Linear stability of the non-extreme Kerr black hole

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Introduction to General Relativity

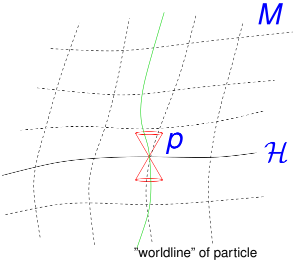
In general relativity, gravity is described by the

geometry of space-time

(M, g): Lorentzian manifold of signature $(+ - - -)$
tangent space T_pM is vector space with indefinite inner product

$$\begin{cases} g(u,u) > 0 & : \quad u \text{ is timelike} \\ g(u,u) = 0 & : \quad u \text{ is lightlike or null} \\ g(u,u) < 0 & : \quad u \text{ is spacelike} \end{cases}$$

this encodes the causal structure



light cone

spacelike hypersurface

The gravitational field is described by the curvature of M

∇ : covariant derivative, Levi-Civita connection,

$$\nabla_i X = \left(\partial_i X^j + \Gamma^j{}_{ik}\, X^k\right) \frac{\partial}{\partial x^j}$$

$R^i{}_{jkl}$: Riemann curvature tensor,

$$\nabla_i \nabla_j X - \nabla_j \nabla_i X = R^l{}_{ijk}\, X^k \,\frac{\partial}{\partial x^l}$$

$R_{ij} = R^l{}_{ilj}$: Ricci tensor, $\quad R = R^i_i$: scalar curvature

Einstein's equations: $\quad R_{jk} - \frac{1}{2}\, R\, g_{jk} = 8\pi\, T_{jk}$

T_{jk} : enery-momentum tensor, describes matter

"matter generates curvature"

vice versa:

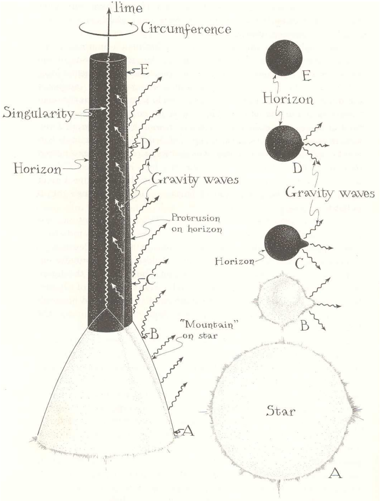
"curvature affects the dynamics of matter"

equations of motion, depend on type of matter:

- classical point particles: geodesic equation
- dust: perfect fluid
- classical waves: wave equations
- quantum mechanical matter:
  equations of wave mechanics
  (Dirac or Klein Gordon equation)
- ......

coupling Einstein equations with equations of motion yields

system of nonlinear PDEs



This Einstein-matter system describes exciting effects like the gravitational collapse of a star to a black hole

but nonlinear system of PDEs, extremely difficult to analyze

figure taken from Kip Thorne,

"Black Holes and Time Warps"

Possible methods and simplifications:

- numerical simulations
- small initial data (Christodoulou-Klainerman, ...)
- analytical work in spherical symmetry and a massless scalar field (Christodoulou, ...)

In this talk:

- consider late-time behavior of gravitational collapse, system has nearly settled down to a stationary black hole consider linear perturbations of a stationary black hole

no symmetry assumptions for perturbation!

Special Solutions to Einstein's equations

describe a "star", no matter outside
vacuum solutions: $R_{jk} = 0$

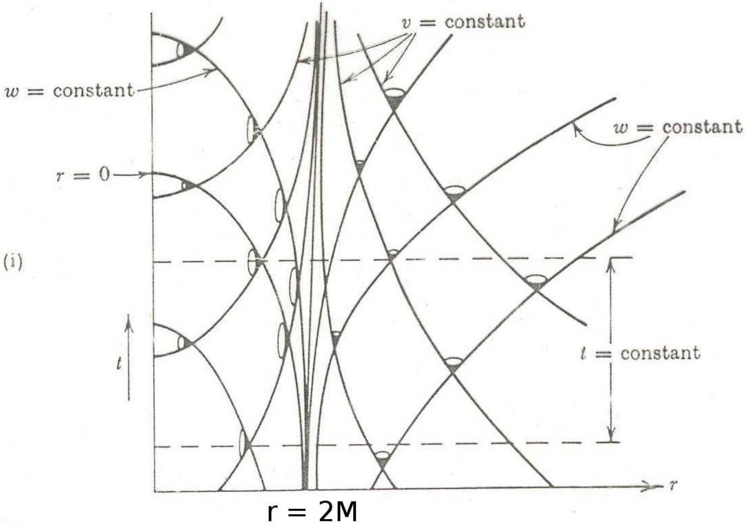
Schwarzschild solution (1916)

spherically symmetric, static, asymptotically flat
polar coordinates (r, ϑ, φ), time t

$$\begin{aligned} ds^2 &= \left(1 - \frac{2M}{r}\right) dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 \\ &\quad -r^2 \left(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2\right) \end{aligned}$$

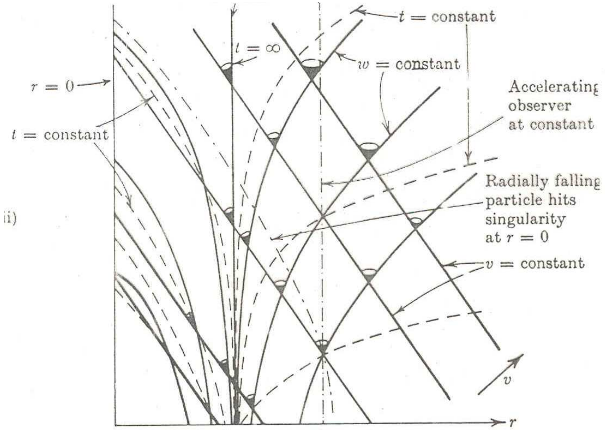
here M is the mass of the star

Schwarzschild solution



(figures taken from Hawking/Ellis, “The Large-Scale Structure of Space-Time”)

Schwarzschild solution in Finkelstein coordinates



Kerr solution (1965)

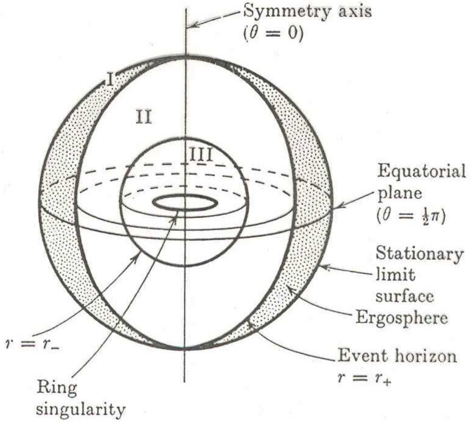
again asymptotically flat, but only axisymmetric, stationary
Boyer-Lindquist coordinates $(t, r, \vartheta, \varphi)$

$$\begin{aligned} ds^2 &= \frac{\Delta}{U} (dt - a \sin^2 \vartheta \, d\varphi)^2 - U \left(\frac{dr^2}{\Delta} + d\vartheta^2 \right) \\ &\quad - \frac{\sin^2 \vartheta}{U} (a \, dt - (r^2 + a^2) \, d\varphi)^2 \\ U(r, \vartheta) &= r^2 + a^2 \cos^2 \vartheta \\ \Delta(r) &= r^2 - 2Mr + a^2 , \end{aligned}$$

M = mass, aM = angular momentum

we always consider non-extreme case $M^2 > a^2$

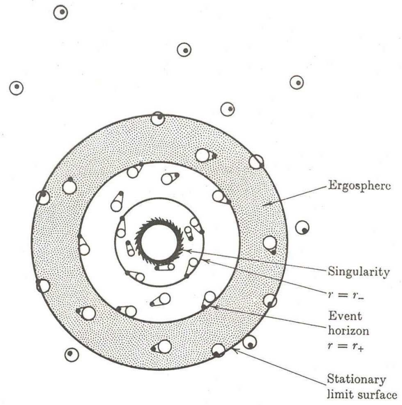
The Kerr solution



two horizons: the Cauchy horizon and the event horizon

ergosphere: annular region outside the event horizon

(figures taken from Hawking/Ellis, “The Large-Scale Structure of Space-Time”) view from north pole



Black hole uniqueness theorem (Israel, Carter, Robinson, in 1970s)

Assume the following:

- *time orientability, topology* $\mathbb{R}^2 \times S^2$
- *weak asymptotic simplicity, causality condition*
- *existence of event horizon with spherical topology*
- *axi-symmetry, pseudo-stationarity*

Every such solution of the vacuum Einstein equations is the non-extreme Kerr solution.

Thus the Kerr solution is the mathematical model of a rotating black hole in equilibrium

Linear Hyperbolic Equations in Kerr

Questions of Physical Interest

- gravitational wave detectors (LIGO, LISA)
  What signals can one expect?
  general interest in propagation of gravitational waves
- Hawking radiation
  Do black holes emit Dirac particles?
- superradiance
  Can one extract energy from rotating black holes using gravitational or electromagnetic waves?
- problem of stability of black holes under electromagnetic or gravitational perturbations

general problem: understand long-time dynamics

Methods

- Vector field method: Rodnianski, Dafermos, ...
- Strichartz estimates, local decay estimates:
  Tataru, Sterbenz, ...
- microlocal analysis, quasi-normal modes:
  Zworski, Vasy, Dyatlov, ...
- Analysis of the Maxwell equations:
  Tataru, Metcalfe, Tohaneanu, ...
  Anderson and Blue

Here focus on spectral methods in the Teukolsky formulation

Newman-Penrose formalism: characterize by spin

spin s	massless	massive
0	scalar waves	Klein-Gordon field
$\frac{1}{2}$	neutrino field	Dirac field
1	electromagnetic waves	vector bosons
$\frac{3}{2}$	Rarita-Schwinger field	
2	gravitational waves	

Structural Results

massless equations of spin s:
system of $(2s+1)$ first order PDEs, write symbolically as

$$\mathcal{D} \begin{pmatrix} \Psi_s \\ \vdots \\ \Psi_{-s} \end{pmatrix} = 0$$

Teukolsky (1972): After multiplying by first-order operator, the first and last components decouple,

$$D\mathcal{D} = \begin{pmatrix} T_s & 0 & \cdots & 0 & 0 \\ * & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & * \\ 0 & 0 & \cdots & 0 & T_{-s} \end{pmatrix}$$

Structural Results

- gives one second order complex equation
- other components obtained by differentiation (Teukolsky-Starobinsky identity)
- combine equations for different s into one so-called Teukolsky master equation, s enters as a parameter

this method does not work for massive equations

- All the equations (massive and massless) can be separated into ODEs:

Carter (1968)	scalar waves, Klein-Gordon field
Güven	neutrino field
Teukolsky (1972)	massless eqns, general spin
Chandrasekhar (1976)	Dirac field

Separation of Variables

explain for the Teukolsky Equation

$$\left(\frac{\partial}{\partial r} \, \Delta \, \frac{\partial}{\partial r} - \frac{1}{\Delta} \left[(r^2 + a^2) \frac{\partial}{\partial t} + a \frac{\partial}{\partial \varphi} - (r - M) s \right]^2 \right.$$

$$-4s \, (r + ia \cos\vartheta) \, \frac{\partial}{\partial t} + \frac{\partial}{\partial \cos\vartheta} \, \sin^2\vartheta \, \frac{\partial}{\partial \cos\vartheta}$$

$$\left. + \frac{1}{\sin^2\vartheta} \left(a \sin^2\vartheta \frac{\partial}{\partial t} + \frac{\partial}{\partial \varphi} + i \cos\vartheta \, s \right]^2 \right) \phi = 0$$

metric function $\Delta(r) = r^2 - 2Mr - a^2$
spin parameter $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$

standard separation ansatz:

$$\phi(t, r, \vartheta, \varphi) = e^{-i\omega t}\, e^{-ik\varphi}\, \Phi(\vartheta, r)$$

yields

$$\bigg(\frac{\partial}{\partial r} \,\Delta\, \frac{\partial}{\partial r} + \frac{1}{\Delta} \Big[(r^2 + a^2)\omega + ak + i(r - M)s \Big]^2$$
$$+4is\,(r + ia\cos\vartheta)\,\omega + \frac{\partial}{\partial \cos\vartheta}\; \sin^2\vartheta\, \frac{\partial}{\partial\cos\vartheta}$$
$$-\frac{1}{\sin^2\vartheta} \Big(a\sin^2\vartheta\,\omega + k - \cos\vartheta\; s \Big]^2 \bigg)\, \Phi = 0$$

Separation of r and ϑ possible,

$$\Phi(r, \vartheta) = X(r)\; Y(\vartheta)$$

last separation does not correspond to space-time symmetry!

A lot of work has been done on the separated equations (= ODEs for fixed separation constants ω, k, λ)

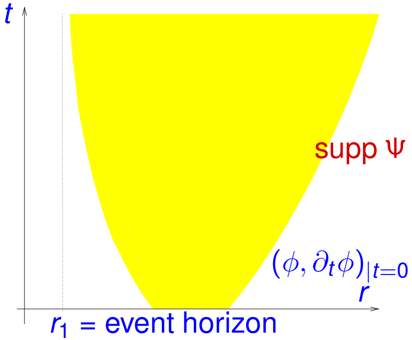
Mode Analysis

- Regge & Wheeler (1957):
  metric perturbations of Schwarzschild
  mode stability: rule out **complex** ω
- Starobinsky (1973)
  superradiance for scalar waves (see later)
- Teukolsky & Press (1973): perturbations of Weyl tensor
  show mode stability in Kerr numerically
  superradiance for higher spin
- Whiting (1989): mode stability in Kerr
  show mode stability analytically for general s

many calculations and numerics in Chandrasekhar's book

mode analysis involves no dynamics
next step: time-dependent analysis

consider the The Cauchy problem,
for simplicity for initial data with compact support



Support stays compact due to finite propagation speed

The Problem of Linear Stability for the Kerr Black Hole

Linear Stability Problem

Decay in L^∞_{loc} for solutions of the Teukolsky equation in Kerr for spin $s = 1$ or $s = 2$.

Frolov and Novikov in “Black Hole Physics” (1998):

This is one of the few truly outstanding problems that remain in the field of black hole perturbations

This problem has been solved in June this year!

- F.F., J. Smoller, “Linear Stability of the Non-Extreme Kerr Black Hole,” arXiv:1606.08005 [math-ph]
- F.F., J. Smoller, “Linear Stability of Rotating Black Holes: Outline of the Proof,” arXiv:1609.03171 [math-ph]

Formulation of Linear Stability

- Consider smooth initial data,
  compactly supported outside the event horizon, i.e.

$$\phi|_{t=0} = \phi_0 \,, \qquad \partial_t \phi|_{t=0} = \phi_1$$

  with $\phi_{0,1} \in C_0^\infty((r_1, \infty) \times S^2)$
- Decompose into azimuthal modes,

$$\phi_{0,1}(r, \vartheta, \varphi) = \sum_{k \in \mathbb{Z}} e^{-ik\varphi} \, \phi_{0,1}^{(k)}(r, \vartheta) \,.$$

Theorem (F.F., Joel Smoller (2016))

For any $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$ and every $k \in \mathbb{Z}/2$, the solution of the Teukolsky equation with initial data $\phi_{0,1}^{(k)}$ decays to zero in $L^\infty_{\text{loc}}((r_1, \infty) \times S^2)$.

The Cauchy Problem for Scalar Waves

In preparation consider the case $s = 0$ of the scalar wave equation

$$\Box \Phi = 0$$

(F-Kamran-Smoller-Yau 2005 and subsequent papers)

Is Euler-Lagrange equation corresponding to the action

$$S = \int_{-\infty}^{\infty} dt \int_{r_1}^{\infty} dr \int_{-1}^{1} d(\cos \vartheta) \int_0^{\pi} d\varphi \, \mathcal{L}(\Phi, \nabla \Phi)$$

with the Lagrangian

$$\begin{aligned} \mathcal{L} \;&=\; -\Delta |\partial_r \Phi|^2 + \frac{1}{\Delta} \Big| ((r^2 + a^2)\partial_t + a\partial_\varphi)\Phi \Big|^2 \\ &\quad - \sin^2 \vartheta \, |\partial_{\cos \vartheta} \varphi|^2 - \frac{1}{\sin^2 \vartheta} \Big| (a \sin^2 \vartheta \partial_t + \partial_\varphi)\Phi \Big|^2 \end{aligned}$$

The Cauchy Problem for Scalar Waves

Noether's theorem: symmetries of $\mathcal{L} \Longleftrightarrow$ conserved quantities
symmetry under time translations gives energy

$$E = \int_{r_1}^{\infty} dr \int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} d\varphi \, \mathcal{E}$$

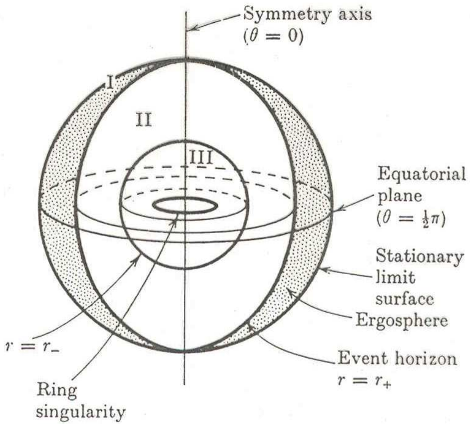
with the energy density $\mathcal{E}$,

$$\begin{aligned} \mathcal{E} = & \left(\frac{(r^2+a^2)^2}{\Delta} - a^2 \sin^2\vartheta \right) |\partial_t \Phi|^2 + \Delta \, |\partial_r \Phi|^2 \\ & + \sin^2\vartheta \, |\partial_{\cos\vartheta} \Phi|^2 + \left(\frac{1}{\sin^2\vartheta} - \frac{a^2}{\Delta} \right) |\partial_\varphi \Phi|^2 . \end{aligned}$$

the energy density may be negative!

ergosphere: $\Delta - a^2 \sin^2\vartheta < 0$, Killing field ∂_t is space-like,

The Ergosphere in the Kerr Geometry

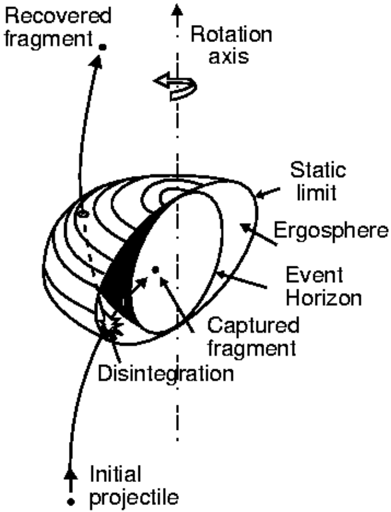


ergosphere: annular region outside the event horizon

The Penrose Process for Point Particles

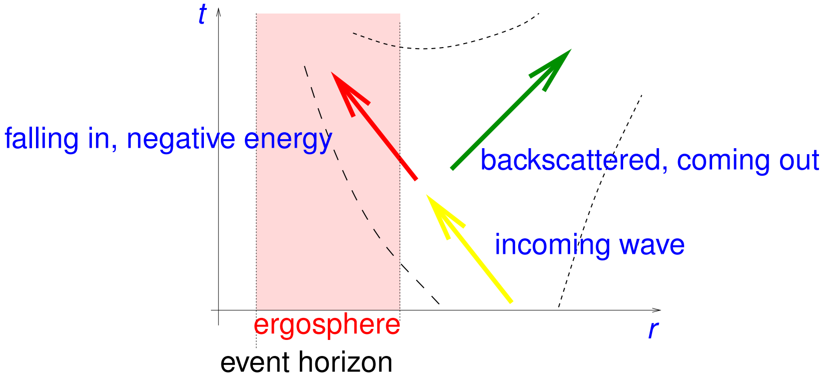
Also for classical point particles the energy can be negative

Penrose (1969), Christodoulou (1970)



Superradiance for Scalar Waves

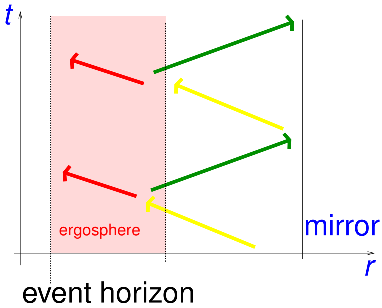
similar effect for waves: Superradiance



- mode analysis: Starobinsky (1973)
- dynamical analysis of “wave packets”:
  F.F, Niky Kamran, Joel Smoller, Shing-Tung Yau (2008)

Stability and Superradiance

superradiance is closely related to stability problem
to explain this consider
Scenario of "black hole bomb" (proposed by Cardoso)



exponential increase of amplitude leads to explosion

Is there a similar effect even without the mirror?

Now return to general spin $s \geq 0$.

Main additional difficulties:

- The Teukolsky equation for $s > 0$ cannot be derived from an action principle
- Noether's theorem cannot be applied, thus no simple form of conserved energy
- The coefficients in the PDE are complex

Proof of Linear Stability

Teukolsky equation for fixed azimuthal mode $\sim e^{-ik\varphi}$:

$$\Bigg(\frac{\partial}{\partial r} \,\Delta\, \frac{\partial}{\partial r} - \frac{1}{\Delta} \left[(r^2 + a^2) \frac{\partial}{\partial t} - iak - (r - M)s \right]^2$$
$$-4s\,(r + ia\cos\vartheta)\,\frac{\partial}{\partial t} + \frac{\partial}{\partial \cos\vartheta}\, \sin^2\vartheta\, \frac{\partial}{\partial \cos\vartheta}$$
$$+ \frac{1}{\sin^2\vartheta} \left(a\sin^2\vartheta \frac{\partial}{\partial t} - ik + is\cos\vartheta \right]^2 \Bigg)\, \phi = 0$$

partial differential equation in t, r, ϑ

Proof of Linear Stability

- Hamiltonian formulation

$$\Psi = \sqrt{r^2 + a^2} \begin{pmatrix} \phi \\ i\partial_t \phi \end{pmatrix}$$

$$i\partial_t \Psi = H\Psi$$

H is a non-symmetric operator on a Hilbert space $\mathcal{H}$.

- idea: use contour methods

$$\Psi(t) = e^{-itH}\,\Psi_0 = -\frac{1}{2\pi i} \oint_\Gamma e^{-i\omega t}\,(H-\omega)^{-1}\,\Psi_0\,d\omega$$

- Resolvent estimates

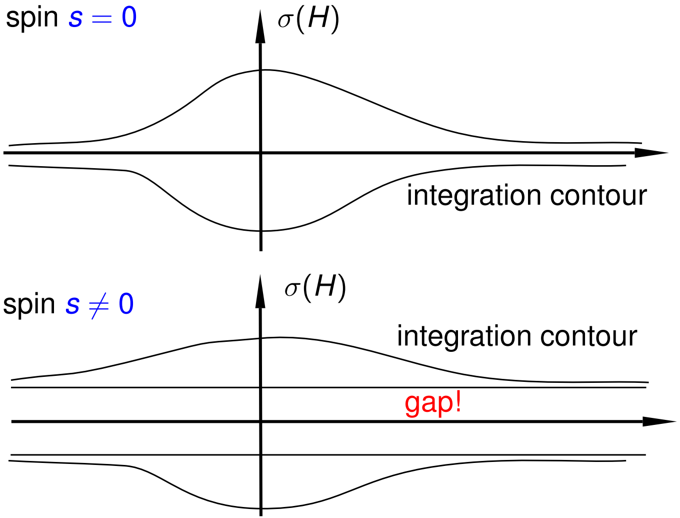
$$R_\omega := (H-\omega)^{-1}$$

exists and is bounded if

$$|\operatorname{Im}\omega| > c$$

Proof of Linear Stability

- Gives contour integral representation for $\Psi(t)$



Proof of Linear Stability

- Spectral decomposition of the angular operator

$$\mathcal{A}_k := -\frac{\partial}{\partial \cos\vartheta} \sin^2\vartheta \frac{\partial}{\partial \cos\vartheta} + \frac{1}{\sin^2\vartheta} \Big(a\omega \sin^2\vartheta + k - s\cos\vartheta \Big)^2$$

non-symmetric operator on $L^2((0,\pi), \sin\vartheta \, d\vartheta)$

Theorem (F-Smoller 2015, arXiv:1507.05756, MAA (2016))

$$|\mathrm{Im}\,\omega| < c\,.$$

- *Decomposition into invariant subspaces,*

$$\sum_{n=0}^{\infty} Q_n = \mathbb{1}\,.$$

- *Q_n are idempotent and mutually orthogonal,*

$$Q_n\, Q_{n'} = \delta_{n,n'}\, Q_n \qquad \text{for all } n, n' \in \mathbb{N} \cup \{0\}\,.$$

- *Uniform bounds: $\|Q_n\| \leq c_2$ for all ω.*
- *Uniform control of dimensions of invariant subspaces.*

Proof based on delicate ODE estimates.

- "glue together" special functions with WKB approximations
- control the error using invariant circle estimates for the Riccati equation with complex potential

$$\phi'' = V\phi$$

$$y = \frac{\phi'}{\phi} \qquad \text{satisfies} \qquad y' = V - y^2$$

- F.F., J. Smoller, "Error estimates for approximate solutions of the Riccati equation with real or complex potentials," *Arch. Rational Mech. Anal.* **197** (2010) 985-1009
- F.F., J. Smoller, "Absence of zeros and asymptotic error estimates for Airy and parabolic cylinder functions," *Commun. Math. Sci.* **12** (2014) 175-200
- F.F., J. Smoller, "Refined error estimates for the Riccati equation with applications to the angular Teukolsky equation," *Methods and Applications of Analysis* **22** (2015) 67-100

Proof of Linear Stability

- Separation of the resolvent

$$\Psi(t) = -\frac{1}{2\pi i} \int_C \sum_{n=0}^{\infty} e^{-i\omega t} \, \frac{1}{(\omega + 3ic)^p} \, \Big(R_\omega \, Q_n^\omega \, (H + 3ic)^p \, \Psi_0 \Big) \, d\omega$$

- main difficulty: control sum over angular momentum modes (note: angular equation involves ω!)
  here again use ODE methods
- deep result is Whiting's mode stability: The separated ODEs have no normalizable solutions for complex ω.
- move contour onto real axis
- rule out "radiant modes" (causality argument)
- finally apply Riemann-Lebesgue lemma:

$$f \in L^1(\mathbb{R}) \quad \Longrightarrow \quad \lim_{t \to \pm\infty} \int_{-\infty}^{\infty} f(\omega) \, e^{-i\omega t} \, d\omega = 0$$

Outlook

- next challenge: nonlinear stability
- probably requires an improvement of our linear stability result:
  - k-dependence of estimates
  - decay in weighted Sobolev spaces
  - optimal regularity assumptions on initial data
  - $\ldots, \ldots$

Thank you for your attention!