoguchi in [1]. As is known that, a helicoidal surface is a kind of some ruled surfaces and rotation surfaces in and there are remarkable studies on helicoidal surfaces in in [3], [4], [9], [12], [16].

In these regards, we have investigated spacelike and timelike helicoidal surfaces with non-null axis as Bonnet surfaces and obtained the necessary and sufficient conditions for the existence of these surfaces.

2 Preliminaries

Let $\mathbb{R}^3_1$ be a Lorentzian 3-space with the nondegenerate metric tensor

$$g = -dx_1^2 + dx_2^2 + dx_3^2$$

where $\{x_1, x_2, x_3\}$ is a system of the canonical coordinates in $\mathbb{R}^3$, [2].

Firstly, we introduce the basic knowledge and notions about the geometry of surfaces in Lorentzian spaces. Suppose $X : M \to \mathbb{R}^3_1$ is an isometric immersion of a surface M into $\mathbb{R}^3_1$. A surface is said to be spacelike (resp. timelike) if the induced metric on M is positive definite (resp. indefinite). Throughout in this paper, we denote by $\{e_1, e_2, e_3\}$ a local orthonormal frame on M, such that e_3 is a unit normal vector field on M. The unit normal vector field e_3 can be regarded as a map $e_3 : M \to H^2_+$ if M is spacelike and as a map $e_3 : M \to S^2_1$ if M is timelike, where $H^2_+ = \left\{x \in \mathbb{R}^3_1 : \langle x, x\rangle = -1,\ x_1 > 0\right\}$ is the hyperbolic space and $S^2_1 = \left\{x \in \mathbb{R}^3_1 : \langle x, x\rangle = 1\right\}$ is the de-Sitter space. Also, $\{e_1, e_2\}$ comprise an orthonormal basis of the tangent space of M at x. Let μ_i be the dual 1-form of e_i defined by $\mu_i(e_j) = \langle e_i, e_j\rangle = \varepsilon_i \delta_{ij}$, $1 \leqslant i, j \leqslant 3$, then

$$dx = \varepsilon_1 \mu_1 e_1 + \varepsilon_2 \mu_2 e_2$$

where $\varepsilon_i = \langle e_i, e_i\rangle = \pm 1$. The connection forms μ_{ij} are defined by

$$de_i = \sum_{j=1}^{3} \varepsilon_j \mu_{ij} e_j \tag{2.1}$$

which satisfy $\mu_{ij} + \mu_{ji} = 0$. Then the structure equations become

$$d\mu_i = \sum_{j=1}^{3} \varepsilon_j \mu_j \wedge \mu_{ji} \tag{2.2}$$

and

$$d\mu_{ij} = \sum_{k=1}^{3} \varepsilon_k \mu_{ik} \wedge \mu_{kj}.$$

Since μ_3 is a zero form on M the exterior derivate of μ_3 gives

$$\varepsilon_1 \mu_1 \wedge \mu_{13} + \varepsilon_2 \mu_2 \wedge \mu_{23} = 0. \tag{2.3}$$

It follows from (2.3) and the Cartan Lemma that there are symmetric tensors h_{ij} such that

$$\mu_{i3}=\sum_{j=1}^{3}\varepsilon_i h_{ij}\mu_j,\quad h_{ij}=h_{ji}. \tag{2.4}$$

The mean curvature H of M is defined

$$H=\frac{1}{2}\varepsilon_3\left(\varepsilon_1 h_{11}+\varepsilon_2 h_{22}\right).$$

A helicoidal motion group is a non-trivial one-parameter group of rigid motions of $\mathbb{R}_1^3$ and any element of such a group is called a helicoidal motion of $\mathbb{R}_1^3$. Here, trivial cases are pure translation groups. Every helicoidal motion group is completely determined by an axis l and a pitch h. Depending on the line l being spacelike, timelike or null, there are three types of motion. If the axis is spacelike (resp. timelike), then l is transformed to the x_3-axis or x_2-axis (resp. x_1-axis). Therefore we can always suppose that l is the x_3-axis (resp. x_1-axis) if is spacelike (resp. timelike). If the axis l is null, then we may assume that l is the line spanned by $(1,0,1)$. If $G_{l,h}=\{\phi_t:\ t\in\mathbb{R}\}$ denotes the helicoidal motion group with axis l and pitch h, for $p=(a,b,c)\in\mathbb{R}_1^3$, the image of p under any helicoidal motion of one parameter group $G_{l,h}$ is (see [4], [9], [11], [12])

$$\phi_t\left(p\right)=\begin{pmatrix}\cosh t & \sinh t & 0\\ \sinh t & \cosh t & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+h\begin{pmatrix}0\\ 0\\ t\end{pmatrix}\text{ if } l \text{ is spacelike,}$$

$$\phi_t\left(p\right)=\begin{pmatrix}1 & 0 & 0\\ 0 & \cos t & -\sin t\\ 0 & \sin t & \cos t\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+h\begin{pmatrix}t\\ 0\\ 0\end{pmatrix}\text{ if } l \text{ is timelike,}$$

$$\phi_t\left(p\right)=\begin{pmatrix}1-\frac{t^2}{2} & t & \frac{t^2}{2}\\ -t & 1 & t\\ -\frac{t^2}{2} & t & 1+\frac{t^2}{2}\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}+h\begin{pmatrix}\frac{t}{3}^3-t\\ t^2\\ \frac{t}{3}^3+t\end{pmatrix}\text{ if } l \text{ is null}$$

where in any case $t\in\mathbb{R}$. If we take $h=0$, then we obtain a rotations group related to axis l.

A helicoidal surface in Lorentzian space $\mathbb{R}_1^3$ is a surface invariant by uniparametric group $G_{l,h}$ of helicoidal motion and a helicoidal surface is given by immersion $X:M\to\mathbb{R}_1^3$ by (s,t) parameters where the images by X of the $t-$curves are the trajectories of the helicoidal motions, while the $s-$curves are their orthogonal trajectories parameterized by arc length in the induced metric such a local parameterization will be called natural parameterization of the helicoidal surface. Notice that the first fundamental form in such parameters can be written as $d\theta^2=\varepsilon_1 ds^2+\varepsilon_2 U^2 dt^2$, where $\varepsilon_1=\langle X_s,X_s\rangle,\varepsilon_2 U^2=\langle X_t,X_t\rangle$ and $U=U\left(s\right)$ is a function depends only s. If a helicoidal surface invariant under $G_{l,h}$ meets the axis l at some point and if l is non-null, then the whole axis is contained in the surface. In this paper, we have omitted the case of helicoidal surfaces with null axis.

3 SPACELIKE HELICOIDAL SURFACES AS BONNET SURFACE

Let M be a spacelike surface with unit normal e_3 which is a map from M to H^2_+ and $\{e_1, e_2\}$ are spacelike tangent vector with corresponding coframe $\{\mu_1, \mu_2\}$. Suppose another typical field of orthonormal principal frame on M such that $\{x; e'_1, e'_2, e'_3 = e_3\}$ with the corresponding connection forms $\{\omega_{ij}\}$, $1 \leqslant i, j \leqslant 3$. Also, we can establish a principal coframe $\{\omega_1, \omega_2\}$ corresponding to $\{e'_1, e'_2\}$. Then the first fundamental form of M is

$$I = (\mu_1)^2 + (\mu_2)^2 = (\omega_1)^2 + (\omega_2)^2.$$

There exists a function ϕ on M such that

$$e'_1 = \cos\phi e_1 + \sin\phi e_2, \quad e'_2 = -\sin\phi e_1 + \cos\phi e_2$$

and

$$\omega_1 = \cos\phi\mu_1 + \sin\phi\mu_2, \quad \omega_2 = -\sin\phi\mu_1 + \cos\phi\mu_2. \tag{3.1}$$

Since e'_1, e'_2 are spacelike principal vector fields, $h_{12} = h_{21}$ defined by (2.4) vanish. Also, by denoting the principal curvatures by $h_{11} = k_1$ and $h_{22} = k_2$, we can say

$$\omega_{13} = k_1\omega_1, \quad \omega_{23} = k_2\omega_2. \tag{3.2}$$

Obviously, the mean and Gauss curvature are

$$H = -\frac{k_1 + k_2}{2}, \;\; K = -k_1 k_2,$$

respectively. From now on, we assume $H^2 + K > 0$ and call $J = \sqrt{H^2 + K} = \frac{k_1 - k_2}{2}$, that is, we consider M is free of the umbilic points.

Theorem 3.1 *Let M be a spacelike surface with no umbilic points. Then the Codazzi-Mainardi equations are equal to*

$$dH = H_1\mu_1 + H_2\mu_2,$$
$$d\sigma = \sin\sigma\left(\tfrac{H_1}{J}\mu_1 - \tfrac{H_2}{J}\mu_2\right) - \cos\sigma\left(\tfrac{H_2}{J}\mu_1 + \tfrac{H_1}{J}\mu_2\right) - *d\ln J - 2\mu_{12} \tag{3.3}$$

*where * is the Hodge operator whose action on the 1-form is described by*

$$*\mu_1 = \mu_2, \quad *\mu_2 = -\mu_1, \quad (*)^2 = -1.$$

Proof. The exterior differentiation of the principal coframe (3.1) gives

$$d\omega_1 = \omega_2 \wedge (-d\phi - \mu_{12}), \quad d\omega_2 = \omega_1 \wedge (d\phi + \mu_{12}).$$

From the structure equations given by (2.2) the connection form associated to the principal coframe is $\omega_{12} = d\phi + \mu_{12}$ so that

$$d\phi = \omega_{12} - \mu_{12}. \tag{3.4}$$

Assume $\omega_{12} = p\omega_1 + q\omega_2$ then p and q are determined uniquely by the structure equations (2.2). Thus, by the aid of the equations (2.2) and (3.2) the Codazzi-Mainardi equations can be reduced to

$$(dk_1 - p\,(k_1 - k_2)\,\omega_2) \wedge \omega_1 = 0, \quad (dk_2 - q\,(k_1 - k_2)\,\omega_1) \wedge \omega_2 = 0.$$

On the other hand, if we set

$$dH = -\frac{dk_1 + dk_2}{2} = u\omega_1 + v\omega_2$$

then we obtain

$$dk_1 = (-2Jq - 2u)\,\omega_1 + 2Jp\omega_2 = 0, \quad dk_2 = 2Jq\omega_1 + (-2v - 2Jp)\,\omega_2 = 0.$$

Dividing $dk_1 - dk_2$ by $2J$ gives us

$$d\ln J = -\frac{u}{J}\omega_1 + \frac{v}{J}\omega_2 + 2\,(-q\omega_1 + p\omega_2)\,.$$

If we apply the Hodge operator to the last equation, we get

$$\omega_{12} = -\tfrac{1}{2J}\,(v\omega_1 + u\omega_2) - \tfrac{1}{2} * d\ln J. \tag{3.5}$$

Since $dH = H_1\mu_1 + H_2\mu_2 = u\omega_1 + v\omega_2$, the relationship between $u,\ v$ and $H_1,\ H_2$ is

$$u = H_1 \cos\phi + H_2 \sin\phi, \qquad v = -H_1 \sin\phi + H_2 \cos\phi.$$

From the last equation and the equations (3.4) and (3.5), we find

$$d2\phi = \sin 2\phi \left(\frac{H_1}{J}\mu_1 - \frac{H_2}{J}\mu_2\right) - \cos 2\phi \left(\frac{H_2}{J}\mu_1 + \frac{H_1}{J}\mu_2\right) - *d\ln J - 2\mu_{12}.$$

Finally, substituting $\sigma = 2\phi$ into the above equation completed the proof. ■

Now, let us establish a principal coframe of a spacelike helicoidal surface with non-null axis such that

$$\mu_1 = ds, \quad \mu_2 = q\,(s)\,dt \tag{3.6}$$

where s is the arc length of curves orthogonal to orbits measured from a fixed orbit and t is time along orbits from a fixed $t = t_0$. The t =constant curves are carried along the orbits by helicoidal motions and foliate the surface. An orthonormal frame $\{e_1, e_2\}$ is determined along these coordinate curves with e_2 tangent to the orbits. From the equations (2.2) and (3.6), it is easily seen that

$$\mu_{12} = \frac{q'\,(s)}{q\,(s)}\mu_2 = \eta\,(s)\,\mu_2.$$

Hence the μ_1-curves are geodesic curve and the μ_2-curves have the geodesic curvature equal to

$$\eta\,(s) = \frac{d}{ds}\ln\left(|q\,(s)|\right).$$

Moreover, $dJ = \frac{dk_1 - dk_2}{2}$, so we can put $dJ = -J_1\mu_1 + J_2\mu_2$ and we obtain

$$d\ln J = -\frac{J_1}{J}\mu_1 + \frac{J_2}{J}\mu_2.$$

Along the each orbit, $k_1,\ k_2,\ \mu$ and ϕ depends on s then $H_2 = J_2 = 0$. Hence the relation (3.3) becomes

$$d\sigma = \sin\sigma\left(\frac{H_1}{J}\right)\mu_1 - \cos\sigma\left(\frac{H_1}{J}\right)\mu_2 + \frac{J_1}{J}\mu_2 - 2\eta\left(s\right)\mu_2.$$

Since $\sigma = \sigma\left(s\right)$, this implies

$$\tfrac{d\sigma}{ds} = \sin\sigma\left(\tfrac{\frac{dH}{ds}}{J}\right) \tag{3.7}$$

and

$$\eta\left(s\right) = \frac{1}{2}\left(\frac{\frac{dJ}{ds}}{J} - \cos 2\phi\frac{\frac{dH}{ds}}{J}\right).$$

Theorem 3.2 *A spacelike helicoidal surface M with $H^2 + K > 0$ in $\mathbb{R}^3_1$ has a one parameter family of non-trivial isometric deformation preserving the mean curvature i.e., M is a spacelike Bonnet surfaces if and only if the following relation is satisfied*

$$\tfrac{d}{ds}\left(\tfrac{\frac{dH}{ds}}{J}\right) + \cos\sigma\left(s\right)\left(\tfrac{\frac{dH}{ds}}{J}\right)^2 + \left(\tfrac{\frac{dH}{ds}}{J}\right)\tfrac{d\ln(|q(s)|)}{ds} = 0 \tag{3.8}$$

where $H = H\left(s\right)$ is the non-constant mean curvature.

Proof. The criterion of Chern given in [13] for the existence of Bonnet surface is $d\alpha_1 = 0$ and $d\alpha_2 = \alpha_1 \wedge \alpha_2$ where $\alpha_1 = \frac{u}{J}\omega_1 - \frac{v}{J}\omega_2,\ \ \alpha_2 = \frac{v}{J}\omega_1 + \frac{u}{J}\omega_2$ and $dH = u\omega_1 + v\omega_2$. By substituting u and v in the equations $\omega_1 = \cos\sigma\left(s\right)ds + \sin\sigma\left(s\right)q\left(s\right)dt,\ \ \omega_2 = -\sin\sigma\left(s\right)ds + \cos\sigma\left(s\right)q\left(s\right)dt$, we get

$$\begin{aligned}\alpha_1 &= \left(\frac{H_1}{J}\cos 2\phi + \frac{H_2}{J}\sin 2\phi\right)ds + \left(\frac{H_1}{J}\sin 2\phi - \frac{H_2}{J}\cos 2\phi\right)q\left(s\right)dt,\\ \alpha_2 &= \left(-\frac{H_1}{J}\sin 2\phi + \frac{H_2}{J}\cos 2\phi\right)ds + \left(\frac{H_1}{J}\cos 2\phi + \frac{H_2}{J}\sin 2\phi\right)q\left(s\right)dt.\end{aligned}$$

By substituting the equations (3.7) into the exterior derivative of these last two equations it is seen that Chern's criterion is verified if and only if the relation (3.8) satisfied. This completes the proof. ∎

From the equation (3.7), it is easily seen that

$$\left(\frac{\frac{dH}{ds}}{J}\right) = \frac{1}{\sin\sigma\left(s\right)}\frac{d\sigma}{ds}.$$

If we differentiate the last equation with respect to s, we find

$$\frac{d}{ds}\left(\frac{\frac{dH}{ds}}{J}\right)=\frac{d}{ds}\left(\frac{d\sigma}{ds}\right)\frac{1}{\sin\sigma\left(s\right)}-\left(\frac{d\sigma}{ds}\right)^{2}\frac{\cos\sigma\left(s\right)}{\sin^{2}\sigma\left(s\right)}.$$

By comparing the last equation with (3.8), we obtain

$$\frac{d}{ds}\left(\frac{d\sigma}{ds}\right)\frac{1}{\sin\sigma\left(s\right)}+\left(\frac{d\sigma}{ds}\right)\frac{1}{\sin\sigma\left(s\right)}\frac{q'\left(s\right)}{q\left(s\right)}=0.$$

The solution of this ordinary differential equation gives us the following remark.

Remark 3.1 *The ordinary differential equation (3.8) is equivalent to*

$$q\left(s\right)\left(\frac{\frac{dH}{ds}}{J}\right)\sin\sigma\left(s\right)=constant$$

with non-constant mean curvature $H=H\left(s\right).$

4 TIMELIKE HELICOIDAL SURFACE AS BONNET SURFACE

Let M be a timelike surface with unit normal e_3 which is a map from M to de Sitter space S_1^2. We can choose a local orthonormal frame field $\{x;e_1,e_2,e_3\}$ on M, such that e_1 is a spacelike tangent vector and e_2 is a timelike tangent vector at x. Obviously, the normal vector e_3 is spacelike at x. If we take into consideration another field of orthonormal principal frame on M such $\{X;e'_1,e'_2,e'_3=e_3\}$ with $\{\omega_1,\omega_2\}$ principal coframe corresponding to $\{e'_1,e'_2\}$, then the first fundamental form of M is

$$I=\left(\mu_1\right)^2-\left(\mu_2\right)^2=\left(\omega_1\right)^2-\left(\omega_2\right)^2. \tag{4.1}$$

The function ϕ exists on M as follows,

$$e'_1=\cosh\phi e_1+\sinh\phi e_2,\ \ e'_2=\sinh\phi e_1+\cosh\phi e_2$$

and

$$\omega_1=\cosh\phi\mu_1+\sinh\phi\mu_2,\ \ \omega_2=\sinh\phi\mu_1+\cosh\phi\mu_2. \tag{4.2}$$

The Weingarten map has real eigen vector if and only if $H^2+K>0$. From now on, we suppose $H^2+K>0$, that is M has no umbilic points. If we define $J=\sqrt{H^2+K}$ then $J=\frac{k_1+k_2}{2}>0$. Since $e'_1,\ e'_2$ are principal vector fields, we can give

$$\omega_{13}=k_1\omega_1,\ \omega_{23}=-k_2\omega_2,\ h_{12}=h_{21}=0$$

where the principal curvatures are $h_{11}=k_1$ and $h_{22}=k_2$ in (2.4). The mean and Gauss curvature of M are

$$H=\frac{k_1-k_2}{2},\ K=-k_1k_2,$$

respectively.

In these regards, we can give the following theorem related to the Codazzi-Mainardi equations for timelike surface.

Theorem 4.1 *The Codazzi-Mainardi equations for timelike surface with $H^2 + K > 0$ are*

$$dH = H_1\mu_1 - H_2\mu_2$$

and

$$d\sigma = -\sinh\sigma\left(\tfrac{H_1}{J}\mu_1 + \tfrac{H_2}{J}\mu_2\right) - \cosh\sigma\left(\tfrac{H_2}{J}\mu_1 + \tfrac{H_1}{J}\mu_2\right) - *d\ln J + 2\mu_{12} \tag{4.3}$$

*where * is the Hodge operator such that*

$$*\mu_1 = -\mu_2,\ *\mu_2 = \mu_1,\ (*)^2 = 1.$$

Proof. From (4.2), we obtain

$$d\omega_1 = -\omega_2 \wedge (d\phi - \mu_{12}),\ d\omega_2 = \omega_1 \wedge (-d\phi + \mu_{12}).$$

Thus, the connection form associated to the principal coframe is $\omega_{12} = -d\phi + \mu_{12}$. This implies

$$d\phi = -\omega_{12} + \mu_{12}. \tag{4.4}$$

From the equations (2.2) the Codazzi-Mainardi equations for M are

$$d\omega_{13} = -\omega_{12} \wedge \omega_{23},\ d\omega_{23} = \omega_{21} \wedge \omega_{12}$$

and they can be reduced to

$$(dk_1 + p(k_1 + k_2)\omega_2) \wedge \omega_1 = 0, \quad (-dk_2 - q(k_1 + k_2)\omega_1) \wedge \omega_2 = 0$$

where $\omega_{12} = p\omega_1 + q\omega_2$. Considering $\frac{dk_1 - dk_2}{2} = u\omega_1 - v\omega_2$, we get

$$dk_1 = (2u - 2Jq)\omega_1 - 2Jp\omega_2 = 0, \quad dk_2 = -2Jq\omega_1 + (2v - 2Jp)\omega_2 = 0.$$

If we divide the addition of the last two relation by $2J$, we obtain

$$d\ln J = \frac{u}{J}\omega_1 + \frac{v}{J}\omega_2 + 2(-q\omega_1 - p\omega_2).$$

By use of Hodge operator, we have

$$\omega_{12} = \tfrac{1}{2J}(v\omega_1 + u\omega_2) + \tfrac{1}{2} * d\ln J. \tag{4.5}$$

If we compare $dH = u\omega_1 - v\omega_2$ and $dH = H_1\mu_1 - H_2\mu_2$ with $\omega_1,\ \omega_2$ expressed in terms of μ_1 and μ_2, we see

$$u = H_1\cosh\phi + H_2\sinh\phi,\ v = H_1\sinh\phi + H_2\cosh\phi.$$

The equation (4.4), (4.5) and the last equations gives us

$$d2\phi = -\sinh 2\phi \left(\frac{H_1}{J}\mu_1 + \frac{H_2}{J}\mu_2 \right) - \cosh 2\phi \left(\frac{H_2}{J}\mu_1 + \frac{H_1}{J}\mu_2 \right) - *d\ln J + 2\mu_{12}.$$

If we take $\sigma = 2\phi$, then we complete the proof. ∎

For a timelike helicoidal surface with non-null axis, there is the relation

$$\mu_{12} = \frac{q'(s)}{q(s)}\mu_2 = \eta(s)\mu_2$$

where the orthonormal corresponding coframe is defined by

$$\mu_1 = ds,\ \mu_2 = q(s)\,dt.$$

Since $dJ = \frac{dk_1 + dk_2}{2}$, we can consider $dJ = -J_1\mu_1 - J_2\mu_2$, then

$$*d\ln J = \frac{J_2}{J}\mu_1 + \frac{J_1}{J}\mu_2.$$

By the fact that, $k_1,\ k_2,\ \mu$ and ϕ depends on s we can say that $H_2 = J_2 = 0$. So we can rewrite the equation (4.3) as follows,

$$d\sigma = -\sinh\sigma \left(\frac{H_1}{J} \right)\mu_1 - \cosh\sigma \left(\frac{H_1}{J} \right)\mu_2 - \frac{J_1}{J}\mu_2 - 2\eta(s)\mu_2.$$

From last equation it is obtained that

$$\frac{d\sigma}{ds} = -\sinh 2\phi \left(\frac{\frac{dH}{ds}}{J} \right)$$

and

$$\eta(s) = \frac{1}{2}\left(\cosh 2\phi \frac{\frac{dH}{ds}}{J} + \frac{\frac{dJ}{ds}}{J} \right).$$

Theorem 4.2 *Let M be a timelike helicoidal with no umbilic point in $\mathbb{R}^3_1$. Then M is a Bonnet surface if and only if there exists a relation as follows*

$$\frac{d}{ds}\left(\frac{\frac{dH}{ds}}{J} \right) - \cosh\sigma(s) \left(\frac{\frac{dH}{ds}}{J} \right)^2 + \left(\frac{\frac{dH}{ds}}{J} \right) \frac{d\ln(|q(s)|)}{ds} = 0 \qquad (4.6)$$

where H is a non-constant mean curvature.

Proof. In [15], it is proved that every timelike constant mean curvature surface with no umbilic points is a timelike Bonnet surface and it is mentioned that M with non-constant mean curvature is timelike Bonnet surface if and only if

$$d\alpha_1 = 0,\ d\alpha_2 = \alpha_1 \wedge \alpha_2$$

where$\alpha_1 = \frac{u}{J}\omega_1 + \frac{v}{J}\omega_2,\ \alpha_2 = \frac{v}{J}\omega_1 + \frac{u}{J}\omega_2$ and $dH = u\omega_1 - v\omega_2$. If we substitute the relations

$$\omega_1 = \cosh\sigma(s)\,ds + \sinh\sigma(s)\,q(s)\,dt,\ \ \omega_2 = \sinh\sigma(s)\,ds + \cosh\sigma(s)\,q(s)\,dt$$

and (4.1) into the exterior derivative of α_1 and α_2, then we verify the necessary and sufficient condition of timelike helicoidal surface being timelike Bonnet surface. The solution of the differential equation (4.6) can be obtained by similar manner. ■

Remark 4.1 *The ordinary differential equation (4.6) is equivalent to*

$$q(s)\left(\frac{\frac{dH}{ds}}{J}\right)\sinh 2\phi = constant$$

where H is non-constant.

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