On one class of holonomy groups in pseudo-Riemannian geometry

Alexey Bolsinov and Dragomir Tsonev
Dept. of Math. Sciences, Loughborough University
Loughborough, LE11 3TU UK

May 31, 2011

Holonomy groups were introduced by Élie Cartan [3] in 1926 in order to study the Riemannian symmetric spaces and since then the classification of holonomy groups has remained one of the classical problems in differential geometry.

Definition 1. *Let M be a smooth manifold and ∇ an affine connection on TM. The holonomy group of ∇ is a subgroup* $\mathrm{Hol}(\nabla) \subset \mathrm{GL}(T_xM)$ *that consists of the linear operators $A : T_xM \to T_xM$ being "parallel transport transformations" along closed loops γ with $\gamma(0) = \gamma(1) = x$.*

In the case of Levi-Civita connections on Riemannian manifolds, the classification of holonomy groups is due to M. Berger [1]. In the pseudo-Riemannian case, the description of holonomy groups is a very difficult problem which still remains open and even particular examples are of interest. We refer to [4], [5] for more information on recent development in this field.

Since we deal with Levi-Civita connections only, we consider subgroups of the (pseudo)-orthogonal group $\mathrm{SO}(g)$, where g is a non-degenerate inner product on a finite-dimensional real vector space V. The main result of our paper is

Theorem 1. *For every g-symmetric operator $L : V \to V$, the identity component of its centraliser in* $\mathrm{SO}(g)$

$$G_L = \{X \in \mathrm{SO}(g) \mid XL = LX\}$$

is a holonomy group for some (pseudo)-Riemannian metric.

In the Riemannian case this theorem becomes trivial as L is diagonalisable and G_L is isomorphic to the direct product $\mathrm{SO}(k_1) \times \cdots \times \mathrm{SO}(k_m) \subset \mathrm{SO}(n)$, which is, of course, a holonomy group. In the pseudo-Riemannian case, L may have non-trivial Jordan blocks and the structure of G_L becomes more complicated.

As usual, instead of G_L we consider its Lie algebra $\mathfrak{g}_L$. The first step is to prove that $\mathfrak{g}_L$ is a Berger algebra. To that end, it is sufficient to construct a formal curvature tensor $R : \Lambda^2(V) \to \mathrm{so}(g)$ such that the image of R coincides with $\mathfrak{g}_L$. The next step is to realise $\mathfrak{g}_L$ as a holonomy Lie algebra for a suitable metric.

Our proof is based on unexpected relationship between curvature tensors for some special metrics and the theory of integrable systems on semi-simple Lie algebras (see [2]) . The following two "magic formulas" play crucial role in our construction:

$$R(X) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t\cdot X), \quad X \in so(g) \simeq \Lambda^2 V, \tag{1}$$

$$C = R(\otimes) = \frac{d}{dt}\Big|_{t=0} p_{\min}(L + t\cdot \otimes), \tag{2}$$

where $p_{\min}(\lambda)$ is the minimal polynomial of L. The first formula gives a map $R : \Lambda^2 V \to \mathrm{gl}(V)$. The second one defines a tensor C of type $(2,2)$ whose meaning can be understood from the following example: if $p_{\min}(\lambda) = \lambda^m$, then

$$R(X) = \sum_{k=1}^{m} L^{k-1} X L^{m-k} \quad \text{and} \quad C = R(\otimes) = \sum_{k=1}^{m} L^{k-1} \otimes L^{m-k}$$

These two algebraic objects possess the following remarkable properties:

Proposition 1. *R satisfies the Bianchi identity and its image is contained in $\mathfrak{g}_L$. In other words, R is a formal curvature tensor for the Lie algebra $\mathfrak{g}_L$.*

So far L and g were defined on a fixed vector space V which now will be considered as $T_0\mathbb{R}^n$. We now extend them onto a neighborhood of $0 \in \mathbb{R}^n$.

Proposition 2. *In local coordinates $(x^1, \dots, x^n)$, we set $L = L(0) = \mathrm{const}$ and*

$$g = g_{ij}(x) = g_{ij}(0) + \frac{1}{2}\sum g_{i\alpha}(0) g_{p\beta}(0) C^{\alpha,\beta}_{j\ ,q} x^p x^q$$

Then L is g-symmetric, $\nabla L = 0$ and the curvature tensor for g at the origin $0 \in \mathbb{R}^n$ is $R(X)$ defined by (1).

Using a kind of "block-wise" modification of formulas (1) and (2), it is not hard to construct a formal curvature tensor $R(X)$ and $(2,2)$-tensor C for which Propositions 1 and 2 still hold, but R is, in addition, an "onto" map. This obviously completes the proof.

References

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