THE NONLINEAR STABILITY OF MINKOWSKI SPACE FOR SELF-GRAVITATING MASSIVE MATTER

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GENERAL RELATIVITY

- **Global geometry of spacetimes** $(M, g_{\alpha\beta})$ with signature $(-,+,+,+)$
- **Einstein equations** for self-gravitating matter $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$
  - Einstein curvature $G_{\alpha\beta} = R_{\alpha\beta} - (R/2) g_{\alpha\beta}$
  - Riccci curvature $R_{\alpha\beta} = \partial^2 g + \partial \star \partial g$
  - scalar curvature $R := R_\alpha^\alpha = g^{\alpha\beta} R_{\alpha\beta}$

CAUCHY PROBLEM

- **Global nonlinear stability of Minkowski spacetime**
  - initial data prescribed on a spacelike hypersurface
  - small perturbation of an asymptotically flat slice in Minkowski space
- **Vacuum spacetimes** $T_{\alpha\beta} = 0$ or massless matter fields
  - Christodoulou - Klainerman (1993), Lindblad - Rodnianski (2010)
- **Massive matter fields** massive matter $T_{\alpha\beta}$, open since 1993
  - LeFloch - Yue Ma (2016)

CHALLENGES

- **Gravitational waves**
  - Weyl curvature (vacuum), Ricci curvature (matter)
- **Nonlinear wave interactions**
  - exclude dynamical instabilities, self-gravitating massive modes
  - avoid gravitational collapse (trapped surfaces, black holes)
- **Global dynamics**
  - (small) perturbation disperses in timelike directions
  - asymptotic convergence to Minkowski spacetime
  - future timelike geodesically complete spacetime

MAIN STRATEGY

- **Nonlinear wave systems**
  - Einstein equations in wave gauge
  - PDE system which couples wave and Klein-Gordon equations
  - no longer scale-invariant, time-asymptotics drastically different
- **Hyperboloidal Foliation Method** PLF-YM, Monograph, 2014
  - foliation of the spacetime by asymptotically hyperbolidal slices
  - sharp time-decay estimates (metric, matter fields) for wave and Klein-Gordon equations
  - quasi-null structure of the Einstein equations

OUTLINE of the lecture

- Einstein gravity and f(R)-gravity
- Nonlinear global stability: geometric statements
- Overview of the Hyperboloidal Foliation Method
- Nonlinear global stability: statements in wave coordinates
- Quasi-null hyperboloidal structure of the Einstein equations

ELEMENTS of proof

- Second-order formulation of the f(R)-gravity theory
- Wave-Klein-Gordon systems
  - null interactions (simplest setup, better energy bounds)
  - weak metric interactions (simplest setup)
  - strong metric interactions (simpler setup)
- Sharp pointwise bounds for wave-Klein-Gordon equations on curved space

EINSTEIN GRAVITY AND F(R)-GRAVITY

Self-gravitating massive fields

Massive scalar field with potential $U(\phi)$, for instance $U(\phi) = \frac{c^2}{2}\phi^2$, described by the energy-momentum tensor

$$T_{\alpha\beta} := \nabla_\alpha \phi \nabla_\beta \phi - \Big(\frac{1}{2} g_{\alpha'\beta'} \nabla_{\alpha'} \phi \nabla^{\beta'} \phi + U(\phi) \Big) g_{\alpha\beta}$$

Einstein-Klein-Gordon system for the unknown $(M, g_{\alpha\beta}, \phi)$

$$R_{\alpha\beta} - 8\pi \Big(\nabla_\alpha \phi \nabla_\beta \phi + U(\phi)\, g_{\alpha\beta} \Big) = 0$$

$$\Box_g \phi - U'(\phi) = 0$$

Field equations of the $f(R)$-modified gravity

Generalized Hilbert-Einstein functional

- Gravitation mediated by additional fields
- Functional

$$\int_M \Big(f(R_g) + 16\pi L[\phi, g] \Big)\, dV_g$$

- $f(R) = R + \frac{\kappa}{2} R^2 + \kappa^2 \mathcal{O}(R^3)$ and $\kappa > 0$
- long history in physics: Weyl 1918, Pauli 1919, Eddington 1924, ...

Critical point equation $N_{\alpha\beta} = 8\pi T_{\alpha\beta}$

$$N_{\alpha\beta} = f'(R_g)\, G_{\alpha\beta} - \frac{1}{2}\Big(f(R_g) - R_g f'(R_g)\Big) g_{\alpha\beta} + \Big(g_{\alpha\beta}\Box_g - \nabla_\alpha\nabla_\beta\Big)\big(f'(R_g)\big)$$

- If f linear, $N_{\alpha\beta}$ reduces to $G_{\alpha\beta}$.
- Vacuum Einstein solutions are vacuum f(R)-solutions
- *Fourth-order* derivatives of g

Gravity/matter coupling

Bianchi identities (geometry) $\nabla^\alpha R_{\alpha\beta} = \frac{1}{2}\nabla_\beta R$

- imply $\nabla^\alpha G_{\alpha\beta} = 0$, but also $\nabla^\alpha N_{\alpha\beta} = 0$.
- Euler equations $\nabla^\alpha T_{\alpha\beta} = 0$

Energy-momentum tensor of a massive field

- Jordan's coupling

$$T_{\alpha\beta} := \nabla_\alpha\phi\nabla_\beta\phi - \Big(\frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi + U(\phi)\Big) g_{\alpha\beta} \qquad \Box_g\phi - U'(\phi) = 0$$

- Einstein's coupling

$$T_{\alpha\beta} := f'(R_g)\Big(\nabla_\alpha\phi\nabla_\beta\phi - \Big(\frac{1}{2}\nabla_\gamma\phi\nabla^\gamma\phi + U(\phi)\Big) g_{\alpha\beta}\Big)$$

ill-posed PDE for ϕ

WAVE-KLEIN-GORDON FORMULATION

Field equations in coordinates

Einstein equations $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

- Second-order system with no specific PDE type
- *Wave coordinates* $\Box_g x^\alpha = 0$
- Second-order system of 11 *nonlinear wave-Klein-Gordon equations*
- Hamiltonian-momentum Einstein's contraints

Modified gravity equations $N_{\alpha\beta} = 8\pi T_{\alpha\beta}$

- Fourth-order system with no specific PDE type
- The augmented formulation
  - conformal transformation $g^\dagger{}_{\alpha\beta} := f'(R_g) g_{\alpha\beta}$
  - evolution equation for the scalar curvature $\rho := \frac{1}{\kappa} \ln f'(R_g)$ (new degree of freedom)
- *Wave coordinates* $\Box_{g^\dagger} x^\alpha = 0$
- Second-order system of 12 *nonlinear wave-Klein-Gordon equations*
- More involved algebraic structure, and additional constraints

Ricci curvature in wave gauge

$2\, g^{\dagger\alpha\beta}\, \partial_\beta g^\dagger{}_{\alpha\gamma} = g^{\dagger\alpha\beta}\, \partial_\gamma g^\dagger{}_{\alpha\beta}$ (following Lindblad-Rodnianski, 2005)

$$2\, R^\dagger_{\alpha\beta} = -\widetilde{\Box}_{g^\dagger} g^\dagger{}_{\alpha\beta} + Q_{\alpha\beta} + P_{\alpha\beta}$$
$$= -g^{\dagger\alpha'\beta'}\, \partial_{\alpha'}\partial_{\beta'} g^\dagger{}_{\alpha\beta} + Q_{\alpha\beta} + P_{\alpha\beta}$$

(i) terms satisfying Klainerman's null condition (good decay in time)

$$Q_{\alpha\beta} := g^{\dagger\lambda\lambda'} g^{\dagger\delta\delta'}\, \partial_\delta g^\dagger{}_{\alpha\lambda'} \partial_{\delta'} g^\dagger{}_{\beta\lambda}$$
$$- g^{\dagger\lambda\lambda'} g^{\dagger\delta\delta'} \left(\partial_\delta g^\dagger{}_{\alpha\lambda'} \partial_\lambda g^\dagger{}_{\beta\delta'} - \partial_\delta g^\dagger{}_{\beta\delta'} \partial_\lambda g^\dagger{}_{\alpha\lambda'}\right)$$
$$+ g^{\dagger\lambda\lambda'} g^{\dagger\delta\delta'} \left(\partial_\alpha g^\dagger{}_{\lambda'\delta'} \partial_\delta g^\dagger{}_{\lambda\beta} - \partial_\alpha g^\dagger{}_{\lambda\beta} \partial_\delta g^\dagger{}_{\lambda'\delta'}\right) + \ldots\ldots\ldots\ldots$$

(ii) "quasi-null terms" (need again the gauge conditions)

$$P_{\alpha\beta} := -\tfrac{1}{2} g^{\dagger\lambda\lambda'} g^{\dagger\delta\delta'}\, \partial_\alpha g^\dagger{}_{\delta\lambda'} \partial_\beta g^\dagger{}_{\lambda\delta'} + \tfrac{1}{4} g^{\dagger\delta\delta'} g^{\dagger\lambda\lambda'}\, \partial_\beta g^\dagger{}_{\delta\delta'} \partial_\alpha g^\dagger{}_{\lambda\lambda'}$$

Notation

- Functions $V_\kappa = V_\kappa(\rho)$ and $W_\kappa = W_\kappa(\rho)$
  - defined from $f(R) \simeq R + \frac{\kappa}{2} R^2$
  - quadratic in ρ
- Quadratic potential $U(\phi) = \frac{c^2}{2}\phi^2$

f(R)-gravity for a self-gravitating massive field

$$\tilde{\Box}_{g^\dagger} g^\dagger{}_{\alpha\beta} = F_{\alpha\beta}(g^\dagger, \partial g^\dagger) - 3\kappa^2 \partial_\alpha \rho \partial_\beta \rho + \kappa V_\kappa(\rho) g^\dagger{}_{\alpha\beta}$$
$$- 8\pi \left(2 e^{-\kappa\rho} \partial_\alpha \phi \partial_\beta \phi + c^2 \phi^2 e^{-2\kappa\rho} g^\dagger{}_{\alpha\beta}\right)$$

$$3\kappa \tilde{\Box}_{g^\dagger} \rho - \rho = \kappa W_\kappa(\rho) - 8\pi \left(g^{\dagger\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + \frac{c^2}{2} e^{-\kappa\rho} \phi^2 \right)$$

$$\tilde{\Box}_{g^\dagger} \phi - c^2 \phi = c^2 \left(e^{-\kappa\rho} - 1\right) \phi + \kappa g^{\dagger\alpha\beta} \partial_\alpha \phi \partial_\beta \rho$$

- wave gauge conditions $g^{\dagger\alpha\beta} \Gamma^{\dagger\lambda}_{\alpha\beta} = 0$
- curvature compatibility $e^{\kappa\rho} = f'(R_{e^{-\kappa\rho} g^\dagger})$
- Hamiltonian and momentum constraints

(propagating from a Cauchy hypersurface)

In the limit $\kappa \to 0$ one has $g^\dagger \to g$ and $\rho \to 8\pi\left(g^{\alpha\beta} \nabla_\alpha \phi \nabla_\beta \phi + \frac{c^2}{2} \phi^2\right)$

Einstein system for a self-gravitating massive field

$$\tilde{\Box}_g g_{\alpha\beta} = F_{\alpha\beta}(g, \partial g) - 8\pi \left(2 \partial_\alpha \phi \partial_\beta \phi + c^2 \phi^2 g_{\alpha\beta}\right)$$
$$\tilde{\Box}_g \phi - c^2 \phi = 0$$

Main issues

- *Time-asymptotic decay (energy, sup-norm), global existence theory*
- *Dependence in f and singular limit $f(R) \to R$*

NONLINEAR GLOBAL STABILITY: geometric statements

Earlier works on vacuum spacetimes or massless matter

- Christodoulou-Klainerman 1993
  - fully geometric proof, Bianchi identities, geometry of null cones
  - null foliation, maximal foliation
- Lindblad-Rodnianski 2010
  - first global existence result in coordinates
  - wave coordinates (despite an "instability" result by Choquet-Bruhat)
  - asymptotically flat foliation
  - their proof relies strongly on the **scaling field** $r\partial_r + t\partial_t$ of Minkowski spacetime
- Extensions to massless models
  - same time asymptotics, same Killing fields
  - Bieri (2009), Zipser (2009), Speck (2014)

Initial value problem

- geometry of the initial hypersurface $(M_0 \simeq \mathbb{R}^3, g_0, k_0)$
- matter fields ϕ_0, ϕ_1
- initial data sets close to a spacelike, asympt. flat slice in Minkowski spacetime

Dynamics of self-gravitating massive matter

- Spatially compact problem
  - compactly supported massive scalar field
- Positive mass theorem
  - no solution can be exactly Minkowski "at infinity"
  - coincides with a slice of Schwarzschild near space infinity, with ADM mass $m << 1$
- Compact Schwarzschild perturbation

Theorem 1. Nonlinear stability of Minkowski spacetime with self-gravitating massive fields

Consider the Einstein-massive field system when the initial data set $(M_0 \simeq \mathbb{R}^3, g_0, k_0, \phi_0, \phi_1)$ is a compact Schwarzschild perturbation satisfying the Einstein constraint equations. Then, the initial value problem

- admits a globally hyperbolic Cauchy development,
- which is foliated by asymptotically hyperbolic hypersurfaces.
- Moreover, this spacetime is future causally geodesically complete and asymptotically approaches Minkowski spacetime.

Theorem 2. Nonlinear stability of Minkowski spacetime in f(R)-gravity

Consider the field equations of $f(R)$-modified gravity when the initial data set $(M_0 \simeq \mathbb{R}^3, g_0, k_0, R_0, R_1, \phi_0, \phi_1)$ is a compact Schwarzschild perurbation satisfying the constraint equations of modified gravity. Then, the initial value problem

- admits a globally hyperbolic Cauchy development,
- which is foliated by asymptotically hyperbolic hypersurfaces.
- Moreover, this spacetime is future causally geodesically complete and asymptotically approaches Minkowski spacetime.

Limit problem $\kappa \to 0$

- relaxation phenomena for the spacetime scalar curvature
- passing from a second-order wave equation to an algebraic equation

Theorem 3. f(R)-spacetimes converge toward Einstein spacetimes

The Cauchy developments of modified gravity in the limit $\kappa \to 0$

> when the nonlinear function $f = f(R)$ (the integrand in the Hilbert-Einstein action) approaches the scalar curvature function R

converge (in every bounded time interval, in a sense specified quantitatively in Sobolev norms) to Cauchy developments of Einstein's gravity theory.

OVERVIEW OF THE HYPERBOLOIDAL FOLIATION METHOD

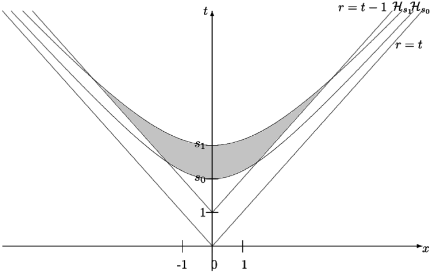
- LeFloch & Ma, monograph published by World Scientific, 2014
- Earlier work by Klainerman and Hormander for Klein-Gordon

Foliations by asymptotically hyperboloidal hypersurfaces

- global coordinate chart $(x^\alpha) = (t, x^a)$ with $a = 1, 2, 3$
- boosts $L_a := x^a \partial_t + t\partial_a$ associated with the Minkowski metric $g_M = -dt^2 + \sum_{a=1,2,3} dx^a$
- foliation of the interior of the light cone by hyperboids

Notation

- foliation of the future light cone from $(t, x) = (1, 0)$
- level sets of constant Lorentzian distance from the origin $(0, 0)$
- hyperboloids $\mathcal{H}_s := \left\{(t, x) \,/\, t > 0;\ t^2 - |x|^2 = s^2\right\}$
- parametrized by their hyperbolic radius $s \geqslant 1$
- data prescribed on $\mathcal{H}_{s_0}$ for some $s_0 > 1$



Vector frames in addition to the frame (∂_t, ∂_a)

Semi-hyperboloidal frame	$\underline{\partial}_0 := \partial_t$	$\underline{\partial}_a := \dfrac{L_a}{t} = \dfrac{x^a}{t}\partial_t + \partial_a$
Hyperboloidal frame	$\overline{\partial}_0 := \partial_s$	$\overline{\partial}_a = \underline{\partial}_a$
Change of frame	$\underline{\partial}_\alpha = \underline{\Phi}_\alpha^{\alpha'}\partial_{\alpha'}$	$\partial_\alpha = \underline{\Psi}_\alpha^{\alpha'}\underline{\partial}_{\alpha'}$
Tensor components	$\underline{T}_{\alpha\beta} = T_{\alpha'\beta'}\underline{\Phi}_\alpha^{\alpha'}\underline{\Phi}_\beta^{\beta'}$	

Energy functional on hyperboloids of Minkowski spacetime

- Semi-hyperboloidal decomposition of the wave operator

$$\Box u = -\frac{s^2}{t^2}\underline{\partial}_0\underline{\partial}_0 u - \frac{3}{t}\partial_t u - \frac{x^a}{t}\left(\underline{\partial}_0\underline{\partial}_a u + \underline{\partial}_a\underline{\partial}_0 u\right) + \sum_a \underline{\partial}_a\underline{\partial}_a u$$

- For instance, for the linear Klein-Gordon operator in Minkowski space

$$\Box u - c^2\, u$$

$u = u(t,x) = u(s,x)$ with $s^2 = t^2 - r^2$ and $r^2 = \sum_a (x^a)^2$

$$E_{m,c}[s,u] := \int_{\mathcal{H}_s} \left(\frac{s^2}{t^2}(\partial_t u)^2 + \sum_{a=1}^{3} \left(\frac{x^a}{t}\partial_t u + \partial_a u \right)^2 + \frac{c^2}{2}\, u^2 \right) dx$$
$$= \int_{\mathcal{H}_s} \left(\frac{s^2}{s^2 + r^2}(\underline{\partial}_0 u)^2 + \sum_{a=1}^{3} (\underline{\partial}_a u)^2 + \frac{c^2}{2}\, u^2 \right) dx$$

Functional analysis for hyperboloidal foliations

- Decompose the wave operators, the metric, etc. in various frames
- Good commutator properties
- Weighted norms based on the translations ∂_α and the Lorentzian boosts L_a only

Weighted norms

- On each hypersurface

$$\|u\|_{\mathcal{H}^n[s]} := \sum_{|J|\leqslant n} \sum_{a=1,2,3} \Big(\int_{\mathcal{H}_s \simeq \mathbb{R}^3} |L_a^J u|^2 \, dx \Big)^{1/2}$$

  - completion of smooth and spatially compacted functions

- In spacetime

$$\|u\|_{\mathcal{H}^N[s_0,s_1]} := \sup_{s\in[s_0,s_1]} \sum_{|I|+n\leqslant N} \left\|\partial^I u\right\|_{\mathcal{H}^n[s]}$$

  - Here, $1 \leqslant s_0 < s_1 < +\infty$ and $N \geqslant 0$
  - for each $s \in [s_0, s_1]$ and for all multi-index $|I| = m \leqslant N$, one has $\partial^I u(s,\cdot) \in \mathcal{H}^{N-m}[s]$.

Klainerman (1985), Hormander (1997)

For functions u defined on $\mathcal{H}_s \subset \mathbb{R}^{3+1}$ (with $t^2 = s^2 + |x|^2$):

$$\sup_{(t,x)\in\mathcal{H}_s} t^{3/2}\,|u(t,x)| \lesssim \|u\|_{\mathcal{H}^2[s]} \simeq \sum_{|I|\leqslant 2} \|L^I u\|_{\mathcal{H}_s}$$

LeFloch-Ma (2014)

For all functions u defined on a hyperboloid $\mathcal{H}_s$:

$$\left\|\frac{u}{r}\right\|_{L^2_f(\mathcal{H}_s)} \lesssim \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)} \qquad \text{with } \underline{\partial}_a = t^{-1} L_a$$

For all functions defined on the hyperboloidal foliation

$$\left\|\frac{u}{s}\right\|_{L^2_f(\mathcal{H}_s)} \lesssim \left\|\frac{u}{s_0}\right\|_{L^2_f(\mathcal{H}_{s_0})} + \sum_a \|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_s)}$$
$$+ \sum_a \int_{s_0}^{s} \Big(\|\underline{\partial}_a u\|_{L^2_f(\mathcal{H}_{s'})} + \|(s'/t)\partial_a u\|_{L^2_f(\mathcal{H}_{s'})} \Big) \frac{ds'}{s'}$$

Remark.

- Compute the divergence of the vector field $\left(0, \frac{t\,x^a u^2}{(1+r^2)s^2}\chi(r/t)\right)$ for some smooth cut-off function concentrated near the light cone $\chi(y) = \begin{cases} 0 & 0 \leqslant y \leqslant 1/3 \\ 1 & 2/3 \leqslant y \end{cases}$
- Similar to integrating $\partial_r\big(u^2/r\big)$ for the classical Hardy inequality

Initial value problem

- Initial data prescribed on an asymptotically hyperbolic hypersurface, identified with $\mathcal{H}_{s_0}$ in our coordinates
- Energy estimates expressed in domains limited by two hyperboloids

Bootstrap

- time-integrability of the source terms
- total contribution of the interaction terms contribute only a finite amount to the growth of the total energy

Model formally extracted from the Einstein-massive field system

- handle "strong interactions" between the metric and the matter, say

$$\begin{aligned} -\Box u &= P^{\alpha\beta}\partial_\alpha v \partial_\beta v + R v^2 \\ -\Box v + u\, H^{\alpha\beta}\partial_\alpha\partial_\beta v + c^2 v &= 0 \end{aligned}$$

- hierarchy of energy bounds of various order of differentiation / growth in s
- successive improvements of the sup-norm bounds, via successive applications of sup-norm estimates

Pointwise decay of solutions

- sup-norm bounds for nonlinear wave equations and nonlinear Klein-Gordon equations on curved space
- **sharp rates** required
- several L^∞-L^∞ estimates
  - integration along radial rays (curved space, Klein-Gordon)
  - integration along characteristics (curved space, wave equation)
  - Kirchhoff explicit formula (flat space, wave equation)

1. A sharp L^∞ estimate for Klein-Gordon equations on curved space

- Introduce the vector field $\underline{\partial}_\perp := \partial_t + \frac{x^a}{t}\partial_a$
- orthogonal to the hyperboloids and proportional to the scaling vector field
- second-order ODE's along radial rays from the origin
- Klainerman (1985), but not optimal time rate; Delort etal. (2004) in two spatial dimensions

Proposition

Solutions of the Klein-Gordon equation $3\kappa\tilde{\Box}_{g^\dagger}\rho - \rho = \sigma$

$$\kappa^{-1/2}s^{3/2}\left|(\rho-\sigma)(t,x)\right| + (s/t)^{-1}s^{1/2}\left|\underline{\partial}_\perp(\rho-\sigma)(t,x)\right| \lesssim V(t,x)$$

- the implied constant *independent* of $\kappa \in [0,1]$
- $V = V(s,x)$ determined from the metric $g^\dagger$ and the right-hand side σ

2. A sharp L^∞ estimate for wave equations on curved space

- Start with a decomposition of the flat wave operator $\Box$ in the semi-hyperboloidal frame as

$$\Box = -\frac{s^2}{t^2}\partial_t\partial_t - \frac{3}{t}\partial_t + \underline{\partial}^a\underline{\partial}_a - \frac{x^a}{t}\left(\partial_t\underline{\partial}_a + \underline{\partial}_a\partial_t\right)$$

- Given the curved metric $g^{\dagger\alpha\beta} = g_M^{\alpha\beta} + H^{\alpha\beta}$, consider the modified wave operator

$$\tilde{\Box}_{g^\dagger} = \Box + H^{\alpha\beta}\partial_\alpha\partial_\beta$$

- Decomposition for the curved wave operator

$$\Big((\partial_t + \partial_r) - t^2(t+r)^{-2}\underline{H}^{00}(\partial_t - \partial_r)\Big)\Big((\partial_t - \partial_r)(ru)\Big)$$
$$= -r\tilde{\Box}_g u + r\sum_{a<b}\left(r^{-1}\Omega_{ab}\right)^2 u + \underline{H}^{00}X[u] + r\,Y[u,H]$$

- $\underline{H}^{00}$ is the $(0,0)$-component of the metric perturbation in the semi-hyperboloidal frame
- $X[u]$ and $Y[u,H]$ involve
  - derivatives tangential to the hyperboloids
  - metric component $\underline{H}^{a\alpha}$
  - independent of $\underline{H}^{00}$

Solutions to the wave equation $-\Box u - H^{\alpha\beta}\partial_\alpha\partial_\beta u = F$

- in which $|\underline{H}^{00}(t,x)| \leqslant \epsilon \frac{t-r+1}{t}$
- given any point $(s_1, x_1) \equiv (t_1, x)$ on an arbitrary hyperboloid $\mathcal{H}_{s_1}$
- denote by $s \in [1, s_1] \mapsto \big(s, \varphi(s; s_1, x_1)\big)$ the characteristic integral curve leaving from (s_1, x_1) associated with the vector field

$$\partial_t + \frac{(t+r)^2 + t^2\underline{H}^{00}(t,x)}{(t+r)^2 - t^2\underline{H}^{00}(t,x)}\,\partial_r.$$

- Integration along this curve

(Lindblad and Rodnianski used a different vector field)

Proposition

$$|(\partial_t - \partial_r)u(t,x)| \lesssim t^{-1}\sup_{\mathcal{H}_1}\Big(|(\partial_t - \partial_r)(ru)|\Big) + t^{-1}|u(t,x)|$$
$$+ t^{-1}\int_{t_0}^{t}\big(|F| + |M|\big)(\tau, \varphi(\tau; t, x))\,d\tau$$

- t_0 is the initial time reached on the hyperboloid $\mathcal{H}_1$ from (t,x)
- $M := r\sum_{a<b}\big(r^{-1}\Omega_{ab}\big)^2 u + \underline{H}^{00}X[u] + rY[u,H]$

3. A sharp L^∞ estimate for the wave equation in Minkowski space

- in Minkowski spacetime, we can rely on Kichhoff formula

Proposition

Let u be a spatially compactly supported to the wave equation

$$-\Box u = f,$$
$$u|_{t=2} = 0, \qquad \partial_t u|_{t=2} = 0,$$

where the source f is spatially compactly supported and satisfies

$$|f| \leqslant C_f t^{-2-\nu}(t-r)^{-1+\mu}$$

for some constants $C_f > 0$, $0 < \mu \leqslant 1/2$, and $0 < |\nu| \leqslant 1/2$. Then, one has

$$|u(t,x)| \lesssim \begin{cases} \frac{C_f}{\nu\mu}(t-r)^{\mu-\nu}t^{-1}, & 0 < \nu \leqslant 1/2 \\ \frac{C_f}{|\nu|\mu}(t-r)^{\mu}t^{-1-\nu}, & -1/2 \leqslant \nu < 0 \end{cases}$$

The Wave-Klein-Gordon Model

$$-\Box u = P^{\alpha\beta}\partial_\alpha v \partial_\beta v + Rv^2 \qquad -\Box v + u\, H^{\alpha\beta}\partial_\alpha\partial_\beta v + c^2 v = 0$$

Theorem. Global existence theory for the wave-Klein-Gordon model

Given any integer $N \geqslant 8$, there exists a positive constant $\epsilon_0 = \epsilon_0(N) > 0$ such that if the compactly supported initial data satisfy

$$\|(u_0, v_0)\|_{H^{N+1}(\mathbb{R}^3)} + \|(u_1, v_1)\|_{H^N(\mathbb{R}^3)} \leqslant \epsilon_0,$$

then the associated Cauchy problem admits a global-in-time solution.

- **hierarchy of energy bounds**

$$E_m(s, \partial^I L^J u)^{1/2} \leqslant C_1 \epsilon s^{k\delta}, \qquad |J| = k, \quad |I| + |J| \leqslant N \text{ (wave / high-order)}$$

$$E_m(s, \partial^I L^J u)^{1/2} \leqslant C_1 \epsilon, \qquad |I| + |J| \leqslant N - 4 \text{ (wave / low-order)}$$

$$E_{m,c^2}(s, \partial^I L^J v)^{1/2} \leqslant C_1 \epsilon s^{1/2 + k\delta}, \qquad |J| = k, \quad |I| + |J| \leqslant N \text{ (KG / high-order)}$$

$$E_{m,c^2}(s, \partial^I L^J v)^{1/2} \leqslant C_1 \epsilon s^{k\delta}, \qquad |J| = k, \quad |I| + |J| \leqslant N - 4 \text{ (KG / low-order)}$$

- **successive improvements** of the sup-norm bounds, via successive applications of the sup-norm estimates above

- **refined pointwise estimate** $|L^J h_{\alpha\beta}| \lesssim \epsilon t^{-1} s^{C\epsilon^{1/2}}$

$$(s/t)^{-2+3\delta}|\partial^I L^J \phi| + (s/t)^{-3+3\delta}|\partial^I L^J \underline{\partial}_\perp \phi| \lesssim \epsilon s^{-3/2 + C\epsilon^{1/2}}$$

Structure of the Einstein equations

- Quadratic nonlinearities in ∂g: involved algebraic structure
- Einstein and f(R)-gravity do not satisfy the null condition (Lindblad-Rodnianski, 2005)
- detailled analysis of the field equations
- here, we work with a hyperboloidal foliation
  - The quasi-null structure $(\underline{P}_{00}, \underline{P}_{a\beta})$
  - The wave gauge condition used to control the component $\underline{\partial_t h}^{00}$
- Further details given below.

Recall the notation

Semi-hyperboloidal frame	$\underline{\partial}_0 := \partial_t$	$\underline{\partial}_a := \frac{L_a}{t} = \frac{x^a}{t}\partial_t + \partial_a$
Hyperboloidal frame	$\overline{\partial}_0 := \partial_s$	$\overline{\partial}_a = \underline{\partial}_a$
Change of frame	$\underline{\partial}_\alpha = \underline{\Phi}_\alpha^{\alpha'} \partial_{\alpha'}$	$\partial_\alpha = \underline{\Psi}_\alpha^{\alpha'} \underline{\partial}_{\alpha'}$
Tensor components	$\underline{T}_{\alpha\beta} = T_{\alpha'\beta'} \underline{\Phi}_\alpha^{\alpha'} \underline{\Phi}_\beta^{\beta'}$	

NONLINEAR STABILITY: Statements in wave coordinates

Theorem 1. Nonlinear stability of Minkowski spacetime for self-gravitating massive fields

Consider the Einstein-massive field system in wave coordinates. Given any sufficiently large integer N, there exist constants $\epsilon_0, \delta, C_0 > 0$ such that the following property holds.

Consider an asymptotically hyperboloidal initial data set $(\mathbb{R}^3, \overline{g}_0, \overline{k}_0, \phi_0, \phi_1)$ coinciding with Schwarzschild outside a compact set and satisfying Einstein's Hamiltonian and momentum constraints together with the smallness conditions $(\epsilon \leqslant \epsilon_0)$

$$\|\overline{\partial}_c(\overline{g}_{0,ab} - \overline{g}_{M,ab})\|_{\mathcal{H}^N[1]} + \|\overline{k}_{0,ab} - \overline{k}_{M,ab}\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$

$$\|\overline{\partial}_a\phi_0, \phi_0, \phi_1\|_{\mathcal{H}^N[1]} \leqslant \epsilon.$$

Then the solution exists globally for all times $s \geqslant 1$

$$\|\overline{\partial}_\gamma(g_{\alpha\beta} - g_{M,\alpha\beta})\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^\delta$$

$$\|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^{\delta+1/2} \qquad \text{(high-order energy)}$$

$$\|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^{N-4}[1,s]} \leqslant C_0\epsilon s^\delta \qquad \text{(low-order energy)}$$

First global stability theory

- large class of spacetimes containing massive matter
- Einstein gravity, as well as f(R)-gravity theory (see below)
- sufficient decay so that the spacetime is future geodesically complete
- smallness conditions on both g, ϕ necessary (gravitational collapse)

Energy may grow in time

- exponent such that $\limsup_{\epsilon \to 0} \delta(\epsilon) = 0$
- t^δ observed by Alinhac (2006) for some semilinear hyperbolic systems
- $s^{\delta+1/2}$ for the scalar field

Work in preparation

- asymptotically Schwarzschild data

$$g_{ab} = \delta_{ab}\Big(1 + \frac{2m}{r}\Big) + O(r^{-1-\delta}), \qquad k_{ab} = O(r^{-2-\delta}), \qquad \phi = O(r^{-1-\delta})$$

- spacetime weight outside the light cone

$w = 1$ for $r \leqslant t$, while $w = (1 + |r - t|)^{1+\delta}$ for $r \geqslant t$

Theorem 2. Nonlinear stability of Minkowski spacetime in modified gravity

Consider the field equations of modified gravity in the augmented conformal formulation and in conformal wave coordinates. Given any sufficiently large integer N and some fixed $\kappa \in [0,1]$, there exist constants $\epsilon_0, \delta, C_0 > 0$ such the following property holds.

Consider an asymptotically hyperboloidal initial data coinciding with Schwarzschild outside a a compact set, $(\mathbb{R}^3, \overline{g}^\dagger{}_0, \overline{k}^\dagger{}_0, \rho_0, \rho_1, \phi_0, \phi_1)$, satisfying the constraints of modified gravity with and

$$\|\overline{\partial}_c\big(\overline{g}^\dagger_{0,ab} - \overline{g}_{M,ab}\big)\|_{\mathcal{H}^N[1]} + \|\overline{k}^\dagger_{0,ab} - \overline{k}_{M,ab}\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$
$$\|\overline{\partial}_a\rho_0, \rho_0, \rho_1\|_{\mathcal{H}^N[1]} + \|\overline{\partial}_a\phi_0, \phi_0, \phi_1\|_{\mathcal{H}^N[1]} \leqslant \epsilon.$$

Then, the solution exists globally for all times $s \geqslant 1$

$$\|\overline{\partial}_\gamma\big(g^\dagger_{\alpha\beta} - g_{M,\alpha\beta}\big)\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^\delta$$
$$\|\overline{\partial}_\alpha\rho, \rho\|_{\mathcal{H}^N[1,s]} + \|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^{\delta+1/2} \qquad \text{(high-order energy)}$$
$$\|\overline{\partial}_\alpha\rho, \rho\|_{\mathcal{H}^{N-4}[1,s]} + \|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^{N-4}[1,s]} \leqslant C_0\epsilon s^\delta \qquad \text{(low-order energy)}$$

with constants possibly *depending upon* κ.

Notation: $\sigma := 8\pi\big(g^{\dagger\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \frac{c^2}{2}\, e^{-\kappa\rho}\phi^2\big)$

Theorem 3. The singular limit problem for the modified gravity equations

Consider a sequence of initial data sets depending upon $\kappa \to 0$, as follows:

$$\|\overline{\partial}_c\big(\overline{g}^{\dagger}_{0,ab} - \overline{g}_{M,ab}\big)\|_{\mathcal{H}^N[1]} + \|\overline{k}^{\dagger}_{0,ab} - \overline{k}_{M,ab}\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$

$$\|\kappa^{1/2}\rho_1, \kappa^{1/2}\overline{\partial}_a\rho_0, \rho_0\|_{\mathcal{H}^N[1]} + \|\overline{\partial}_a\phi_0, \phi_0, \phi_1\|_{\mathcal{H}^N[1]} \leqslant \epsilon$$

$$\|\rho_1 - \sigma_1, \overline{\partial}_a(\rho_0 - \sigma_0), \kappa^{-1/2}(\rho_0 - \sigma_0)\|_{\mathcal{H}^{N-2}[1]} \leqslant \epsilon.$$

Then, the solutions exist for all times $s \geqslant 1$ and all $\kappa \to 0$, with a constant C_0 **independent** of κ

$$\|\overline{\partial}_\gamma\big(g^{\dagger}_{\alpha\beta} - g_{M,\alpha\beta}\big)\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^{\delta}$$

$$\|\kappa^{1/2}\overline{\partial}_\alpha\rho, \rho\|_{\mathcal{H}^N[1,s]} + \|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^N[1,s]} \leqslant C_0\epsilon s^{\delta+1/2}$$

$$\|\kappa^{1/2}\overline{\partial}_\alpha\rho, \rho\|_{\mathcal{H}^{N-4}[1,s]} + \|\overline{\partial}_\alpha\phi, \phi\|_{\mathcal{H}^{N-4}[1,s]} \leqslant C_0\epsilon s^{\delta}$$

$$\|\overline{\partial}_\alpha(\rho - \sigma), \kappa^{-1/2}(\rho - \sigma)\|_{\mathcal{H}^{N-2}[1,s]} \leqslant C_0\epsilon s^{\delta+1/2}$$

$$\|\overline{\partial}_\alpha(\rho - \sigma), \kappa^{-1/2}(\rho - \sigma)\|_{\mathcal{H}^{N-6}[1,s]} \leqslant C_0\epsilon s^{\delta}$$

Moreover, if

- the initial data set $\left(\overline{g}^{\dagger(\kappa)}, \overline{k}^{\dagger(\kappa)}, \rho_0^{\dagger(\kappa)}, \rho_1^{\dagger(\kappa)}, \phi_0^{\dagger(\kappa)}, \phi_1^{\dagger(\kappa)}\right)$ converges to some limit $\left(\overline{g}^{(0)}, \overline{k}^{(0)}, \rho_0^{(0)}, \rho_1^{(0)}, \phi_0^{(0)}, \phi_1^{(0)}\right)$
- in the norms associated with the uniform bounds above

then

- the corresponding solutions $\left(g^{\dagger(\kappa)}, \rho^{\dagger(\kappa)}, \phi^{\dagger(\kappa)}\right)$ to the system of modified gravity **converge to a solution** $\left(g^{(0)}, \phi^{(0)}\right)$ **of the Einstein-massive field system**

with, in particular, in the $\mathcal{H}^{N-2}$ norm on each compact set in time

$$\rho^{(\kappa)} \to R^{(0)} := 8\pi \left(g^{(0)\alpha\beta}\partial_\alpha \phi^{(0)} \partial_\beta \phi^{(0)} + \frac{c^2}{2} (\phi^{(0)})^2\right) \quad \text{as } \kappa \to 0.$$

Remarks.

- The convergence property above relates a *fourth-order system* to a *second-order system* of PDEs.
- The highest ($(N+1)$-th order) derivatives of the scalar curvature are $O(\varepsilon\kappa^{-1/2})$ in L^2 and may blow-up when $\kappa \to 0$, while the N–th order derivatives are solely bounded and need not converge in a strong sense.
- Throughout, the initial data set of modified gravity (and thus the solution to the field equations) satisfies the compatibility condition $e^{\kappa\rho} = f'(R_{e^{-\kappa\rho} g^\dagger})$ relating the augmented variable ρ to the spacetime scalar curvature.

CONCLUDING REMARKS

- **Application of the Hyperboloidal Foliation Method**
  - Encompass a large class of nonlinear wave-Klein-Gordon systems with *quasi-null coupling*

  Alinhac, Lindblad on asymptotically Euclidian foliations

- **Fully geometric construction**
  - Improve the growing rate $s^{1/2}$ for the scalar field
  - Additional arguments, hyperboloidal foliation based on the curved metric

  Q. Wang by generalizing Christodoulou-Klainerman's geometric method

- **Extension to other massive fields**
  - Kinetic models (density), Vlasov equa. (collisionless), Boltzmann equa.

  Fajman, Joudioux, Smulevici

- **Penrose's peeling estimates**
  - Asymptotics for the spacetime curvature along timelike directions

    Penrose, Christodoulou-Klainerman

  - Very challenging open problem in wave coordinates

    Lindblad-Rodnianski

  - Our *Hyperboloidal Foliation Method* provides a possible path to establishing the peeling estimates directly in wave gauge.
  - For instance, for nonlinear wave systems with null forms and without metric coupling
    - proof *simpler* than the standard one based on flat hypersurfaces
    - *uniform energy bound* for the highest-order energy

P.G. LeFloch and Y. Ma

- *The hyperboloidal foliation method,* World Scientific, 2014
- *The nonlinear stability of Minkowski space for self-gravitating massive fields*
  - *A wave-Klein-Gordon model* Comm. Math. Phys. (2016) ArXiv:1507.01143
  - *Analysis of the Einstein equations* ArXiv:1511.03324
  - *Analysis of the $f(R)$-theory of modified gravity* ArXiv:1412.8151 Comptes Rendus Acad. Sc. Paris (2016)

+ article under completion

WAVE EQUATIONS WITH NULL INTERACTIONS

The simplest model

$$\Box u = P^{\alpha\beta} \partial_\alpha u \, \partial_\beta u \qquad u|_{\mathcal{H}_{s_0}} = u_0, \quad \partial_t u|_{\mathcal{H}_{s_0}} = u_1 \qquad (\star)$$

- initial data u_0, u_1 compactly supported in the intersection of the spacelike hypersurface $\mathcal{H}_{s_0}$ and the cone $\mathcal{K} = \{(t,x) \, / \, |x| < t-1\}$ with $s_0 > 1$
- standard null condition: $P^{\alpha\beta}\xi_\alpha\xi_\beta = 0$ for all $\xi \in \mathbb{R}^4$ satisfying $-\xi_0^2 + \sum_a \xi_a^2 = 0$
- hyperboloidal energy $E_M = E_{M,0}$: Minkowski metric and zero K-G mass
- admissible vector fields $Z \in \mathscr{Z}$: spacetime translations ∂_α, boosts L_a

Theorem. Global existence theory for wave equations with null interactions

There exist $\varepsilon_0 > 0$ and $C_1 > 1$ such that for all initial data satisfying

$$\sum_{|I| \leqslant 3} \sum_{Z \in \mathscr{Z}} E_M(s_0, Z^I u)^{1/2} \leqslant \varepsilon \leqslant \varepsilon_0$$

the Cauchy problem $(\star)$ admits a global-in-time solution, satisfying the uniform energy bound

$$\sum_{|I| \leqslant 3} \sum_{Z \in \mathscr{Z}} E_M(s, Z^I u)^{1/2} \leqslant C_1 \varepsilon$$

and the uniform decay estimate $\left|\partial_\alpha u(t,x)\right| \leqslant \frac{C_1 \varepsilon}{t\,(t-|x|)^{1/2}}$.

THE QUASI-NULL HYPERBOLOIDAL STRUCTURE OF THE EINSTEIN EQUATIONS

The hierarchy property of quasi-null terms in the hyperboloidal foliation

Proposition

$$\begin{aligned}\tilde{\Box}_g \underline{h}_{00} \simeq\ & \mathbb{GS}_{hh}(0,0) + \underline{P}_{00} - \left(2\kappa\rho^2 + 8\pi c^2\phi^2\right)\underline{g}_{M,00} - 3\kappa^2\partial_t\rho\partial_t\rho \\ & - 16\pi\partial_t\phi\partial_t\phi + \mathbb{C}(0,0)\end{aligned}$$

$$\tilde{\Box}_g \underline{h}_{0a} \simeq \frac{2}{t}\underline{\partial}_a h_{00} - \frac{2x^a}{t^3}h_{00} + \mathbb{GS}_{hh}(0,0) + \mathbb{S}_{\rho\rho}(0,0) + \mathbb{S}_{\phi\phi}(0,0) + \mathbb{C}(0,0)$$

$$\begin{aligned}\tilde{\Box}_g \underline{h}_{aa} \simeq\ & \frac{4x^a}{t^2}\underline{\partial}_a h_{00} + \left(\frac{2}{t^2} - \frac{6|x^a|^2}{t^4}\right)h_{00} + \frac{4}{t}\underline{\partial}_a h_{0a} - \frac{4x^a}{t^3}h_{0a} \\ & - \left(2\kappa\rho^2 + 8\pi c^2\phi^2\right)\underline{g}_{M,aa} + \mathbb{GS}_{hh}(0,0) + \mathbb{S}_{\rho\rho}(0,0) + \mathbb{S}_{\phi\phi}(0,0) + \mathbb{C}(0,0)\end{aligned}$$

$$\begin{aligned}\tilde{\Box}_g \underline{h}_{ab} \simeq\ & \frac{2}{t^2}\left(x^b\underline{\partial}_a h_{00} + x^a\underline{\partial}_b h_{00}\right) - \frac{6x^ax^b}{t^4}h_{00} + \frac{2}{t}\underline{\partial}_a h_{0b} - \frac{2x^a}{t^3}h_{0b} + \frac{2}{t}\underline{\partial}_a h_{0a} - \frac{2x^b}{t^3}h_{0a} \\ & - \left(2\kappa\rho^2 + 8\pi c^2\phi^2\right)\underline{g}_{M,ab} + \mathbb{GS}_{hh}(0,0) + \mathbb{S}_{\rho\rho}(0,0) + \mathbb{S}_{\phi\phi}(0,0) + \mathbb{C}(0,0)\end{aligned}$$

with $a \neq b$.

Observations

- The quasi-null terms $P_{\alpha\beta}$ in the semi-hyperboloidal frame read

$$\underline{P}_{\alpha\beta} = \frac{1}{4} g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \underline{\partial}_\alpha h_{\gamma\delta} \underline{\partial}_\beta h_{\gamma'\delta'} - \frac{1}{2} g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \underline{\partial}_\alpha h_{\gamma\gamma'} \underline{\partial}_\beta h_{\delta\delta'}.$$

- Null terms: $g^{\dagger\alpha\beta}\partial_\alpha u \partial_\beta v$ and $\partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v$
- All the components $\underline{P}_{\alpha\beta}$ *except* $\underline{P}_{00}$ are "good terms" (null terms):
  - they involve at least one derivative tangential to the hyperboloids,
  - plus terms having more decay in time (change of frame).

Proposition

Up to irrelevant multiplicative coefficients, one has

$$\begin{aligned}
\underline{P}_{a\beta} &\simeq \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0) \\
\underline{P}_{00} &\simeq \underline{g}^{\dagger\gamma\gamma'} \underline{g}^{\dagger\delta\delta'} \partial_t \underline{h}_{\gamma\gamma'} \partial_t \underline{h}_{\delta\delta'} + \overline{g}_M^{\gamma\gamma'} \overline{g}_M^{\delta\delta'} \partial_t \underline{h}_{\gamma\delta} \partial_t \underline{h}_{\gamma'\delta'} \\
&\quad + \mathbb{GS}_{hh}(0,0) + \mathbb{F}(0,0) + \mathbb{C}(0,0)
\end{aligned}$$

Proof. Observe that

$$\underline{P}_{00} = \frac{1}{4} g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \partial_t h_{\gamma\delta} \partial_t h_{\gamma'\delta'} - \frac{1}{2} g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \partial_t h_{\gamma\gamma'} \partial_t h_{\delta\delta'}$$
$$\simeq \frac{1}{4} \underline{g}^{\dagger\gamma\gamma'} \underline{g}^{\dagger\delta\delta'} \partial_t \underline{h}_{\gamma\delta} \partial_t \underline{h}_{\gamma'\delta'} - \frac{1}{2} \underline{g}^{\dagger\gamma\gamma'} \underline{g}^{\dagger\delta\delta'} \partial_t \underline{h}_{\gamma\gamma'} \partial_t \underline{h}_{\delta\delta'} + \mathbb{F}(0,0),$$

where we have changed the frame (for instance for the first term):

$$\begin{aligned} g^{\dagger\gamma\gamma'} g^{\delta\delta'} \partial_t h_{\gamma\delta} \partial_t h_{\gamma'\delta'} =& \underline{g}^{\dagger\gamma\gamma'} \underline{g}^{\delta\delta'} \partial_t \underline{h}_{\gamma\delta} \partial_t \underline{h}_{\gamma'\delta'} \\ &+ g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \underline{h}_{\gamma''\delta''} \partial_t \left(\underline{\Psi}_\gamma^{\gamma''} \underline{\Psi}_\delta^{\delta''} \right) \partial_t \left(\underline{\Psi}_{\gamma'}^{\gamma'''} \underline{\Psi}_{\delta'}^{\delta'''} \right) \underline{h}_{\gamma'''\delta'''} \\ &+ g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \underline{\Psi}_\gamma^{\gamma''} \underline{\Psi}_\delta^{\delta''} \partial_t \underline{h}_{\gamma''\delta''} \partial_t \left(\underline{\Psi}_{\gamma'}^{\gamma'''} \underline{\Psi}_{\delta'}^{\delta'''} \right) \underline{h}_{\gamma'''\delta'''} \\ &+ g^{\dagger\gamma\gamma'} g^{\dagger\delta\delta'} \partial_t \left(\underline{\Psi}_\gamma^{\gamma''} \underline{\Psi}_\delta^{\delta''} \right) \underline{h}_{\gamma''\delta''} \underline{\Psi}_{\gamma'}^{\gamma''} \underline{\Psi}_{\delta'}^{\delta''} \partial_t \underline{h}_{\gamma'''\delta'''}. \end{aligned}$$

Moreover, modulo cubic terms, we can also replace the curved metric by the flat one

$$\frac{1}{4} \overline{g}_M^{\gamma\gamma'} \overline{g}_M^{\delta\delta'} \partial_t \underline{h}_{\gamma\delta} \partial_t \underline{h}_{\gamma'\delta'} - \frac{1}{2} \underline{g}^{\dagger\gamma\gamma'} \underline{g}^{\dagger\delta\delta'} \partial_t \underline{h}_{\gamma\gamma'} \partial_t \underline{h}_{\delta\delta'} + \mathbb{F}(0,0) + \mathbb{C}(0,0).$$

□

The (0,0)-component of the metric $\partial_t \underline{h}^{00}$

- Notation

$$h^{\alpha\beta} = g^{\dagger\,\alpha\beta} - g_M^{\alpha\beta} \qquad h_{\alpha\beta} = g^{\dagger}{}_{\alpha\beta} - g_{M,\alpha\beta}$$

$$\underline{h}^{\alpha\beta} = \underline{g}^{\dagger\alpha\beta} - \underline{g}_M^{\alpha\beta} \qquad \underline{h}_{\alpha\beta} = \underline{g}^{\dagger}_{\alpha\beta} - \underline{g}_{M,\alpha\beta}$$

$$\underline{h}^{\alpha\beta} = h^{\alpha'\beta'}\underline{\Psi}^{\alpha}_{\alpha'}\underline{\Psi}^{\beta}_{\beta'} \qquad \underline{h}_{\alpha\beta} = h_{\alpha'\beta'}\underline{\Phi}^{\alpha'}_{\alpha}\underline{\Phi}^{\beta'}_{\beta}.$$

- The wave gauge condition $g^{\dagger\,\alpha\beta}\Gamma^{\dagger\gamma}_{\alpha\beta} = 0$ reads

$$g^{\dagger}{}_{\beta\gamma}\partial_\alpha g^{\dagger\,\alpha\beta} = \frac{1}{2}g^{\dagger}{}_{\alpha\beta}\partial_\gamma g^{\dagger\,\alpha\beta}.$$

Proposition. Key component of the metric

$$\partial_t \underline{h}^{00} \simeq (s/t)^2 \partial \underline{h} + \underline{\partial}\,\underline{h} + t^{-1}\underline{h} + \underline{h}\,\partial\underline{h} + t^{-1} h\,\underline{h}$$

Corollary. High-order derivative of the key metric component

$$|\partial_t \underline{h}^{00}| \lesssim (s/t)^2|\partial \underline{h}| + |\underline{\partial} \underline{h}| + t^{-1}|\underline{h}| + |\partial \underline{h}|\,|\underline{h}|$$

$$\begin{aligned}
&|\partial^I L^J \partial_t \underline{h}^{00}| + |\partial_t \partial^I L^J \underline{h}^{00}| \\
&\lesssim \sum_{\substack{|I'|+|J'|\leqslant|I|+|J| \\ |J'|\leqslant|J|}} \Big((s/t)^2|\partial\partial^{I'} L^{J'} \underline{h}| + |\partial^{I'} L^{J'} \underline{\partial} \underline{h}| + t^{-1}|\partial^{I'} L^{J'} \underline{h}|\Big) \\
&\quad + \sum_{\substack{|I_1|+|I_2|\leqslant|I| \\ |J_1|+|J_2|\leqslant|J|}} |\partial^{I_1} L^{J_1} \underline{h}|\,|\partial\partial^{I_2} L^{J_2} \underline{h}|.
\end{aligned}$$

Observations

- The "bad" derivative of $\underline{h}^{00}$ is bounded by the "good" derivatives arising in the right-hand side.
- The "bad" term $|\partial \underline{h}|$ still arise, but it is multiplied by the factor $(s/t)^2$ which provides us with extra decay and turns this term into a "good" term.

The reduced form of the quasi-null terms

Proposition

$$\underline{P}_{00} \simeq \partial_t \underline{h}_{a\alpha} \partial_t \underline{h}_{b\beta} + \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0) + \mathbb{F}(0,0)$$
$$\underline{P}_{a\beta} \simeq \mathbb{GS}_{hh}(0,0) + \mathbb{C}(0,0) + \mathbb{F}(0,0)$$

Observations

- Hence, it only remains to analyze the term $\partial_t \underline{h}_{a\alpha} \partial_t \underline{h}_{b\beta}$, which is done within the bootstrap argument.
- This proposition follows from
  - $\underline{P}_{00} \simeq \underline{g}^{\dagger\gamma\gamma'} \underline{g}^{\dagger\delta\delta'} \partial_t \underline{h}_{\gamma\gamma'} \partial_t \underline{h}_{\delta\delta'} + \overline{g}_M^{\gamma\gamma'} \overline{g}_M^{\delta\delta'} \partial_t \underline{h}_{\gamma\delta} \partial_t \underline{h}_{\gamma'\delta'} + \dots$
  - $\partial_t \underline{h}^{00} \simeq (s/t)^2 \partial \underline{h} + \underline{\partial}\, \underline{h} + t^{-1} \underline{h} + \underline{h}\, \partial \underline{h} + t^{-1} h\, \underline{h}$
  - and a technical calculation leading to

$$\underline{g}^{\dagger\alpha\alpha'} \underline{g}^{\dagger\beta\beta'} \partial_t \underline{g}^{\dagger}_{\alpha\alpha'} \partial_t \underline{g}^{\dagger}_{\beta\beta'} \simeq \underline{g}^{\dagger 0a} \underline{\partial}_0 \underline{g}^{\dagger}_{0a} \underline{g}^{\dagger 0b} \underline{\partial}_0 \underline{g}^{\dagger}_{0b} + \mathbb{GS}_{hh}(0,0) + \mathbb{F}(0,0) + \mathbb{C}(0,0)$$