Characterization and structure of second-order symmetric Lorentzian manifolds

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- **Simple examples of rth-symmetric Lorentzian spaces (within the class of plane waves) :**

  $$g_A = -2du\left(dv + \sum_{i,j=2}^{n-1}\left(\mathbf{a_{ij}u^{r-1}} + \ldots + \mathbf{b_{ij}u} + \mathbf{c_{ij}}\right)x^i x^j du\right) + \sum_{i,j=2}^{n-1}\delta_{ij}dx^i dx^j$$

Purpose of this talk

- **to determine** the *Lorentzian 2nd-order symmetric spaces of arbitrary dimension*, i.e.,

$$\textit{Lorentzian manifolds with } \nabla\nabla R = 0$$

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     - Preliminary Characterization
     - Local and Global Structure theorems
  2. **Local study of Brinkmann spaces**
     - Motivation
     - Geometric developments
  3. **Steps for the proof** of the Local Structure Theorem

Main results.

Preliminary characterization *[OFB, Sánchez M, Senovilla J M M, 2011]*

The following statements are equivalent in a n-dimensional Lorentzian manifold (M, g):

(I) (M, g) is 2**nd-symmetric** $(\nabla\nabla R = 0)$.

(II) X, Y, Z, V are parallelly propagated vector fields along any curve $\alpha \Longrightarrow (\nabla_V R)(X, Y)Z$ is parallelly propagated along α.

(III) (M, g) is semi-symmetric ($R_{XY} R = 0$, or equiv., $\nabla_X(\nabla_Y R) - \nabla_{\nabla_X Y} R$ is symmetric in X, Y) and satisfies the equivalent conditions in the auxiliary Lemma below.

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Auxiliary Lemma

The following conditions are equivalent in a n-dimensional Lorentzian manifold :

(I) $\nabla_X(\nabla_Y R) - \nabla_{\nabla_X Y}R$ is skew-symmetric in X, Y.

(II) Let X, Y, Z **be parallelly propagated vector fields** along some **geodesic** γ (γ' its tangent vector field). Then, the vector field $(\nabla_{\gamma'} R)(X, Y)Z$ is itself parallelly propagated along γ.

(III) Let Π **be a non-degenerate tangent plane and** $K(\Pi)$ **its sectional curvature**. Then, the parallel transport Π_γ of Π along any **geodesic** γ (γ' its tangent vector field) satisfies that $\nabla_{\gamma'}(K(\Pi_\gamma))$ remains constant along γ.

Main results.

Structure Theorem *[OFB, Sánchez M, Senovilla J M M, 2011]*

Let (M,g) be a 2nd-symmetric non-symmetric n-dimensional Lorentzian manifold.

LOCAL CLASSIFICATION:

(M,g) is locally isometric to $(\mathbb{R}^{d+2}\times N, g_A \oplus g_N)$, where:

- $g_A = -2du\left(dv + \sum_{i,j=2}^{n-1}(a_{ij}u + b_{ij})x^i x^j du\right) + \sum_{i,j=2}^{n-1}\delta_{ij}dx^i dx^j$ and at least one of $a_{ij} \neq 0$, and
- (N, g_N) is a symmetric (non-flat) Riemannian space.

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GLOBAL CLASSIFICATION:

If (M, g) is geodesically complete and simply connected, then (M, g) is globally isometric to $(\mathbb{R}^{d+2} \times N, g_A \oplus g_N)$.

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Elements of the Proof (local result):

- exploit the implicit symmetries of the equations of 2nd-symmetry,
- develop different technical tools for some classes of PDE, and
- transform the problem into two: one elementary, the other reducible to the symmetric case

Local study of Brinkmann spaces

Motivation

For arbitrary dimension, when $\nabla\nabla R = 0 \not\Longrightarrow \nabla R = 0$?

THEOREM *[JMM Senovilla, 2008]*

Let $D \subseteq M$ be a simply-connected domain of an n-dimensional 2nd-symmetric Lorentzian manifold (M, g). Then, **if there is NO parallel lightlike vector field** on D, (D, g) is in fact *locally symmetric* (i.e., $\nabla R = 0$).

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Therefore:

a 2-symmetric non-locally symmetric Lorentzian manifold must have a parallel lightlike vector field and locally, it is a Brinkmann space

$$ds^2 = -2du(dv + Hdu + \sum_{i,j=2}^{n-1} W_i dx^i) + \sum_{i,j=2}^{n-1} g_{ij} dx^i dx^j, \quad i, j \in \{2, \ldots, n-1\}$$

where H, W_i and $g_{ij} = g_{ji}$ are functions independent of v, otherwise arbitrary.

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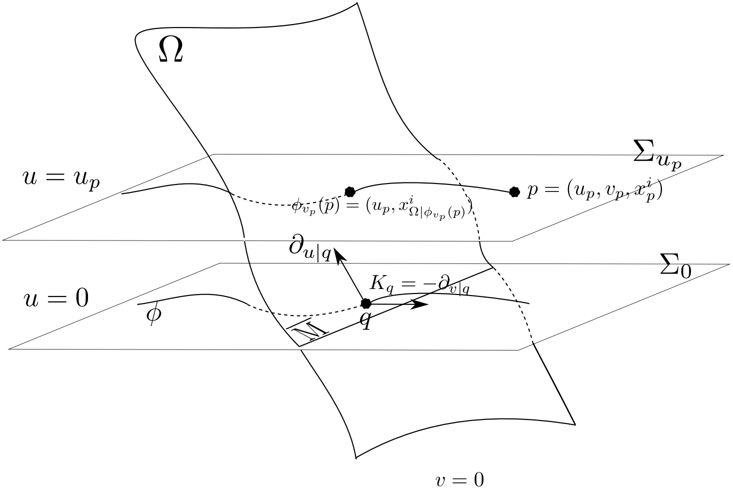
Task: write the equations of 2nd-symmetry ($\nabla_\rho \nabla_\sigma R^\alpha{}_{\beta\lambda\mu} = 0$ in an appropriate basis) **for Brinkmann spaces** in a manageable way and solve them.

Local study of Brinkmann spaces

Geometric developments

- **Metric**:
  $g = -2du\left(dv + H(u,x^k)du + \sum_{i,j=2}^{n-1} W_i(u,x^k)dx^i\right) + \sum_{i,j=2}^{n-1} g_{ij}(u,x^k)dx^i dx^j$
- Spacelike $(n-2)$-**foliation** $\overline{\mathcal{M}}$: $\{u = u_0, v = v_0\}$



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$$\bar{\ } : \quad \Gamma(T^r_s M) \longrightarrow \Gamma(T^r_s \overline{\mathcal{M}}) \quad \Big| \quad \mathring{\ } : \quad \Gamma(T^r_s \overline{\mathcal{M}}) \longrightarrow \Gamma(T^r_s M)$$

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  1. **The (symmetric) covariant derivative** $\overline{\nabla}$ on $\overline{\mathcal{M}}$
     $$\begin{array}{rccc} \overline{\nabla} : & \Gamma(T\overline{\mathcal{M}}) \times \Gamma(T\overline{\mathcal{M}}) & \longrightarrow & \Gamma(T\overline{\mathcal{M}}) \\ & (X, Y) & \rightarrow & \overline{\nabla}_X Y \end{array}$$
  2. **The transverse tensor derivation** D_0 on $\overline{\mathcal{M}}$
     $$\begin{array}{rccc} D_0 : & \Gamma(T^r_s \overline{\mathcal{M}}) & \longrightarrow & \Gamma(T^r_s \overline{\mathcal{M}}) \\ & T & \rightarrow & D_0 T = \overline{(\nabla_{(\partial_u - H\partial_v)} \mathring{T})} \end{array}$$

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- The **curvature tensor** $\overline{\mathcal{R}}$ of the foliation $\overline{\mathcal{M}}$

$$\overline{\mathcal{R}}(X,Y)Z = (\overline{\nabla}_X\overline{\nabla}_Y - \overline{\nabla}_Y\overline{\nabla}_X - \overline{\nabla}_{[X,Y]})Z \in \Gamma(T\overline{\mathcal{M}}),\ \forall X,Y,Z \in \Gamma(T\overline{\mathcal{M}})$$

and its derived **Ricci tensor** $\overline{\mathcal{R}ic}$ and **scalar curvature** $\overline{\mathcal{S}}$.

Definitions

- $\overline{\mathcal{M}}$ is **flat** (respectively, **locally symmetric**) if $\overline{\mathcal{R}} = 0$ (resp. $\overline{\nabla}\,\overline{\mathcal{R}} = 0$)
- $\overline{\mathcal{M}}$ is *u*-**Einstein** if $\overline{\mathcal{R}\mathrm{ic}} = \mu \overline{g}$ for some function μ such that $d\mu \wedge du = 0$.
  1. If $\mu = const.$, $\overline{\mathcal{M}}$ is **Einstein**
  2. If $\mu \equiv 0$, $\overline{\mathcal{M}}$ is **Ricci-flat**.

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Proposition

Let (M, g) be a Brinkmann space with a spatial foliation $\overline{\mathcal{M}}$. Then,

1. $\overline{\nabla}^r\overline{\mathcal{R}} = 0$ (**rth-symmetric**)$\Longrightarrow \overline{\nabla}\,\overline{\mathcal{R}} = 0$ (**locally symmetric**).
2. $\overline{\nabla}\,\overline{\mathcal{R}} = 0$(**locally symmetric**) and $\overline{\mathcal{R}ic} = 0$ (**Ricci-flat**) $\Longrightarrow \overline{\mathcal{R}} = 0$ (**flat**)
3. If $\overline{\mathcal{M}}$ is **flat**, the Brinkmann space admits a chart $\{u, v, y^i\}$ such that the metric g becomes:

$$g = -2du(dv + Hdu + \sum_{i=2}^{n-1} W_i dy^i) + \sum_{i,j=2}^{n-1} \delta_{ij} dy^i dy^j.$$

Classical version of Eisenhart theorem

If a Riemannian manifold (N, g_R) admits a symmetric two-covariant tensor field $L \in \Gamma(T^0_2 N)$ not proportional to the metric g_R such that $\nabla^{g_R} L = 0$, then

- g_R is reducible: $g_R = g_R^{(1)} \oplus g_R^{(2)} \oplus \ldots \oplus g_R^{(s)}$ (with each $g_R^{(m)}$ not necessarily irreducible).
- $L = \sum_{m=1}^{s} \lambda_m \, g_R^{(m)}$ for some constants λ_m.

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Generalized Eisenhart theorem *[OFB, Sánchez M, Senovilla J M M, 2011]*

Let (M, g) be a Brinkmann space and fix a Brinkmann chart $\{u, v, x^i\}$. Assume that there exists a symmetric section $\overline{L} \in \Gamma(T_2^0 \overline{\mathcal{M}})$ not proportional to $\overline{g}$ which is:

- invariant under the flow of ∂_v, $\overline{\nabla}$-parallel and D_0-parallel,

then, the decomposition $\{u, v\}$ admits a Brinkmann chart $\{u, v, y^i\}$ such that:

1. $\overline{g}$ is reducible: $\overline{g} = (\sum_{i,j=2}^{n-1} g_{ij}(u, x^k) dx^i dx^j =)\ \overline{g}^{(1)} \oplus \ldots \oplus \overline{g}^{(s)}$ for some $s \geq 2$.
2. $\overline{L} = \sum_{m=1}^{s} \lambda_m \, \overline{g}^{(m)}$ for some constants $\lambda_m \in \mathbb{R}$.

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- Geometric elements for R: $A \in T_2\overline{\mathcal{M}}$, $B \in T_3\overline{\mathcal{M}}$, $\overline{R} \in T_3^1\overline{\mathcal{M}}$

$$A_{ij} = R^1{}_{i0j}; \quad B_{ijk} = R^1{}_{ijk}; \quad \overline{\mathcal{R}}^i{}_{jkl} = R^i{}_{jkl}$$

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- Geometric elements for ∇R: $\widetilde{A} \in T_2\overline{\mathcal{M}}$, $\widehat{A}, \widetilde{B} \in T_3\overline{\mathcal{M}}$, $\widehat{B}, \widetilde{R} \in T_3^1\overline{\mathcal{M}}$

$$\widetilde{A}_{ij} = \nabla_0 R^1{}_{i0j}; \ \widehat{A}_{sij} = \nabla_s R^1{}_{i0j}; \ \widetilde{B}_{ijk} = \nabla_0 R^1{}_{ijk}; \ \widehat{B}_{sijk} = \nabla_s R^1{}_{ijk}; \ \widetilde{R}^i{}_{jkl} = \nabla_0 R^i{}_{jkl}$$

- Simplified expressions for $\nabla\nabla R$: with $t = -\frac{1}{2}(\dot{\overline{g}} + \overline{d}W) \in T_2\overline{\mathcal{M}}$ and $h = \overline{d}H - \dot{W} \in T_1\overline{\mathcal{M}}$.

$$\nabla_s\nabla_0 R^i{}_{jkl} = \overline{\nabla}_s \widetilde{R}^i{}_{jkl} + \sum_{r=2}^{n-1} t^r{}_s \overline{\nabla}_r \overline{\mathcal{R}}^i{}_{jkl}, \qquad \nabla_0\nabla_0 R^i{}_{jkl} = D_0 \widetilde{R}^i{}_{jkl} - \sum_{r=2}^{n-1} h^r \overline{\nabla}_r \overline{\mathcal{R}}^i{}_{jkl},$$

$$\nabla_n\nabla_s R^1{}_{ijk} = \overline{\nabla}_n \widehat{B}_{sijk} - \sum_{r=2}^{n-1} t_{rn} \overline{\nabla}_s \overline{\mathcal{R}}^r{}_{ijk}, \qquad \nabla_0\nabla_s R^1{}_{ijk} = D_0 \widehat{B}_{sijk} + \sum_{r=2}^{n-1} h_r \overline{\nabla}_s \overline{\mathcal{R}}^r{}_{ijk},$$

$$\nabla_s\nabla_0 R^1{}_{ijk} = \overline{\nabla}_s \widetilde{B}_{ijk} - \sum_{r=2}^{n-1} t^r{}_s (\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \qquad \nabla_0\nabla_0 R^1{}_{ijk} = D_0 \widetilde{B}_{ijk} + \sum_{r=2}^{n-1} h^r (\widetilde{R}_{rijk} - \widehat{B}_{rijk}),$$

$$\nabla_k\nabla_s R^1{}_{i0j} = \overline{\nabla}_k \widehat{A}_{sij} - 2\sum_{r=2}^{n-1} t^r{}_k \widehat{B}_{s(ij)r}, \qquad \nabla_0\nabla_s R^1{}_{i0j} = D_0 \widehat{A}_{sij} + 2\sum_{r=2}^{n-1} h^r \widehat{B}_{s(ij)r},$$

$$\nabla_k\nabla_0 R^1{}_{i0j} = \overline{\nabla}_k \widetilde{A}_{ij} - \sum_{r=2}^{n-1} t^r{}_k (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}), \qquad \nabla_0\nabla_0 R^1{}_{i0j} = D_0 \widetilde{A}_{ij} + \sum_{r=2}^{n-1} h^r (2\widetilde{B}_{(ij)r} - \widehat{A}_{rij}),$$

$$\nabla_m\nabla_s R^i{}_{jkl} = \overline{\nabla}_m \overline{\nabla}_s \overline{\mathcal{R}}^i{}_{jkl}, \qquad \nabla_0\nabla_s R^i{}_{jkl} = D_0 \overline{\nabla}_s \overline{\mathcal{R}}^i{}_{jkl}$$

Steps for the proof

Step 2. Introduce the 2nd-symmetry condition to simplify the given geometric elements.

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$$\nabla_s\nabla_0 R^1{}_{ijk} = \overline{\nabla}_s\widetilde{B}_{ijk} - \sum_{r=2}^{n-1} t^r{}_s(\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \qquad \nabla_0\nabla_0 R^1{}_{ijk} = D_0\widetilde{B}_{ijk} + \sum_{r=2}^{n-1} h^r(\widetilde{R}_{rijk} - \widehat{B}_{rijk}),$$

$$\nabla_k\nabla_s R^1{}_{i0j} = \overline{\nabla}_k\widehat{A}_{sij} - 2\sum_{r=2}^{n-1} t^r{}_k\widehat{B}_{s(ij)r}, \qquad \nabla_0\nabla_s R^1{}_{i0j} = D_0\widehat{A}_{sij} + 2\sum_{r=2}^{n-1} h^r\widehat{B}_{s(ij)r},$$

$$\nabla_k\nabla_0 R^1{}_{i0j} = \overline{\nabla}_k\widetilde{A}_{ij}\ , \qquad \nabla_0\nabla_0 R^1{}_{i0j} = D_0\widetilde{A}_{ij}\ ,$$

$$\nabla_m\nabla_s R^i{}_{jkl} = \overline{\nabla}_m\overline{\nabla}_s\overline{\mathcal{R}}^i{}_{jkl}, \qquad \nabla_0\nabla_s R^i{}_{jkl} = D_0\overline{\nabla}_s\overline{\mathcal{R}}^i{}_{jkl}.$$

Steps for the proof

Step 2. Introduce the 2nd-symmetry condition to simplify the given geometric elements.

The non-vanishing components of $\nabla\nabla R$ up to now:

$$\nabla_s\nabla_0 R^i{}_{jkl} = \overline{\nabla}_s\widetilde{R}^i{}_{jkl} + \sum_{r=2}^{n-1} t^r{}_s\overline{\nabla}_r\overline{\mathcal{R}}^i{}_{jkl}, \qquad \nabla_0\nabla_0 R^i{}_{jkl} = D_0\widetilde{R}^i{}_{jkl} - \sum_{r=2}^{n-1} h^r\overline{\nabla}_r\overline{\mathcal{R}}^i{}_{jkl},$$

$$\nabla_n\nabla_s R^1{}_{ijk} = \overline{\nabla}_n\widehat{B}_{sijk} - \sum_{r=2}^{n-1} t_{rn}\overline{\nabla}_s\overline{\mathcal{R}}^r{}_{ijk}, \qquad \nabla_0\nabla_s R^1{}_{ijk} = D_0\widehat{B}_{sijk} + \sum_{r=2}^{n-1} h_r\overline{\nabla}_s\overline{\mathcal{R}}^r{}_{ijk},$$

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$$\nabla_k\nabla_s R^1{}_{i0j} = \overline{\nabla}_k\widehat{A}_{sij} - 2\sum_{r=2}^{n-1} t^r{}_k\widehat{B}_{s(ij)r}, \qquad \nabla_0\nabla_s R^1{}_{i0j} = D_0\widehat{A}_{sij} + 2\sum_{r=2}^{n-1} h^r\widehat{B}_{s(ij)r},$$

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1. **First simplification:** $\nabla_m\nabla_s R^i{}_{jkl} = \overline{\nabla}_m\overline{\nabla}_s\overline{\mathcal{R}}^i{}_{jkl} = 0 \Longrightarrow \overline{\nabla}_s\overline{\mathcal{R}}^i{}_{jkl} = 0$

$\overline{\nabla}^r\overline{\mathcal{R}} = 0$ (**rth-symmetric**)$\Longrightarrow \overline{\nabla}\,\overline{\mathcal{R}} = 0$ (**locally symmetric**).

Steps for the proof

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The non-vanishing components of $\nabla\nabla R$ up to now:

$$\nabla_s\nabla_0 R^i{}_{jkl} = \overline{\nabla}_s\widetilde{R}^i{}_{jkl} \cancel{+ \sum_{r=2}^{n-1} t^r{}_s\overline{\nabla}_r\overline{\mathcal{R}}^i{}_{jkl}}, \qquad \nabla_0\nabla_0 R^i{}_{jkl} = D_0\widetilde{R}^i{}_{jkl} \cancel{- \sum_{r=2}^{n-1} h^r\overline{\nabla}_r\overline{\mathcal{R}}^i{}_{jkl}},$$

$$\nabla_n\nabla_s R^1{}_{ijk} = \overline{\nabla}_n\widehat{B}_{sijk} \cancel{- \sum_{r=2}^{n-1} t_{rn}\overline{\nabla}_s\overline{\mathcal{R}}^r{}_{ijk}}, \qquad \nabla_0\nabla_s R^1{}_{ijk} = D_0\widehat{B}_{sijk} \cancel{+ \sum_{r=2}^{n-1} h_r\overline{\nabla}_s\overline{\mathcal{R}}^r{}_{ijk}},$$

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$$\nabla_k\nabla_s R^1{}_{i0j} = \overline{\nabla}_k\widehat{A}_{sij} - 2\sum_{r=2}^{n-1} t^r{}_k\widehat{B}_{s(ij)r}, \qquad \nabla_0\nabla_s R^1{}_{i0j} = D_0\widehat{A}_{sij} + 2\sum_{r=2}^{n-1} h^r\widehat{B}_{s(ij)r},$$

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$$\nabla_m\nabla_s R^i{}_{jkl} = 0, \qquad \nabla_0\nabla_s R^i{}_{jkl} = 0.$$

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$$\nabla_n\nabla_s R^1{}_{ijk} = \overline{\nabla}_n \widehat{B}_{sijk}, \qquad \nabla_0\nabla_s R^1{}_{ijk} = D_0 \widehat{B}_{sijk},$$

$$\nabla_s\nabla_0 R^1{}_{ijk} = \overline{\nabla}_s \widetilde{B}_{ijk} - \sum_{r=2}^{n-1} t^r{}_s(\widetilde{R}_{rijk} - \widehat{B}_{rijk}), \qquad \nabla_0\nabla_0 R^1{}_{ijk} = D_0 \widetilde{B}_{ijk} + \sum_{r=2}^{n-1} h^r(\widetilde{R}_{rijk} - \widehat{B}_{rijk}),$$

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2. **Second simplification:** **all the geometric elements for ∇R except $\widetilde{A}$ vanish, i.e,** $\widehat{B} = 0$; $\widetilde{B} = 0$; $\widehat{A} = 0$; $\widetilde{R} = D_0\overline{\mathcal{R}} = 0$.

Steps for the proof

Step 2. Introduce the 2nd-symmetry condition to simplify the given geometric elements.

The non-vanishing components of $\nabla\nabla R$ up to now:

$$\nabla_s \nabla_0 R^i{}_{jkl} = \cancel{\overline{\nabla}_s \widetilde{R}^i{}_{jkl}}, \qquad \nabla_0 \nabla_0 R^i{}_{jkl} = \cancel{D_0 \widetilde{R}^i{}_{jkl}},$$

$$\nabla_n \nabla_s R^1{}_{ijk} = \cancel{\overline{\nabla}_n \widehat{B}_{sijk}}, \qquad \nabla_0 \nabla_s R^1{}_{ijk} = \cancel{D_0 \widehat{B}_{sijk}},$$

$$\nabla_s \nabla_0 R^1{}_{ijk} = \cancel{\overline{\nabla}_s \widetilde{B}_{ijk} - \sum_{r=2}^{n-1} t^r{}_s (\widetilde{R}_{rijk} - \widehat{B}_{rijk})}, \qquad \nabla_0 \nabla_0 R^1{}_{ijk} = \cancel{D_0 \widetilde{B}_{ijk} + \sum_{r=2}^{n-1} h^r (\widetilde{R}_{rijk} - \widehat{B}_{rijk})},$$

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   Proof. Exploit the integrability conditions and combine them with technical algebraic results as the following:

Technical algebraic result

Let $\mathcal{V}$ be a l-dimensional vector space with a positive definite inner product and T_{ijk} a (0,3)-tensor such that:

(A) it is skew-symmetric in the last two indices: $T_{ijk} = -T_{ikj}$,

(B) it satisfies a cyclic identity: $T_{ijk} + T_{jki} + T_{kij} = 0$

If $T_{(ij)k}$ satisfies that $\sum_{r=2}^{n-1} T_{(ij)}{}^{r}\, T_{rnm} = 0$, then $T_{ijk} = 0$.

Steps for the proof

Step 2. Introduce the 2nd-symmetry condition to simplify the given geometric elements.

The non-vanishing components of $\nabla\nabla R$ up to now:

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Consequently, the expressions for the components of $\nabla\nabla R$ are also simplified...

Steps for the proof

Step 2. Introduce the 2nd-symmetry condition to simplify the given geometric elements.

The non-vanishing components of $\nabla\nabla R$ up to now:

$$\nabla_k \nabla_0 R^1{}_{i0j} = \overline{\nabla}_k \widetilde{A}_{ij} \; , \qquad \nabla_0 \nabla_0 R^1{}_{i0j} = D_0 \widetilde{A}_{ij}$$

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Consequently, the expressions for the components of $\nabla\nabla R$ are also simplified...

...and the equations of 2nd-symmetry are simplified to:

$$\boxed{\begin{array}{ll} \overline{\nabla}\widetilde{A} = 0, & D_0 \widetilde{A} = 0 \\ \overline{\nabla}\; \overline{\mathcal{R}} = 0, & D_0 \overline{\mathcal{R}} = 0 \\ \widehat{B} = 0;\; \widetilde{B} = 0;\; \widehat{A} = 0 & \end{array}}$$

Steps for the proof

Step 3. Reduce the equations into two simpler and solvable parts.

Equations of 2nd-symmetry:

$$\overline{\nabla}\,\widetilde{A}=0,\quad D_0\widetilde{A}=0\quad \widehat{B}=0;\ \widetilde{B}=0;\ \widehat{A}=0$$
$$\overline{\nabla}\,\overline{\mathcal{R}}=0,\quad D_0\overline{\mathcal{R}}=0\quad (\Longrightarrow \overline{\nabla}\,\overline{\mathcal{R}ic}=0;\quad D_0\overline{\mathcal{R}ic}=0)$$

Generalized Eisenhart theorem

If there exists a symmetric section $\overline{L}\in\Gamma(T^0_2\overline{\mathcal{M}})$ not proportional to $\overline{g}$ which is invariant under the flow of ∂_v, $\overline{\nabla}$-parallel and D_0-parallel, then, there exists a chart $\{u,v,y^i\}$ such that:

1. $\overline{g}$ is reducible: $\overline{g} =(\sum_{i,j=2}^{n-1} g_{ij}(u,x^k)dx^i dx^j =)\ \overline{g}^{(1)}\oplus\ldots\oplus\overline{g}^{(s)}$ for some $s\geq 2$.
2. $\overline{L}=\sum_{m=1}^{s}\lambda_m\,\overline{g}^{(m)}$ for some **constants** $\lambda_m\in\mathbb{R}$.

Steps for the proof

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Equations of 2nd-symmetry:

$$\begin{array}{lll} \overline{\nabla}\, \widetilde{A} = 0, & D_0 \widetilde{A} = 0 & \widehat{B} = 0;\ \widetilde{B} = 0;\ \widehat{A} = 0 \\ \overline{\nabla}\, \overline{\mathcal{R}} = 0, & D_0 \overline{\mathcal{R}} = 0 & (\Longrightarrow \overline{\nabla}\, \overline{\mathcal{R}ic} = 0;\quad D_0 \overline{\mathcal{R}ic} = 0) \end{array}$$

Generalized Eisenhart theorem

If there exists a symmetric section $\overline{L} \in \Gamma(T^0_2 \overline{\mathcal{M}})$ not proportional to $\overline{g}$ which is invariant under the flow of ∂_v, $\overline{\nabla}$-parallel and D_0-parallel, then, there exists a chart $\{u, v, y^i\}$ such that:

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- **Applying this Generalized Eisenhart theorem to $\widetilde{A}$ and $\overline{\mathcal{R}ic}$:** the equations are reorganized into **two Lorentzian problems**:
  1. A locally symmetric Brinkmann space with a non Ricci-flat foliation, i.e., equations for *locally symmetric Lorentzian manifolds with a parallel lightlike vector field* $\longrightarrow$ known solution (Cahen-Wallach)
  2. A Brinkmann space with a Ricci-flat $\Longrightarrow$ Flat foliation with equations for *2nd-symmetric plane waves* $\longrightarrow$ solvable, giving the simple example presented before:

$$g_A = -2du\left(dv + \sum_{i,j=2}^{n-1} (a_{ij}u + b_{ij}) x^i x^j du\right) + \sum_{i,j=2}^{n-1} \delta_{ij} dx^i dx^j$$

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