Closed Geodesics in Lorentzian Surfaces

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History

Theorem (Tipler'79)

Let (M, g) be a compact spacetime with a covering space containing a compact Cauchy hypersurface. Then (M, g) contains a closed timelike geodesic.

Theorem (Galloway'86)

Every closed (compact with empty boundary) Lorentzian surface contains a closed timelike or periodic (C^1-closed up to suitable parameterization) lightlike geodesic.

Result

Theorem

Every closed Lorentzian surface contains two closed geodesics (closed orbits of the geodesic flow), one of which is nonlightlike. This lower bound is optimal.

Optimality: Consider $(\mathbb{R}^2, \overline{g})$ where

$$\overline{g} := \cos^2(x)[dx^2 - dy^2] + 2\sin(x)dxdy.$$

$\overline{g}$ is invariant under

$$\Gamma := \langle (x,y) \mapsto (x, y+1), (x,y) \mapsto (x+\pi, -y) \rangle.$$

$\rightsquigarrow \mathbb{R}^2/\Gamma = \mathbb{K}^2$ with the induced metric contains exactly 2 closed geodesics $\{x = 0\}, \{x = \pi/2\}$.

Asymptotic Cycles

For every spacetime (T^2, g) the local transversal lightlike foliations are globally well defined and orientable $\mathfrak{F}^{\pm}$.

The homology classes of the **asymptotic cycles** of the foliations $\mathfrak{F}^+$ and $\mathfrak{F}^-$ lie on on two lines $m^+, m^- \subseteq H_1(T^2, \mathbb{R})$

Let $f\colon [0,\infty) \to M$ be C^1 (e.g. flowline of a vector field).

$$h_T(\omega) := \frac{1}{T}\int_0^T f^*\omega$$

If $\text{w-lim}_{T\to\infty} h_T = h_\infty$ exists, h_∞ vanishes on exact forms $\rightsquigarrow$ $[h_\infty] \in H_1(M, \mathbb{R})$ is well defined.
h_∞ is the asymptotic cycle of f. For oriented 1-foliations $\mathfrak{F}$ asymptotic cycles are defined via the flows of the vector fields tangent to $\mathfrak{F}$.

Class A/class B spacetimes

(M^2, g) is class A $:\Leftrightarrow$ There exist a finite time orientable covering (T^2, g') with $m^+ \neq m^-$.

(M^2, g) is class B $:\Leftrightarrow$ All other cases.

(M^2, g) is class A $\Leftrightarrow$ $(\widetilde{M}^2, \widetilde{g})$ and $(\widetilde{M}^2, -\widetilde{g})$ are globally hyperbolic.

Proposition

Let (M^2, g) be class A. Then any free homotopy class contains a closed geodesic. Especially class A surfaces contain ∞-many closed geodesics.

Class B surfaces (M^2, g) contain compact leafs of either foliation $\mathfrak{F}^+$ and $\mathfrak{F}^-$.
The complement of the compact leafs is not empty and consists of cylinders or Möbius strips Z, such that $(\widetilde{Z}, \widetilde{g}|_{\widetilde{Z}})$ and $(\widetilde{Z}, -\widetilde{g}|_{\widetilde{Z}})$ are globally hyperbolic.

Proposition

Every such cylinder contains a closed geodesic.

If a class B surface contains only one such Z, then the lightlike geodesic bounding Z is complete, i.e. closed.

Remark

One can be more precise on the number of closed geodesics depending on the "rotation number" of the metric of the metric g, i.e. the connected component of the space of Lorentzian metrics g belongs to.

Literature: –. Closed Geodesics in Lorentzian Surfaces. arXiv:1011.4878, accepted to the Trans. of the AMS.