Geodesic connectedness on Gödel type spacetimes: a “static” variational set-up

joint work with Anna Maria Candela and José Luis Flores

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Gödel type spacetimes

In A.M. Candela and M. Sánchez, *Differential Geom. Appl.* 2000:

A semi-Riemannian manifold $(M, \langle \cdot, \cdot \rangle_L)$ is a *Gödel type spacetime* if $M = M_0 \times \mathbb{R}^2$, where $(M_0, \langle \cdot, \cdot \rangle_R)$ is a Riemannian manifold and

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where $x \in M_0$, the variables (y, t) are the natural coordinates of $\mathbb{R}^2$ and A, B, C are C^1 scalar fields on M_0 satisfying

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Other references: A. K. Raychaudhuri and S. N. G. Thakurta, *Phys. Rev.* 1980; M.O. Calvão, I. Damião Soares and J. Tiomno, *Gen. Relat. Gravit.* 1990; A. Melfo, L. A. Núñez, U. Percoco and V. Villalba, *J. Math. Phys* 1992; H.L. Carrión, M.J. Rebouças and A.F.F. Teixeira, *J. Math. Phys.* 1999; P. Piccione and D.V. Tausk, *Calc. Var.* 2002

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$(\mathbb{R}^4, ds^2)$ is geodesically complete, geodesically connected, has closed timelike curves

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* if $A(x)C(x) > 0$ on M_0 the spacetime is *standard stationary* (and if $B \equiv 0$ it is *standard static*)

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Variational methods and critical point theory give results on geodesic connectedness, even if f is unbounded and its critical points have infinite Morse index

Main references

Seminal papers:

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- V. Benci and D. Fortunato, *Adv. Math.* 1994
- V. Benci, D. Fortunato and F. Giannoni, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 1991
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- M. Sánchez, *Nonlinear Anal.* 2001
- A. M. Candela and M. Sánchez, *Special Volume in the ESI-Series on Mathematics and Physics* 2008

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  F. Giannoni and P. Piccione, *Comm. Anal. Geom.* 1999; F. Giannoni, P. Piccione and R. Sampalmieri, *J. Math. Anal. Appl.* 2000; R. B., A.M. Candela, J.L. Flores and M. Sánchez, *Adv. Nonlinear Stud.* 2003; E. Caponio, A. Masiello, P. Piccione, *Math. Z.* 2003; R. B., A.M. Candela and J.L. Flores, *J. Geom. Phys.* 2006; A.M. Candela, J.L. Flores and M. Sánchez, *Adv. Math.* 2008

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with $\Delta_y := y_q - y_p, \Delta_t := t_q - t_p$ and $\bar{y} = \phi_y(\bar{x}), \bar{t} = \phi_t(\bar{x})$

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$\ast$ Morse relations, P. Piccione and D.V. Tausk, *Calc. Var.* 2002

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admits two real (non-null) eigenvalues

$$\lambda_\pm(x) = \frac{a(x) - c(x) \pm \sqrt{(a(x)+c(x))^2 + 4b^2(x)}}{2}$$

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I is almost $\mathcal{J}$ in the static case!

The variational principle on standard static spacetimes

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with $\Delta_t := t_p - t_q$ and $\bar{t} = \phi_t(\bar{x})$

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Moreover the last result is optimal: there exists a family of geodesic disconnected spacetimes for $\alpha = 2 + \epsilon, \epsilon > 0$

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- if $|\mathcal{L}(x)| \geq \nu > 0$ on $\Omega^1(x_p, x_q)$ $\mathcal{J}$ satisfies the Palais-Smale condition

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Then, $(M, \langle \cdot, \cdot \rangle_L)$ *is geodesically connected.*

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- The cases $A \equiv C$, $a \equiv c$, $\lambda_+ = \lambda_-$ are included in previous ones

Other cases

- If $\mathcal{L}(x) < 0$ and $a(x) - c(x) > 0$: $\lambda_+(x) \geq \lambda_-(x) > 0$, then

$$\mathcal{J}(x) \geq \frac{1}{2}\,\|\dot{x}\|^2 - \frac{1}{2}\frac{(\Delta_+^2(x) + \Delta_-^2(x))}{\lambda_-(x)}$$

  and conditions for $\lambda_-(x)$ on $\Omega^1(x_p, x_q)$ ensuring the coercivity of $\mathcal{J}$ can be given
- If $\mathcal{L}(x) < 0$, $a(x) - c(x) < 0$: $\lambda_-(x) < \lambda_+(x) < 0$, hence M is geodesically connected if there exists $\nu > 0$ such that $\mathcal{L}(x) \leq -\nu < 0$ for all $x \in H^1(I, M_0)$ and $A(x) - C(x) < 0$ for all $x \in M_0$
- If $\mathcal{L}(x) > 0$ and $a(x) - c(x) < 0$?
- The cases $A \equiv C$, $a \equiv c$, $\lambda_+ = \lambda_-$ are included in previous ones
- Variational methods do not yield optimal results (the classical Gödel Universe does not verify our assumptions)