On one class of holonomy groups in pseudo-Riemannian geometry

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Holonomy groups were introduced by Élie Cartan [3] in 1926 in order to study the Riemannian symmetric spaces and since then the classification of holonomy groups has remained one of the classical problems in differential geometry.

Definition 1. Let $M$ be a smooth manifold and $\nabla$ an affine connection on $TM$. The holonomy group of $\nabla$ is a subgroup $\text{Hol}(\nabla) \subset \text{GL}(T_xM)$ that consists of the linear operators $A : T_xM \rightarrow T_xM$ being “parallel transport transformations” along closed loops $\gamma$ with $\gamma(0) = \gamma(1) = x$.

In the case of Levi-Civita connections on Riemannian manifolds, the classification of holonomy groups is due to M. Berger [1]. In the pseudo-Riemannian case, the description of holonomy groups is a very difficult problem which still remains open and even particular examples are of interest. We refer to [4], [5] for more information on recent development in this field.

Since we deal with Levi-Civita connections only, we consider subgroups of the (pseudo)-orthogonal group $\text{SO}(g)$, where $g$ is a non-degenerate inner product on a finite-dimensional real vector space $V$. The main result of our paper is

Theorem 1. For every $g$-symmetric operator $L : V \rightarrow V$, the identity component of its centraliser in $\text{SO}(g)$

$$G_L = \{ X \in \text{SO}(g) \mid XL = LX \}$$

is a holonomy group for some (pseudo)-Riemannian metric.

In the Riemannian case this theorem becomes trivial as $L$ is diagonalisable and $G_L$ is isomorphic to the direct product $\text{SO}(k_1) \times \cdots \times \text{SO}(k_m) \subset \text{SO}(n)$, which is, of course, a holonomy group. In the pseudo-Riemannian case, $L$ may have non-trivial Jordan blocks and the structure of $G_L$ becomes more complicated.

As usual, instead of $G_L$ we consider its Lie algebra $\mathfrak{g}_L$. The first step is to prove that $\mathfrak{g}_L$ is a Berge algebra. To that end, it is sufficient to construct a formal curvature tensor $R : \Lambda^2(V) \rightarrow \mathfrak{so}(g)$ such that the image of $R$ coincides
with $g_L$. The next step is to realise $g_L$ as a holonomy Lie algebra for a suitable metric.

Our proof is based on unexpected relationship between curvature tensors for some special metrics and the theory of integrable systems on semi-simple Lie algebras (see [2]). The following two “magic formulas” play crucial role in our construction:

$$R(X) = \frac{d}{dt} \bigg|_{t=0} p_{\min}(L + t \cdot X), \quad X \in so(g) \simeq \Lambda^2 V,$$

(1)

$$C = R(\otimes) = \frac{d}{dt} \bigg|_{t=0} p_{\min}(L + t \cdot \otimes),$$

(2)

where $p_{\min}(\lambda)$ is the minimal polynomial of $L$. The first formula gives a map $R : \Lambda^2 V \rightarrow gl(V)$. The second one defines a tensor $C$ of type $(2, 2)$ whose meaning can be understood from the following example: if $p_{\min}(\lambda) = \lambda^m$, then

$$R(X) = \sum_{k=1}^{m} L^{k-1} X L^{m-k} \quad \text{and} \quad C = R(\otimes) = \sum_{k=1}^{m} L^{k-1} \otimes L^{m-k}.$$

These two algebraic objects possess the following remarkable properties:

**Proposition 1.** $R$ satisfies the Bianchi identity and its image is contained in $g_L$. In other words, $R$ is a formal curvature tensor for the Lie algebra $g_L$.

So far $L$ and $g$ were defined on a fixed vector space $V$ which now will be considered as $T_0 \mathbb{R}^n$. We now extend them onto a neighborhood of $0 \in \mathbb{R}^n$.

**Proposition 2.** In local coordinates $(x^1, \ldots, x^n)$, we set $L = L(0) = \text{const}$ and

$$g = g_{ij}(x) = g_{ij}(0) + \frac{1}{2} \sum_{\alpha, \beta} g_{i\alpha}(0) g_{j\beta}(0) C^{\alpha, \beta}_{j, q} x^p x^q.$$

Then $L$ is $g$-symmetric, $\nabla L = 0$ and the curvature tensor for $g$ at the origin $0 \in \mathbb{R}^n$ is $R(X)$ defined by (1).

Using a kind of “block-wise” modification of formulas (1) and (2), it is not hard to construct a formal curvature tensor $R(X)$ and $(2, 2)$-tensor $C$ for which Propositions 1 and 2 still hold, but $R$ is, in addition, an “onto” map. This obviously completes the proof.

**References**


