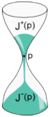
Pinching the Fermat Metric in Stationary Spacetimes

Alexander Dirmeier
(joint work with M Plaue and M Scherfner)

Technische Universität Berlin
Department of Mathematics
(Project Group Lorentzian Geometry)



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Pinching a Randers Metric

- Riemannian manifold (M,g), one-form $b \in \Lambda^1 M$, Finsler metric of Randers type $F: TM \to \mathbb{R}_0^+$,

$$F = \sqrt{g + b^2} + b$$

- Another Riemannian metric g_0 on M; $X, Y : M \to \mathbb{R}^+$
- g_0 complete: $\frac{g_0}{Y^2}$ or $X^2 g_0$ complete depending on growth of X and Y
- $\frac{g_0}{Y^2}$ complete $\Rightarrow F$ complete $\Rightarrow X^2 g_0$ complete

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$$\sqrt{\frac{g_0(x)[v,v]}{Y(x)^2}} \leq \sqrt{g(x)[v,v] + b(x)[v]^2} + b(x)[v] \leq \sqrt{X(x)^2 g_0(x)[v,v]}$$

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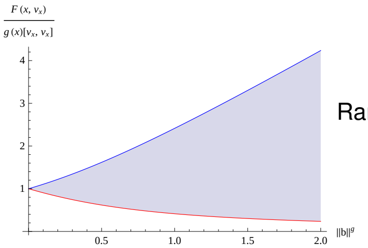

Randers metric depends on direction!

$$\|b\|_x^g = \sup_{v \in T_x M \setminus \{0\}} \frac{|b(x)[v]|}{\sqrt{g(x)[v,v]}}$$

- Find best estimates for X and Y

Known Result [M Sánchez 1997]

g complete and $\|b\|_x^g$ grows at most linearly $\Rightarrow F$ complete

- Get Randers completeness $\leftrightarrow$ global hyperbolicity by Riemannian completeness $\oplus$ growth condition

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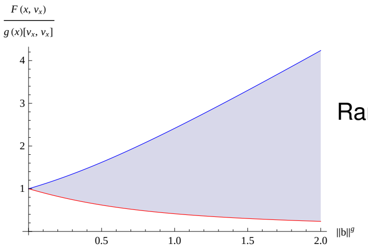

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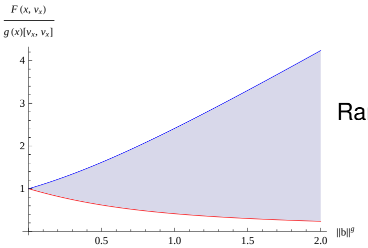

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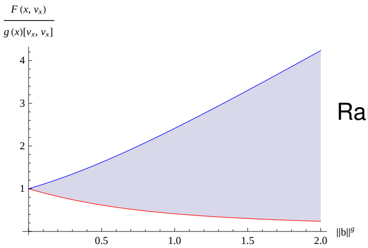

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Standard Stationary Spacetimes

- Spacetime: $(N = \mathbb{R} \times M, l)$ with $(t, x) \in \mathbb{R} \times M$

$$l = -\mathrm{d}t^2 + 2b\mathrm{d}t + g$$

g Riemannian on M and $b \in \Lambda^1 M$

Known Result [E Caponio, M A Javaloyes, A Masiello 2007]

$M_t := \{t\} \times M$ Cauchy surface $\Leftrightarrow F = \sqrt{g + b^2} + b$ complete

- Randers metric $F = \sqrt{g + b^2} + b$ called Fermat metric

Known Result [E Caponio, M A Javaloyes, M Sánchez 2010]

(N, l) globally hyperbolic $\Rightarrow h = g + b^2$ complete Riemannian metric on M

- Independent of the change of slicing:
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Known Result [E Caponio, M A Javaloyes, A Masiello 2007]

$M_t := \{t\} \times M$ Cauchy surface $\Leftrightarrow F = \sqrt{g + b^2} + b$ forward and backward complete

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Conformal Transformations

- (M,g) complete Riemannian manifold
- When is $g' = \psi^2 g$ or $g^* = \frac{g}{A^2}$ with $A, \psi \colon M \to \mathbb{R}^+$ complete, too?
- Depends on growth of A towards g-infinity.
- $A \colon M \to \mathbb{R}^+$ grows at most linearly towards g-infinity $\Leftrightarrow$ for all $x_0 \in M$ there are $c_1, c_2 \in \mathbb{R}$ such that

$$A(x) \leq c_1 d_g(x_0, x) + c_2$$

Theorem [$\sim$ (2010)]

For a complete Riemannian manifold (M,g) and a function $A \colon M \to \mathbb{R}$, $g^* = \frac{g}{A^2}$ is complete if and only if A grows at most linearly towards g-infinity

- c_1 independent of x_0 and $c_2 > A(x_0)$

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Proper Functions

- Recall: $f : M \to \mathbb{R}$ proper $\Leftrightarrow f^{-1}(K)$ compact for all compact $K \subset \mathbb{R}$

Theorem [W B Gordon (1973,1974)]

- (M,g) any Riemannian manifold and $f : M \to \mathbb{R}$ a proper function then $(M,\tilde{g})$ is a complete Riemannian manifold with $\tilde{g} = g + \mathrm{d}f \otimes \mathrm{d}f$
- (M,g) is a complete Riemannian manifold if and only if there is a proper function $f : M \to \mathbb{R}$ such that $\sup_{x\in M} ||\mathrm{d}f||_x^g < \infty$

- By a proper function we get a necessarily complete Riemannian metric

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Pinching with $g + b \otimes b$

Theorem [$\sim$ (2010)]

Let $(N = \mathbb{R} \times M, l = -\mathrm{d}t^2 + 2b\mathrm{d}t + g)$ a standard stationary spacetime and $F = \sqrt{g + b \otimes b} + b$ the Fermat metric on M then

$$\frac{1}{2}\sqrt{\frac{g + b \otimes b}{(1 + (\|b\|^g)^2)^2}} \leq F \leq 2\sqrt{g + b \otimes b}$$

as best estimate.

- Upper bound: $X = 2$ constant $\Rightarrow g + b^2$ complete if M_t Cauchy surfaces as known
- Lower bound: $Y(x) = (\|b\|_x^g)^2$

Corollary [$\sim$ (2010)]

If $g + b \otimes b$ is complete and $(\|b\|_x^g)^2$ grows at most linearly towards $(g + b^2)$-infinity then (N, l) is globally hyperbolic with Cauchy surfaces M_t.

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Pinching with g

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Let $(\mathbb{R}\times M, l = -\mathrm{d}t^2 + 2b\mathrm{d}t + g)$ a standard stationary spacetime and $F = \sqrt{g + b\otimes b} + b$ the Fermat metric on M then

$$\frac{\sqrt{g}}{\sqrt{1+(\|b\|^g)^2}+\|b\|^g} \leq F \leq (\sqrt{1+(\|b\|^g)^2}+\|b\|^g)\sqrt{g}$$

as best estimate.

Corollary [$\sim$ (2010)]

If any two of the following three items hold simultaneously they imply the third:

- The slices M_t are Cauchy surfaces.
- $\|b\|_x^g$ grows at most linearly towards g-infinity.
- g is complete.

Pinching with g

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Pinching with $g + \mathrm{d}f \otimes \mathrm{d}f$

- $f : M \to \mathbb{R}$ proper $\Rightarrow g + \mathrm{d}f^2$ necessarily complete

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$$\frac{1}{2}\sqrt{\frac{g + \mathrm{d}f^2}{(1 + (\|\mathrm{d}f\|^g)^2)(1 + (\|b\|^g)^2)}} \leq F \leq 2\sqrt{(1 + (\|b\|^g)^2)(g + \mathrm{d}f^2)}$$

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Corollary [∼ (2010)]

If the product function $\|\mathrm{d}f\|^g_x \cdot \|b\|^g_x$ grows at most linearly towards $(g + \mathrm{d}f^2)$-infinity for a proper function $f : M \to \mathbb{R}$ then (N, l) is globally hyperbolic with Cauchy surfaces M_t.

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Assume (N, l) is globally hyperbolic then $||b||_x^g$ grows at most linearly towards $(g + \mathrm{d}f^2)$-infinity for any proper function $f\colon M \to \mathbb{R}$.

- Upper bound: $X(x) = \sqrt{(1 + (||b||_x^g)^2)}$ independent of $||\mathrm{d}f||_x^g$ and holds for all slicings

$$t \mapsto t + \varphi(x), \quad b \mapsto b - \mathrm{d}\varphi, \quad F \mapsto F - \mathrm{d}\varphi$$

with $F - \mathrm{d}\varphi$ complete.

Pinching with $g + \mathrm{d}f \otimes \mathrm{d}f$

Corollary [$\sim$ (2010)]

Assume (N, l) is globally hyperbolic then $\|b\|_x^g$ grows at most linearly towards $(g + \mathrm{d}f^2)$-infinity for any proper function $f : M \to \mathbb{R}$.

- Upper bound: $X(x) = \sqrt{(1 + (\|b\|_x^g)^2)}$ independent of $\|\mathrm{d}f\|_x^g$ and holds for all slicings

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Conclusion

- Purely Riemannian and/or analytic criteria for global hyperbolicity/Cauchy surfaces in stationary spacetimes
- Criteria for Randers completeness in terms of proper functions / growth of functions?

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