Conformally flat homogeneous Lorentzian manifolds

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Our Problem

Classify conformally flat homogeneous semi-Riemannian manifolds

a semi-Riemannian manifold (M^n_q, g) is **conformally flat**
$\Leftrightarrow$ $\forall$ $p \in M$, $\exists$ $(V, x_1, \cdots, x_n)$ of p , a C^∞ ft $\rho > 0$ s. t.

$$g = \rho^2(-dx_1^2 - \cdots - dx_q^2 + dx_{q+1}^2 + \cdots + dx_n^2)$$

a semi-Riemannian manifold (M^n_q, g) is **homogeneous**
$\Leftrightarrow$ $\forall$ $p, p' \in M$, $\exists$ an isometry ϕ of M s.t. $\phi(p) = p'$.

Known results

∘ the Riemanninan case: H.Takagi 1975

(1)$M^n(k)$,

(2)$M^m(k) \times M^{n-m}(-k), \ \ k \neq 0, 2 \leq m \leq n-2,$

(3)$M^{n-1}(k) \times \mathbb{R}, \ \ k \neq 0,$

$M^m(k)$ the simply connected complete Riemannian manifold of constant curvature k

∘ 3-dim Lorentzian manifolds : Honda and Tsukada 2007

the examples which are not symmetric spaces

Characetrizations of conformally flatness

M_q^n : an $n(\geq 4)$-dim semi-Riem mfd with a metric g of index q.

The following conditions are equivalent:

(1) M is conformally flat

(2) the curvature tensor R of M satisfies the following:

$$R(X,Y) = AX \wedge Y + X \wedge AY, \quad A = \frac{1}{n-2}\left(Q - \frac{S}{2(n-1)} Id\right),$$

where Q :the Ricci operator, S : the scalar curvature

(3) $\exists$ an isometric immersion of M into $\Lambda \subset \mathbb{R}^{n+2}_{q+1}$,

where $\Lambda = \{\mathsf{x} \in \mathbb{R}^{n+2}_{q+1} - \{0\} | \langle \mathsf{x}, \mathsf{x} \rangle = 0\}$: the light cone in $\mathbb{R}^{n+2}_{q+1}$

The key of our approach :

to determine the forms of the operator A

We assume that $\boldsymbol{M}$ is a homogeneous semi-Riemannian manifold. Then the —possibly complex— eigenvalues of $\boldsymbol{A}$ and their algebraic multiplicities are constant on $\boldsymbol{M}$

We have the useful identity of the eigenvalues of $\boldsymbol{A}$.

Theorem 1 M_q^n : a conformally flat homogeneous semi-Riemannian manifold

$\lambda_1, \cdots, \lambda_r$: the distinct eigenvalues of the operator A on M

$m_1, \cdots, m_r$: their algebraic multiplicities.

If for some $i \in \{1, \cdots, r\}$, the eigenvalue λ_i is real and <u>the dimension of its eigenspace coincides with its algebraic multiplicity</u>, then we have

$$\sum_{j \neq i} m_j \frac{\lambda_j + \lambda_i}{\lambda_j - \lambda_i} = 0.$$

The case that A is diagonalizable with real eigenvalues

In this case A has at most two distinct eigenvalues.

Theorem 2 M_q^n : an n-dimensional conformally flat homogeneous semi-Riemannian manifold
If the operator A is diagonalizable with real eigenvalues, then M_q^n is locally isometric to one of the following:
(1)$M_q^n(k)$,
(2)$M_{q'}^m(k) \times M_{q-q'}^{n-m}(-k), k \neq 0, 2 \leq m \leq n-2$,
(3)$M_q^{n-1}(k) \times \mathbb{R}$ or $M_{q-1}^{n-1}(k) \times \mathbb{R}_1$, $k \neq 0$,

where $M_{q'}^m(k)$ is of constant curvature k and index q'

The case of Lorentzian manifolds

From now on we assume that M is an $n(\geq 4)$ dimensional conformally flat homogeneous Lorentzian manifold whose operator A is not diagonalizable with real eigenvalues.

Theorem 3 Under the assumption above, A has exactly one of the following three forms:

Case 1.

$$\begin{pmatrix} a & -b & & & & & & \\ b & a & & & & & & \\ & & \lambda & & & & & \\ & & & \ddots & & & & \\ & & & & \lambda & & & \\ & & & & & -\lambda & & \\ & & & & & & \ddots & \\ & & & & & & & -\lambda \end{pmatrix}$$

$$a^2 + b^2 = \lambda^2$$
$$b \neq 0$$
$$\dim T_\lambda = \dim T_{-\lambda}$$

Case 2.

$$\begin{pmatrix} \lambda & \varepsilon & & & \\ & \lambda & & & \\ & & \lambda & & \\ & & & \ddots & \\ & & & & \lambda \end{pmatrix} \qquad \begin{array}{l} \varepsilon = 1 \text{ or } -1 \\ \lambda \leq 0 \end{array}$$

Case 3

$$\begin{pmatrix} \lambda & 0 & 0 & & & & & & \\ 0 & \lambda & 1 & & & & & & \\ 1 & 0 & \lambda & & & & & & \\ & & & \lambda & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda & & & \\ & & & & & & -\lambda & & \\ & & & & & & & \ddots & \\ & & & & & & & & -\lambda \end{pmatrix}$$

$\lambda < 0$
$\dim T_{-\lambda} \leq \dim T_{\lambda} - 2$

With respect to a semi-orthonormal basis $\langle e_1, e_2 \rangle = 1, \ \langle e_i, e_j \rangle = \delta_{ij} \ (i, j \geq 3)$

Construction and characeterization 1

Example 1 An indefinite inner product $\langle\ ,\ \rangle$ on $\mathbb{R}^m$ $(n = 2m-2)$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_1 y_1 + \sum_{i=2}^{m} x_i y_i \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^m$$

$M(m, 2 : \mathbb{R})$: the linear space of real $m \times 2$ matrices.
For $X, Y \in M(m, 2 : \mathbb{R})$, we define an inner product (X, Y) by

$$(X, Y) = \langle \mathbf{x}_1, \mathbf{y}_1 \rangle + \langle \mathbf{x}_2, \mathbf{y}_2 \rangle$$
$$X = (\mathbf{x}_1, \mathbf{x}_2), \quad Y = (\mathbf{y}_1, \mathbf{y}_2) \quad \mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}^m.$$

$\boldsymbol{K = SO_+(1, m-1) \times SO(2)}$: the product Lie group
The action of $\boldsymbol{K}$ on $\boldsymbol{M(m, 2 : \mathbb{R})}$ by

$$\boldsymbol{(k_1, k_2) \times X \mapsto k_1 X k_2^{-1}}$$

Then $\boldsymbol{K}$ acts as the group of orthogonal transformations with respect to (,).

$\boldsymbol{X_o = c \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}$ $\boldsymbol{(c > 0)}$: a lightlike vector with respect to (,).

$\boldsymbol{M_c}$: the $\boldsymbol{K}$-orbit through $\boldsymbol{X_o}$.

Then M_c is a hypersurface in the lightcone Λ .

Moreover M_c is a $2m - 2(= n)$-dimensional conformally flat Lorentzian manifold whose linear operator A has the form

$$A = \frac{1}{2c^2}\begin{pmatrix} 0 & -1 & & \\ 1 & 0 & & \\ & & I_{m-2} & \\ & & & -I_{m-2} \end{pmatrix}$$

Theorem 4 M of Case 1. Then M is locally isometric to M_c ($c = \frac{1}{\sqrt{2|\lambda|}}$) constructed in Example 1.

Construction and characeterization 2

Example 2 $\mathfrak{k}$:a real linear space with the basis E_i $(3 \le i \le n)$, F_{ij} $(3 \le i < j \le n)$, X_i $(1 \le i \le n)$.
We define a bracket operation [,] on $\mathfrak{k}$ as follows:

$$[E_i, E_j] = 0$$
$$[E_i, F_{jk}] = \delta_{ij} E_k - \delta_{ik} E_j$$
$$[E_i, X_1] = 0$$
$$[E_i, X_2] = -X_i - cE_i$$
$$[E_i, X_j] = \delta_{ij} X_1$$

$$[F_{ij}, F_{kl}] = -\delta_{ik} F_{jl} + \delta_{jk} F_{il} + \delta_{il} F_{jk} - \delta_{jl} F_{ik}$$
$$[F_{ij}, X_1] = 0$$
$$[F_{ij}, X_2] = 0$$
$$[F_{ij}, X_k] = -\delta_{ik} X_j + \delta_{jk} X_i$$

$$[X_1, X_2] = -cX_1$$
$$[X_1, X_j] = 0$$

$$[X_i, X_j] = 0$$

$$[X_2, X_j] = -\varepsilon E_j$$

$i, j, k, l \ge 3$ $\qquad$ $c \in \mathbb{R}$ $\qquad$ $\varepsilon = 1$ or -1

Then [,] satisfies the Jacobi identity and $\mathfrak{k}$ becomes a Lie algebra. Let $\mathfrak{h}$ be a linear subspace of $\mathfrak{k}$ spanned by $\{\boldsymbol{E_i}, \boldsymbol{F_{ij}}\}$. Then $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{k}$.

K : a simply connected Lie group corresponding to $\mathfrak{k}$
H : the connected Lie subgroup of K which corresponds to $\mathfrak{h}$

$M = K/H$
$\pi : K \longrightarrow K/H = M$:the projection, $\quad \pi(H) = o$.
$\pi : \mathfrak{k} \longrightarrow T_oM$: the differential of π at $e \in K$
$\mathfrak{p}$:the subspace spanned by $\{X_i \ (1 \leq i \leq n)\}$, $\quad \mathfrak{p} \simeq T_oM$
an inner product $\langle \ \rangle$ on $\mathfrak{p}$ defined by

$$\langle X_1, X_2 \rangle = 1, \quad \langle X_i, X_j \rangle = \delta_{ij} \ (3 \leq i, j \leq n), \ \text{ otherwise } 0.$$

We can define the K-invariant Lorentzian metric g on M .

(M, g) is conformally flat and its linear operators A has the form case 2 with $\lambda = 0$.

Theorem 5 M of case 2 with $\lambda = 0$. Then M is locally isometric to the model constructed in Example 2.

Remark c : the parameter in $\mathfrak{k}$.

$$(M, g) \text{ is a Lorentzian } \underline{\text{symmetric}} \text{ space} \Leftrightarrow c = 0$$

Case 2 $\lambda < 0$

We can construct examples similarly to Example 2 and characterize them.

Case 3

We can construct examples similarly to Example 2. However we cannot solve the classification problem for this case at the present.

The theory of infinitesimally homogeneous spaces by Singer

Our method is due to the theory above.

M : a semi-Riemannian manifold.

$$\boldsymbol{R, \nabla R, \nabla^2 R, \cdots} \quad \textbf{:} \text{ the essential local invariants of } M$$

If M is conformally flat,

$$\boldsymbol{A, \nabla A, \nabla^2 A, \cdots} \quad \textbf{:} \text{ the essential local invariants of } M$$

Singer's theory : the relation between the homogeneity and the curvature tensor and its covariant derivatives

For a non-negative integer l,

$P(l)$: $\forall p, q \in M$ $\exists$ a linear isometry $\phi : T_pM \to T_qM$ s. t.

$$\phi^*(\nabla^i R)_q = (\nabla^i R)_p \quad i = 0, 1, ..., l.$$

$\mathfrak{so}(T_pM)$:the Lie algebra of the endomorphisms of T_pM which are skew-symmetric with respect to $\langle , \rangle$

For a non-negative integer l,

$$\mathfrak{g}_l(p) = \{ X \in \mathfrak{so}(T_pM) \mid X \cdot (\nabla^i R)_p = 0, \quad i = 0, 1, ..., l \ \}$$

$\exists$ a first integer $s(p)$ s.t.

$$\mathfrak{so}(T_pM) \supseteq \mathfrak{g}_0(p) \supsetneq \mathfrak{g}_1(p) \supsetneq \mathfrak{g}_2(p) \supsetneq \cdots \supsetneq \mathfrak{g}_{s(p)}(p) = \mathfrak{g}_{s(p)+1}(p).$$

Definition M is infinitesimally homogeneous if M satisfies $P(s(p)+1)$ for some point $p \in M$.

If M is infinitesimally homogeneous, we put $s_M = s(p)$ and call it **the Singer invariant** of M.

Theorem S.1 A connected infinitesimally homogeneous semi-Riemannian manifold is locally homogeneous.

Theorem S.2 M ,M' : locally homogeneous semi-Riemannian manifolds , $p \in M$, $p' \in M'$. Suppose that there exists a linear isometry $\phi : T_pM \to T_{p'}M'$ such that

$$\phi^*(\nabla^i R')_{p'} = (\nabla^i R)_p \quad i = 0, 1, ..., s_M + 1,$$

Then there exists a local isometry φ of a neighborhood of p onto a neighborhood of p' which satisfies $\varphi(p) = p'$ and $\varphi_{*p} = \phi$.

Corollary M and M' : conformally flat locally homogeneous semi-Riemannian manifolds , $p \in M$, $p' \in M'$. Suppose that there exists a linear isometry $\phi : T_pM \to T_{p'}M'$ such that

$$\phi^*(\nabla^i A')_{p'} = (\nabla^i A)_p \quad i = 0, 1, ..., s_M + 1.$$

Then there exists a local isometry φ of a neighborhood of p onto a neighborhood of p' which satisfies $\varphi(p) = p'$ and $\varphi_{*p} = \phi$.

Singer's invariants of conformally flat homogeneous

Lorentzian manifolds

	the Singer invariant
Case 1 $dim = 4$	0
Case 1 $dim \geq 6$	1
Case 2 $\lambda < 0$	1
Case 2 $\lambda = 0$	0
Case 3	?