A new class of holonomy groups in the pseudo-Riemannian geometry and integrable systems on Lie algebras

Alexey Bolsinov
joint work with Dragomir Tsonev
Loughborough University, UK

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Definition

Let M^n be a smooth manifold endowed with an affine symmetric connection ∇. The holonomy group of ∇ is a subgroup $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(T_x M)$ that consists of the linear operators $A : T_x M \to T_x M$ being 'parallel transport transformations' along closed loops $\gamma(t)$ with $\gamma(0) = \gamma(1) = x$.

Problem. Given a subgroup $H \subset GL(n, \mathbb{R})$, can it be realised as the holonomy group for an appropriate symmetric connection on M^n?

Riemannian case and irreducible case: the problem is completely solved

Marcel Berger, D. V. Alekseevskii, R. Bryant, D. Joyce, L. Schwahhöfer, S. Merkulov.

Pseudo-Riemannian case: many fundamental results but still open

L. Bérard Bergery, A. Ikemakhen, C. Boubel, D. V. Alekseevskii, T. Leistner, A. Galaev.

We deal with Levi-Civita connections only. In algebraic terms this means that we consider only subgroups of the (pseudo-)orthogonal group $\mathrm{SO}(g)$:

$$H \subset \mathrm{SO}(g) = \{A \in \mathrm{GL}(V) \mid g(Au, Av) = g(u, v),\ u, v \in V\},$$

where g is a non-degenerate bilinear form on V.

Theorem

For every g-symmetric operator $L : V \to V$, the identity connected component of its centraliser in $\mathrm{SO}(g)$

$$G_L = \{X \in \mathrm{SO}(g) \mid XL = LX\}$$

is a holonomy group for a certain (pseudo)-Riemannian metric g.

What is $\mathfrak{g}_L$ in matrix form?

$$L = \begin{pmatrix} L_1 & & & \\ & L_2 & & \\ & & \ddots & \\ & & & L_k \end{pmatrix}, \qquad g = \begin{pmatrix} g_1 & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_k \end{pmatrix}$$

where

$$L_i = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}, \qquad g_i = \pm \begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$$

$$\begin{pmatrix} 0 & M_{12} & \cdots & M_{1k} \\ M_{21} & 0 & & \vdots \\ \vdots & & \ddots & M_{k-1,k} \\ M_{k1} & \cdots & M_{k,k-1} & 0 \end{pmatrix}, \quad \textit{where } M_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \mu_1 & \mu_2 & \cdots & \mu_{n_i} \\ 0 & \cdots & 0 & 0 & \mu_1 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \mu_2 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \mu_1 \end{pmatrix},$$

with $i < j$, $\mu \in \mathbb{R}$ and $M_{ji} = -g_j M_{ij}^\top g_i$.

Definition

A map $R : \Lambda^2 V \to \mathrm{gl}(V)$ is called a *formal curvature tensor* if it satisfies the Bianchi identity

$$R(u \wedge v)w + R(v \wedge w)u + R(w \wedge u)v = 0 \qquad \text{for all } u, v, w \in V.$$

Definition

A Lie subalgebra $\mathfrak{h} \subset \mathrm{gl}(V)$ is called Berger, if $\mathfrak{h}$ is generated as a vector space by the images of the formal curvature tensors $R : \Lambda^2 V \to \mathrm{gl}(V)$ such that $\mathrm{Im}\, R \subset \mathfrak{h}$.

Berger test:

Let ∇ be a symmetric affine connection on TM. Then the Lie algebra $\mathfrak{hol}(\nabla)$ of its holonomy group $\mathrm{Hol}(\nabla)$ is Berger.

Important remark:

Since we consider the Levi-Civita connections only, we can identify $\Lambda^2 V$ and $\mathrm{so}(g)$. Then the curvature operator R can be understood as a linear map:

$$R : \mathrm{so}(g) \longrightarrow so(g).$$

Our construction is based on

$$[R(X), L] = [X, M]$$

where $R : \mathrm{so}(n) \to \mathrm{so}(n)$, and L and M are some fixed symmetric matrices

Integrable systems on Lie algebras: Manakov, Mischenko, Fomenko

Projectively equivalent metrics: Sinjukov + AB, Matveev, Kiosak

Properties of sectional operators on $\mathrm{so}(g)$

Let $R\colon \mathrm{so}(g) \to \mathrm{so}(g)$ satisfy $[R(X), L] = [X, M]$ for g-symmetric L and M. Then:

- M belongs to the center of the centralizer of L and, therefore, can be presented as $M = p(L)$ where $p(t)$ is a polynomial.
- Can we find R explicitly? Yes:

$$R(X) = \frac{d}{dt}|_{t=0}\, p(L + tX). \tag{1}$$

Proof: Differentiate the identity

$$[p(L + tX), L + tX] = 0 \quad \Rightarrow \quad [\frac{d}{dt}|_{t=0}\, p(L + tX), L] + [p(L), X] = 0$$

- The image of R belongs to $\mathrm{so}(g)$.
- R is symmetric w.r.t. the Killing form on $\mathrm{so}(g)$.
- R satisfies the Bianchi identity.
  In other words, R is a formal curvature tensor.
- In our case, M is actually zero! But (1) still makes sense and defines a non-trivial operator if $p(t)$ is the minimal polynomial of L. In particular, the image of R belongs to $\mathfrak{g}_L$.

Step one: Berger test

Consider a non-degenerate bilinear form g on a finite-dimensional real vector space V, and a g-symmetric linear operator $L : V \to V$, i.e.,

$$g(Lv, u) = g(v, Lu), \quad \text{for all } u, v \in V.$$

Our goal is to verify that the centralizer of L in $\mathrm{so}(g)$, i.e.,

$$\mathfrak{g}_L = \{X \in \mathrm{so}(g) \mid XL - LX = 0\}$$

is a Berger algebra.

To that end, we need to describe formal curvature tensors

$$R : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$$

and analyse the subspace in $\mathfrak{g}_L$ spanned by their images.

In particular, if we can find just one single formal curvature tensor R such that $\mathrm{Im}\, R = \mathfrak{g}_L$, then our problem is solved.

Question: How to find R?

Define a linear mapping $R : \mathrm{so}(g) \to \mathrm{so}(g)$ by:

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX), \qquad (2)$$

where $p_{\min}(\lambda)$ is the minimal polynomial of L.

Proposition

Let $L : V \to V$ be a g-symmetric operator. Then (2) defines a formal curvature tensor $R : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$ for the Lie algebra $\mathfrak{g}_L$.

Proposition

Let $L : V \to V$ be a g-symmetric nilpotent operator that consists of two Jordan blocks. Then the image of the formal curvature tensor $R : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$ defined by (2) coincides with $\mathfrak{g}_L$. In particular, $\mathfrak{g}_L$ is Berger.

More generally:

Proposition

Let $L : V \to V$ be a g-symmetric operator such that each eigenvalue has at most two Jordan blocks. Then the image of the formal curvature tensor $R : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$ defined by (2) coincides with $\mathfrak{g}_L$. In particular, $\mathfrak{g}_L$ is Berger.

“Block-wise” modification of formula (2)

We decompose L into Jordan blocks $L_1, \dots, L_k$, then for each pair of blocks L_i and L_j we define an operator $\hat{R}_{ij} : \mathrm{so}(n) \to \mathfrak{g}_L$ by means of (2) on the subspace spanned by these blocks and extend it trivially onto all other blocks.
We finally set:

$$R_{\mathrm{formal}} = \sum_{i,j} \hat{R}_{ij} \tag{3}$$

This operator solves our problem:

Theorem

Let $L : V \to V$ be a g-symmetric operator. Then $R_{\mathrm{formal}} : \Lambda^2 V \simeq \mathrm{so}(g) \to \mathfrak{g}_L$ given by (3) is a formal curvature tensor such that $\mathrm{Im}\, R = \mathfrak{g}_L$. In particular, $\mathfrak{g}_L$ is a Berger algebra.

Step two: Realisation

We need to give an example of g such that $\mathfrak{hol}(\nabla) = \mathfrak{g}_L$, namely:

For a given operator $L : T_{x_0}M \to T_{x_0}M$, we need to find
(pseudo)-Riemannian metric $g(x)$ on M and
$(1,1)$-tensor field $L(x)$ (with the initial condition $L(x_0) = L$)
such that

1. $\nabla L(x) = 0$;
2. $R(x_0)$ coincides with the above formal curvature tensor R_{formal}.

The idea is natural:

- set $L(x) = \mathrm{const}$
- try to find the desired metric $g(x)$ in the form:

$$\text{constant} + \text{quadratic}$$

i.e.,

$$g_{ij}(x) = (g_0)_{ij} + \sum \mathcal{B}_{ij,pq} x^p x^q \tag{4}$$

where $\mathcal{B}$ satisfies obvious symmetry relations, namely, $\mathcal{B}_{ij,pq} = \mathcal{B}_{ji,pq}$ and $\mathcal{B}_{ij,pq} = \mathcal{B}_{ij,qp}$.

Algebraic reformulation for "quadratic" metrics

It is more convenient to replace $\mathcal{B}_{ij,pq}$ by $B_{j\,q}^{i\,p} = g_0^{\alpha i} g_0^{\beta p} \mathcal{B}_{\alpha j,\beta q}$ and to treat the latter as a linear map

$$B : \mathrm{gl}(V) \to \mathrm{gl}(V) \quad \text{defined by } B(X)^i_q = B_{j\,q}^{i\,p} X^j_p.$$

- The condition that L is g-symmetric reads:

$$B(L \cdot X) = L \cdot B(X), \quad X \in \mathrm{gl}(V)$$

- The condition $\nabla L = 0$ amounts to the following equation for B:

$$[B(X), L] + [B(X), L]^* = 0, \quad X \in \mathrm{gl}(V)$$

- The curvature tensor of g at the origin $x_0 = 0$ takes the following form:

$$R(X) = B(X) - B(X)^*, \quad X \in \mathrm{so}(g_0)$$

So we need to solve

$$B(X) - B(X)^* = R_{\mathrm{formal}}(X), \quad X \in \mathrm{so}(g_0).$$

Magic formula 2

Consider the following formal expression:

$$B = \frac{1}{2} \cdot \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t \cdot \otimes), \qquad (5)$$

where $p_{\min}(\lambda) = \sum_{m=0}^{n} a_m \lambda^m$ is the minimal polynomial of L. Equivalently,

$$B = \frac{1}{2} \cdot \sum_{m=0}^{n} a_m \sum_{j=0}^{m-1} L^{m-1-j} \otimes L^j. \qquad (6)$$

Proposition

Consider the quadratic metric $g(x) = g_0 + \mathcal{B}(x, x)$ with B defined by (5) (or, equivalently, by (6)). Then

1) L is g-symmetric;

2) $\nabla L = 0$, where ∇ is the Levi-Civita connection for g;

3) The curvature tensor for g at the origin is defined by 'magic formula 1', i.e.,

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + tX).$$

As we see, the realisation procedure is based on the following 'informal relationship' between R and B:

$$B = \frac{1}{2} R(\otimes).$$

This proposition solves the realisation problem in the most important "two Jordan blocks" case.

To get the realisation for the general case, we proceed just in the same way as we did for the algebraic part.
Namely, we split L into Jordan blocks and define for each pair L_i, L_j of Jordan blocks a formal curvature tensor $\hat{R}_{ij}$ as before. Then by using 'magic formula 2' we realise this formal curvature tensor by an appropriate quadratic metric $g(x) = g_0 + \hat{\mathcal{B}}_{ij}(x, x)$ satisfying $\nabla L = 0$. And finally, we set

$$g(x) = g_0 + \mathcal{B}(x, x), \quad \text{with } B = \sum_{i,j} \hat{B}_{ij}.$$

Then, by linearity, this metric still satisfies $\nabla L = 0$ and its curvature tensor coincides with $R_{\mathrm{formal}} = \sum \hat{R}_{ij}$ consructed in the first part of the proof.