# Calabi-Bernstein type problems in Lorentzian Geometry

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We begin with two examples of nonlinear partial differential equations, which arise in the context of some differential geometric problem.

(i) The minimal hypersurface equation in the Euclidean space $\mathbb{R}^{n+1}$. For a smooth function $u : \Omega \longrightarrow \mathbb{R}$ on a domain $\Omega$ in $\mathbb{R}^n$, the problem is given by

$$\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0, \tag{1}$$

where $D$ and div denote the gradient and divergence operators in $\mathbb{R}^n$ respectively. This equation is elliptic, being the affine functions trivial solutions.

(ii) The maximal spacelike hypersurface equation in the Lorentz-Minkowski spacetime $\mathbb{L}^{n+1}$,

with coordinates $(t, x_1, ..., x_n)$ (and Lorentzian form $g = -dt^2 + \sum_{j=1}^{n} dx_j^2$);

the equation is for $t = u(x_1, ..., x_n)$ to satisfy

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad | Du |^2 < 1. \tag{2}$$

where $D$ and div denote the gradient and divergence operators in $\mathbb{R}^n$ respectively.

The condition $| Du |^2 < 1$ assures that the graph of every solution is spacelike, this is, the fundamental form induced on the graph is definite positive.

Moreover, the problem is elliptic thanks to this extra constraint.

Note that, if we take an unitary normal vector field on the graph $t = u(x_1, ..., x_n)$ in the same time-orientation of $\partial_t$, then its mean curvature is given by

$$H = \operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right).$$

On the other hand, the graph of $u$ is extremal, among functions satisfying the spatial condition under interior variations (with compact support) for the volume integral,

$$V = \int \sqrt{1- \mid Du \mid^2}\, dx_1 \wedge .. \wedge dx_n.$$

Again, trivial solution of equation (2) are (spacelike) affine function.

## Bernstein theorem

The early seminal result of S. Bernstein,[1] amended by E. Hopf,[2] is the well-known following uniqueness theorem,

> *The only entire solutions to the equation (1) in $\mathbb{R}^3$ are the affine functions.*

This result is known as the classical Bernstein theoren.

In 1968, J. Simons[3] proved a result which in combination with theorems of E. De Giorgi[4] and W.H. Fleming[5] yield a proof of the Bernstein theorem for $n \leq 7$. Moreover, there is a counterexample $u \in C^\infty(\mathbb{R}^n)$ to the Bernstein conjecture for each $n \geq 8$.

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[1] S. Bernstein, Sur une théorème de géometrie et ses applications aux équations dérivées partielles du type elliptique, *Comm. Soc. Math. Kharkov*, **15** (1914), 38–45.

[2] E. Hopf, On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature, *Proc. Amer. Math. Soc.*, **1** (1950), 80–85.

[3] J. Simons, Minimal varieties in Riemannian manifolds, *Ann. of Math.*, **88** (1968) 62–105.

[4] E. De Giorgi, Una estensione del teorema di Bernstein, *Ann. Scuola Norm. Sup. Pisa*, **19** (1965), 79–85.

[5] W.H. Fleming, On the oriented Plateau problem, *Rend. Circ. Mat. Palermo*, **11** (1962), 69–90.

## Calabi-Bernstein Theorem

One of the most relevant results in the context of global geometry of spacelike surfaces is the classical Calabi-Bernstein Theorem. This result was established in 1970 by Calabi[6] inspired in the classical Bernstein theorem, via a duality between solutions to equations (1) and (2).

In its non-parametric version, it asserts that the only entire solutions to the maximal surface equation

$$\operatorname{div}\left(\frac{Du}{\sqrt{1-|Du|^2}}\right) = 0, \quad |Du| < 1 \tag{3}$$

in the Lorentz-Minkowski spacetime $\mathbb{L}^3$ are affine functions.

In fact, Calabi also shows that the result holds for the case of maximal hypersurfaces in $\mathbb{L}^4$. Later on, Cheng and Yau[7] extended the Calabi-Bernstein theoren to the general $n+1$-dimensional case.

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[6] E. Calabi, Examples of Bernstein Problem for Some Nonlinear Equations, *Proc. Symp. Pure Math.* **15** (1970), 223–230.

[7] S.Y. Cheng and S.T. Yau, Maximal space-like hypersurfaces in the Lorentz-Minkowski spaces, *Ann. Math.*, **104** (1976), 407–419.

## Parametric vs. non-parametric versions

The Calabi-Bernstein Theorem can also be formulated in a parametric way. In this case, it states that the only complete maximal hypersurfaces in $\mathbb{L}^{n+1}$ are the spacelike planes.

Nevertheless, both versions (parametric and non-parametric ones) are not equivalent a priopri, since there exist examples of spacelike entire graphs in $\mathbb{L}^{n+1}$ which are not complete.[8] This fact, is a notable difference and difficulty with respect to the Riemannian case, where all entire graph in $\mathbb{R}^{n+1}$ must be complete.

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[8]See, for instance L.J. Alías and P. Mira, On the Calabi-Bernstein theorem for maximal hypersurfaces in the Lorentz-Minkowski space, *Proc. of the meeting Lorentzian Geometry-Benalmádena 2001, Spain, Pub. RSME,* **5** (2003), 23–55.

## Some approaches to the classical Calabi-berntein theorem

After the general proof by Cheng and Yau, several authors have approached to the classical version of Calabi-Bernstein theoren from different perspectives, providing diverse extensions and new proofs of the result in $\mathbb{L}^3$. Thus, Kobayashi[9] derived the Calabi-Bernstein Theorem as a consequence of the corresponding Weierstrass-Enneper parameterization for maximal surfaces in $\mathbb{L}^3$.

In the real field, a simple proof, which only requires the Liouville theorem for harmonic functions on the Euclidean plane $\mathbb{R}^2$ was given by Romero. [10] As the author says, the proof is inspired in a proof of the classical Bernstein theorem given by Chern.[11]

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[9] O. Kobayashi, Maximal surfaces in the 3-dimensional Minkowski space $\mathbb{L}^3$, *Tokyo J. Math.* **6** (1983), no. 2, 297–309.

[10] A. Romero, Simple proof of Calabi-Bernstein's theorem on maximal surfaces, *Proc. Amer. Math. Soc.* **124** (1996), no. 4, 1315–1317.

[11] S. S. Chern, Simple proofs of two theorems on minimal surfaces, *Enseign. Math.* **15** (1969), 53–61.

Via a local integral inequality for the Gaussian curvature of a maximal surface, Alías and Palmer[12] provided another new proof for the parametric case. These authors also get a new proof of the non-parametric version based on a duality result.[13] Recently, yet another short proof of both versions has been given by Romero and Rubio[14] making use of the interface between the parabolicity of a Riemannian surface and the capacity of geodesic annuli. Finally, a more recent original new proof has been given by Aledo, Romero and Rubio[15] by using a development inspired by the well-known Bochner's technique.

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[12] L.J. Alías and B. Palmer, On the Gaussian curvature of maximal surfaces and the Calabi-Bernstein theorem, *Bull. London Math. Soc.* **33** (2001), no. 4, 454–458.

[13] L.J. Alías and B. Palmer, A duality result between the minimal surface equation and the maximal surface equation, *An. Acad. Bras. Cienc.*, **73** (2001), 161–164.

[14] A. Romero and R.M. Rubio. New proof of the Calabi-Bernstein theorem, *Geom. Dedicata* **147** (2010), 173–176.

[15] J.A. Aledo, A. Romero and R.M. Rubio. The classical Calabi-Bernstein Theorem revisited *J. Math. Anal. Appl.* **431** (2015) 1172–1177

## Romero-Rubio's proof

Consider the Lorentz-Minkowski space $\mathbb{L}^3$ with its Lorentzian metric

$$\langle , \rangle = -dt^2 + dx^2 + dy^2$$

and let $x : S \longrightarrow \mathbb{L}^3$ be a (connected) immersed spacelike surface in $\mathbb{L}^3$. Observe that $S$ must be orientable and let $N$ be the unitary normal vector field on $S$ such that $\langle N, \partial_t \rangle > 0$. If $\theta(p)$ denotes the hyperbolic angle between $N$ and $-\partial_t$ at $p \in S$, then $\cosh\theta = \langle N, \partial_t \rangle$.

We will denote by $\overline{\nabla}$ and $\nabla$ the Levi-Civita connections of $\mathbb{L}^3$ and $S$, respectively. Then the Gauss and Weingarten formulas for $S$ in $\mathbb{L}^3$ are given, respectively, by

$$\overline{\nabla}_X Y = \nabla_X Y - \langle A(X), Y \rangle N \tag{4}$$

and

$$A(X) = -\overline{\nabla}_X N, \tag{5}$$

for all tangent vector fields $X, Y \in \mathfrak{X}(S)$, where $A : \mathfrak{X}(S) \longrightarrow \mathfrak{X}(S)$ stands for the *shape operator* associated to $N$.

On other hand, the tangential component of $\partial_t$ at any point of $S$ is given by $\partial_t^T = \partial_t + \cosh\theta N$.

We suppose that $S$ is maximal. It is immediate to see that

$$\nabla \cosh\theta = -A\partial_t^T$$

where $A$ denotes the shape operator associated to $N$. It is not difficult to obtain by standard computation the following formulas:

$$|\nabla \cosh\theta|^2 = \frac{1}{2}\mathrm{trace}\,(A^2)\sinh^2\theta \quad \text{and} \quad \Delta \cosh\theta = \mathrm{trace}\,(A^2)\cosh\theta$$

where $\nabla$ and $\Delta$ are respectively the gradient and laplacian relative to the induced Riemannian metric $g$ on $S$.

We will need a technical result, wich is a reformulation[16] of a Lemma by Alías and Palmer.[17]

*Let $S$ be an $n(\geq 2)$-dimensional Riemannian manifold and let $v \in C^2(S)$ which satisfies $v\Delta v \geq 0$. Let $B_R$ be a geodesic ball of radius $R$ in $S$. For any $r$ such that $0 < r < R$ we have*

$$\int_{B_r} |\nabla v|^2 \, dV \leq \frac{4 \sup_{B_R} v^2}{\mu_{r,R}},$$

*where $B_r$ denote the geodesic ball of radius $r$ around $p$ in $S$ and $\frac{1}{\mu_{r,R}}$ is the capacity of the annulus $B_R \setminus \bar{B}_r$.*

[16] A. Romero and R.M. Rubio. New proof of the Calabi-Bernstein theorem, *Geom. Dedicata* **147** (2010), 173–176.

[17] L.J. Alías and B. Palmer, Zero mean curvature surfaces with non-negative curvature in flat Lorentzian 4-spaces, *Proc. R. Soc. London A.* **455** (1999), 631–636.

## The parametric case

Consider the fuction $v : S \longrightarrow (\frac{\pi}{2}, \frac{3\pi}{2})$, $v(p) = \arctan(\cosh\theta(p))$, which has an advantage on the original $\cosh\theta$, this is, $v$ is bounded.

It is inmediate to verify $v\Delta v \geq 0$, from the previous Lemma, and taking into account that

$$\nabla v = \frac{1}{1+\cosh^2\theta}\nabla\cosh\theta,$$

we have

$$\int_{B_r} |\nabla v|^2\, dV \leq \frac{9\pi^2}{\mu_{r,R}},$$

for $0 < r < R$, which easily gives

$$\int_{B_r} |\nabla(\cosh\theta)|^2\, dV \leq \frac{C}{\mu_{r,R}},$$

where $B_r$ denote the geodesic disc of radius $r$ around $p$ in $S$, $\frac{1}{\mu_{r,R}}$ is the capacity of annulus $B_R \setminus \bar{B}_r$ and $C = C(p, r) > 0$ is constant.

Now, the surface $S$ is necessarily non compact and from the Gauss formula it has curvature $K \geq 0$.

If we assume that $S$ is complete, a classical result by Ahlfors and Blanc-Fiala-Huber[18], affirms that a complete 2-dimensional Riemannian manifold with non-negative Gauss curvature is parabolic.

On other hand, it is well know that $S$ will be parabolic if and only if $\lim_{R\to\infty} \frac{1}{\mu_{r,R}} = 0$. We get that $R$ can approach to infinity for a fixed arbitrary point $p$ and a fixed $r$, obtaining that $\cosh\theta$ is constant on $S$.

[18]See for instance, J. L. Kazdan, Parabolicity and the Liouville property on complete Riemannian manifolds, in: Seminar on new results in nonlinear partial differential equations (Bonn, 1984), *Aspects of Math.* E10, Ed. A.J. Tromba, Friedr. Vieweg and Sohn, Bonn, 1987, pp. 153–166.

## The non-parametric case

For each $u \in C^\infty(\Omega)$, note that the induced metric on $\Omega \subset \mathbb{R}^2$, via the graph $\big\{(u(x,y),x,y) : (x,y) \in \Omega\big\} \subset \mathbb{L}^3$, is $g_u := -du^2 + g_0$, where $g_0$ is the usual Riemannian metric of $\mathbb{R}^2$.

The metric $g_u$ is positive definite, if and only if $u$ satisfies $|Du| < 1$, where $Du$ denote the gradient of $u$ in $(\mathbb{R}^2, g_0)$.

The graph of $u$ is spacelike and has zero mean curvature if and only if $u$ is a solution to the maximal surface equation (2) in the Lorentz-Minkowski space.

We consider on $\mathbb{R}^2$ the function $\cosh\theta = \frac{1}{\sqrt{1-|Du|^2}}$ and the conformal metric $g' = (\cosh\theta + 1)^2 g_u$, which taking into account the relation between curvatures for conformal changes[19] is flat.

If the graph is entire, then $g'$ is complete, because $L' \geq L_0$ where $L'$ and $L_0$ denote the lengths of a curve on $\mathbb{R}^2$ with respect to $g'$ and the usual metric of $\mathbb{R}^2$,

[19] See for instance, A.L. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.

Taking into account the invariance of subharmonic functions by conformal changes of metric, we are in position to use the same argument as in the parametric case on the Riemannian surface $(F, g')$ to get the result.

## **Using the Bochner technique, Aledo-Romero-Rubio's proof** (parametric version)

Let $x : S \longrightarrow \mathbb{L}^3$ a (connected) immersed maximal surface in $\mathbb{L}^3$. We choose a unit timelike normal vector field $N$ globally defined on $S$ in the same time-orientation of $\frac{\partial}{\partial t}$.

From the Gauss equation, it is well-known that

$$\text{trace}(A^2) = 2K. \tag{6}$$

The idea of the proof is to choose a suitable function on the maximal surface and to apply the Bochner-Lichnerowicz's Formula.

Recall that the well-known Bochner-Lichnerowicz's Formula[20] states that

$$\frac{1}{2}\triangle\left(|\nabla u|^2\right) = |\mathrm{Hess}\, u|^2 + \mathrm{Ric}(\nabla u, \nabla u) + \langle \nabla u, \nabla(\triangle u)\rangle \tag{7}$$

for $u \in \mathcal{C}^\infty(S)$. Here Ric stands for the Ricci tensor of $S$ and $|\mathrm{Hess}\, u|^2$ is the square algebraic trace-norm of the Hessian of $u$, namely $|\mathrm{Hess}\, u|^2 := \mathrm{trace}(H_u \circ H_u)$ where $H_u$ denotes the operator defined by $\langle H_u(X), Y\rangle := \mathrm{Hess}\,(u)(X, Y)$ for all $X, Y \in \mathfrak{X}(S)$.

Let us choose $a \in \mathbb{L}^3$ a null vector, i.e., a non-zero vector such that $\langle a, a\rangle = 0$, and consider the function $\langle N, a\rangle$ on $S$.

Now, applying Schwarz's inequality (for symmetric square matrix), we have,

$$|\mathrm{Hess}\,\langle N, a\rangle|^2 \geq \frac{1}{2}(\triangle\langle N, a\rangle)^2. \tag{8}$$

[20] I. Chavel, *Eigenvalues in Riemannian Geometry*, Pure Appl. Math. **115**, Academic Press, New York, 1984.

From the Weingarten formula (5) it is easy to obtain the gradient of the function $\langle N, a\rangle$ on $S$,

$$\nabla\langle N, a\rangle = -A(a^\top), \tag{9}$$

where $a^\top = a + \langle N, a\rangle N$ is tangent to $S$ and standard computations allow us to obtain

$$\triangle\langle N, a\rangle = \langle N, a\rangle \mathrm{trace}(A^2). \tag{10}$$

Taking into account that $|a^\top|^2 = \langle N, a\rangle^2$ and that $S$ is maximal, we get

$$|\nabla\langle N, a\rangle|^2 = K\langle N, a\rangle^2 \tag{11}$$

and so

$$\mathrm{Ric}(\nabla\langle N, a\rangle, \nabla\langle N, a\rangle) = K|\nabla\langle N, a\rangle|^2 = K^2\langle N, a\rangle^2. \tag{12}$$

With the previous computations, we can to apply the Bochner-Lichnerowicz's Formula to the choosed function $\langle N, a \rangle$ on $S$ and to obain the following inequlity

$$\triangle K \geq 4K^2. \tag{13}$$

Since the Gauss curvature of $S$ is non-negative and if we assume that $S$ is complete, then we can use the following known result (see, for instance[21]),

**Lemma** *Let $S$ be a complete Riemannian surface whose Gaussian curvature is bounded from below and $u \in \mathcal{C}^\infty(S)$ a non-negative function such that $\triangle u \geq cu^2$ for a positive constant $c$. Then $u$ vanishes identically on $S$.*

As a consequence, $K \equiv 0$ and so $S$ is totally geodesic.

On the other hand, the authors also give, by means a suitable conformal metric a proof of the non-parametric version.

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[21] Y.J. Suh, Generalized maximum principles and their applications to submanifolds and S. S. Chern's conjectures, Proceedings of the Eleventh International Workshop on Differential Geometry, 135–152, Kyungpook Nat. Univ., Taegu, 2007

## Some extension of the classical result

We will begin with a new example of non-parametric Calabi-Bernstein type problems given by Latorre and Romero[22]. We have to say that this paper is the first one dealing with the maximal surface equation for warped Lorentzian products, whose fiber is a complete (non-compact) 2-Riemannian manifold.

The authors introduce a new conformal metric, which has inspired several works later. The warping function is assumed non-locally constant and its fiber is the Euclidean plane. Obviously the Calabi-Bernstein theorem is not included in this case.

A new version of non-parametric Calabi-Bernstein type theorem in the case of a Lorentzian product $\mathbb{R} \times F$, where $F$ denotes a Riemannian 2-manifold, with non-negative curvature, has been given by Albujer and Alías[23] [24].

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[22] J.M. Latorre and A. Romero, New examples of Calabi-Bernstein problem for some nonlinear equations, *Diff. Geom. Appl.* **15** (2001), 153–163.

[23] A.L. Albujer and L.J. Alías, Calabi-Bernstein results for maximal surfaces in Lorentzian product spaces, *J. Geom. Phys.* **59** (2009), 620–631.

[24] A.L. Albujer and L.J. Alías, Parabolicity of maximal surfaces in Lorentzian product spaces, *Math. Z.* **267** (2011) 453–464.

Recently, another Calabi-Bernstein type results in the more general ambient of a warped Lorentzian product are given by Caballero, Romero and Rubio[25] [26]. So, the authors obtain several extension of the classical Calabi-Bernstein theorem to tree-dimensional warped products satisfying suitable energy conditions and whose fiber can be non-necessarily of non-negative Gaussian curvature.

In a different direction, yet another extension of the classical result have been given by Pelegrín, Romero and Rubio. This time, the ambient is a 3-dimensional spacetime obeying a known energy condition and which admits a parallel lightlike vector field.[27] This work, will be detailed by the first author in the poster session.

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[25] M. Caballero, A. Romero and R.M. Rubio, Uniqueness of maximal surfaces in Generalized Robertson-Walker spacetimes and Calabi-Bernstein type problems, *J. Geom. Phys.* **60** (2010), 394–402.

[26] M. Caballero, A. Romero and R.M. Rubio, New Calabi-Bernstein results for some nonlinear equations *Anal. Appl.* **11** (2013), no. 1, 1350002, 13 pp.

[27] J.A. Pelegrín, A. Romero and R.M. Rubio, On maximal hypersurfaces in Lorentz manifolds admitting a parallel lightlike vector field, *Class. Quantum Grav.*, **33** (2016), 055003.

Finally, we will describe with more detail a new extension of the classical Calabi-Bernstein theorem by Rubio-Salamanca.[28]

In this last work, the authors study entire solutions to the maximal surface equation in a Lorentzian 3-dimensional warped product, whose fiber is given by a Riemannian surface with finite total curvature.

Recall that a complete Riemannian surface has finite total curvature if the integral of the absolute value of its Gaussian curvature is finite. Of course, the Euclidean plane has finite total curvature, but note that any complete surface, whose curvature is non-negative outside a compact subset has finite total curvature.

On the other hand, examples of complete minimal surfaces in $\mathbb{R}^3$ with finite total curvature are known.[29] Examples in a different ambient space can be found.[30]

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[28] Rubio, R.M. and J.J. Salamanca, Maximal surface equation on a Riemannan 2-manifold with finite total curvature, *J. Geom. Phys.* **92** (2015), 140–146.

[29] D. Hoffman and H. Karcher, Complete Embedded Minimal Surfaces of Finite Total Curvature, *Geometry V, Encyclopaedia of Mathematical Sciences* **90** (1997), pp 5–93, Springer.

[30] J. Pyo and M. Rodríguez, Simply Connected Minimal Surfaces with Finite Total Curvature in $\mathbb{H}^2 \times \mathbb{R}$, *Int. Math. Res. Not. IMRN* (2013) doi: 10.1093/imrn/rnt017.

## Rubio-Salamanca's extension

The authors deal with the following nonlinear elliptic differential equation, in divergence form:

$$\mathrm{div}\Big(\frac{Du}{f(u)\sqrt{f(u)^2-\mid Du\mid^2}}\Big)=-\frac{f'(u)}{\sqrt{f(u)^2-\mid Du\mid^2}}\Big(2+\frac{\mid Du\mid^2}{f(u)^2}\Big) \qquad \text{(E,1)}$$

$$\mid Du\mid< f(u) \qquad \text{(E,2)}$$

where $f$ is a smooth real-valued function defined on an open interval $I$ of the real line $\mathbb{R}$, the unknown $u$ is a function defined on a domain $\Omega$ of a non-compact complete Riemannian surface $(F,g)$ with finite total curvature, $u(\Omega)\subseteq I$, $D$ and div denote the gradient and the divergence of $(F,g)$ and $\mid Du\mid^2:=g(Du,Du)$.

The constraint (E.2) is the ellipticity condition.

The authors are mainly interested in uniqueness and non-existence results for entire solutions (i.e. defined on all $F$) of equation (E).

Note that a constant functions $u = c$ is a solution to the equation (E), if and only if $f'(c) = 0$.

On the other hand, the solutions of (E) are the extremals under interior variations for the functional

$$u \longmapsto \int f(u)\sqrt{f(u)^2 - \mid Du \mid^2}\, dA,$$

where $dA$ is the area element for the Riemannian metric $g$, which acts on functions $u$ such that $u(\Omega) \subseteq I$ and $\mid Du \mid < f(u)$.

This variational problem naturally arise from Lorentzian geometry. In order to see this, consider the product manifold $M := I \times F$ endowed with the Lorentzian metric

$$\langle\, ,\, \rangle = -\pi_I^*(dt^2) + f(\pi_I)^2 \pi_F^*(g), \tag{14}$$

where $\pi_I$ and $\pi_F$ denote the projections from $M$ onto $I$ and $F$, respectively.

The Lorentzian manifold $(M = I \times_f F, \langle\, ,\, \rangle)$ is known as a warped product, with base $(I, -dt^2)$, fiber $(F, g)$ and warping function $f$.

For each $u \in C^\infty(\Omega)$, $u(\Omega) \subseteq I$, the induced metric on $\Omega$ from the Lorentzian metric (14), via its graph $\Sigma_u = \{(u(p), p) : p \in \Omega\}$ in $M$, is written as follows

$$g_u = -du^2 + f(u)^2 g,$$

and it is positive definite, i.e. Riemannian, if and only if $u$ satisfies $\mid Du \mid < f(u)$ everywhere on $\Omega$.

When $g_u$ is Riemannian, $f(u)\sqrt{f(u)^2 - \mid Du \mid^2}\, dA$ is the area element of $(\Omega, g_u)$.

Therefore (E.1) of (E) is the Euler-Lagrange equation for the area functional, its solutions are spacelike graphs of zero mean curvature in $M$, and this equation is called the maximal surface equation in $M$.

Observe that when $I = \mathbb{R}$, $F = \mathbb{R}^2$ and $f = 1$, the equation (E) is the maximal surface equation in $\mathbb{L}^3$.

If we denote by $N$ the unit normal vector field $N$ on $\Sigma_u$ such that $\langle N, \partial_t \rangle \geq 1$ on $\Sigma_u$, where $\partial_t := \partial/\partial t \in \mathfrak{X}(M)$, then

$$N = \frac{-f(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}} \left(1, \frac{1}{f(u)^2} Du\right),$$

and the hyperbolic angle $\theta$ between $-\partial_t$ and $N$ is given by

$$\langle N, \partial_t \rangle = \cosh\theta = \frac{f(u)}{\sqrt{f(u)^2 - \mid Du \mid^2}}.$$

## Some basic concepts

Any warped product $I \times_f F$ possesses an infinitesimal timelike conformal symmetry which is an important tool in this work. Indeed, the vector field

$$\xi := f(\pi_I)\,\partial_t,$$

which is timelike and, from the relationship between the Levi-Civita connections of $M$ and those of the base and the fiber, satisfies

$$\overline{\nabla}_X \xi = f'(\pi_I)\,X \tag{15}$$

for any $X \in \mathfrak{X}(M)$, where $\overline{\nabla}$ is the Levi-Civita connection of the warped metric.

Thus, $\xi$ is conformal with $\mathcal{L}_\xi \langle\, , \,\rangle = 2\, f'(\pi_I)\,\langle\, , \,\rangle$ and its metrically equivalent 1-form is closed.

## Null Convergence Energy Condition

The Lorentzian warped product spaces considered must satisfy certain natural energy condition, which turns out to have an expression in terms of the curvature of its fiber $(F, g)$ and the warping function $f$.

Recall that a Lorentzian manifold obeys the null convergence condition (NCC) if its Ricci tensor $\overline{\mathrm{Ric}}$ satisfies

$$\overline{\mathrm{Ric}}(Z, Z) \geq 0,$$

for any null vector $Z$, i.e. $Z \neq 0$ such that $\langle Z, Z \rangle = 0$.

Taking into account how the Ricci tensor of $M$ is obtained from the Gaussian curvature of the fiber $K^F$ and the warping function $f$,[31] it is easy to check that a Lorentzian warped product space $I \times_f F$ with a 2-dimensional fiber obeys NCC if and only if

$$\frac{K^F(\pi_{_F})}{f^2} - (\log f)'' \geq 0. \tag{16}$$

[31] See for instance B. O'neill, Corollary 7.43

## Spacelike graphs

Let $\Sigma_u = \{(u(p), p) \; : \; p \in F\}$ be, the graph of $u \in C^\infty(F)$ such that $u(F) \subseteq I$ in the Lorentzian warped product $M = I \times_f F$. Suppose that the graph is spacelike.

Note that $\pi_I(u(p), p) = u(p)$ for any $p \in F$, and so $\pi_I$ on the graph and $u$ can be naturally identified by the isometry between $(\Sigma_u, \langle \, , \rangle)$ and $(F, g_u)$.

Analogously, the differential operators $\nabla$ and $\Delta$ in $(\Sigma_u, \langle \, , \rangle)$ can be identified with those ones $\nabla_u$ and $\Delta_u$ in $(F, g_u)$.

Denote $\partial_t^\top = \partial_t + \langle N, \partial_t \rangle N$ the tangential component of $\partial_t$ on $\Sigma_u$. It is not difficult to see

$$\nabla \pi_{I|_{\Sigma_u}} := \nabla u = -\partial_t^\top .$$

If we suppose the graph maximal and consider the distinguished function

$$\langle N, \xi \rangle = f(u) \cosh \theta,$$

on the graph, we have

$$\nabla \langle N, \xi \rangle = -A\xi^{\top},$$

where $A$ denote the shape operator of the graph. Now taking a orhonormal frame consisting of the eigenvectors of the shape operator, we obtain

$$| \nabla \langle N, \xi \rangle |^2 = \frac{1}{2} \mathrm{trace}\,(A^2) \{ \langle N, \xi \rangle^2 - f(\pi_I)^2 \}. \tag{17}$$

Using the Gauss and Codazzi equations, as well as, the expresion for the Ricci tensor of $M$[32], it is a standard computation to obtain (via the isommetry)

$$\begin{aligned} \Delta_u (f(u) \cosh \theta) &= \left\{ \frac{K^F}{f(u)^2} - (\log f)''(u) \right\} |\nabla_u u|^2 f(u) \cosh \theta \\ &+ \frac{1}{2} \mathrm{trace}(A^2) f(u) \cosh \theta \end{aligned} \tag{18}$$

[32] See for instance B. O'neill, Chapter 7

On the other hand, taking into account the Gauss equation and using again the expresion for the Ricci tensor of $M$, then the Gauss curvature of a maximal graph is

$$K = \frac{f'(u)^2}{f(u)^2} + \left\{ \frac{K^F}{f(u)^2} - (\log f)''(u) \right\} \mid \partial_t^\top \mid^2 + \frac{K^F}{f(u)^2} + \frac{1}{2}\mathrm{trace}(A^2). \qquad (19)$$

As a direct consequence, from (18) we have the following alternative expresion,

$$\Delta_u(f(u)\cosh\theta) = \left\{ K_u - \frac{f'(u)^2}{f(u)^2} - \frac{K^F}{f(u)^2} + \frac{1}{2}\mathrm{trace}(A^2) \right\} f(u)\cosh\theta. \qquad (20)$$

## Conformal metric

On the manifold $F$ we consider the following Riemannian metric

$$g'_u := f(u)^2 \cosh^2\theta\, g_u, \tag{21}$$

where

$$f(u)\cosh\theta = \frac{f(u)^2}{\sqrt{f(u)^2 - \mid Du \mid^2}}$$

and $\mid Du \mid^2 := g(Du, Du)$. Therefore, if $\epsilon := \mathrm{Inf}(f) > 0$ we get the following inequality

$$L' \geq \epsilon^2 L,$$

where $L'$ and $L$ denote the lengths of a curve in $F$ with respect to $g'_u$ and $g$, respectively. Consequently, $g'_u$ is complete whenever $g$ is complete.

Now, suppose that $\mathrm{Sup} f(u) < \infty$. Put $\lambda = \mathrm{Sup} f(u)$ and consider the new Riemannian metric

$$g^*_u := (f(u)\cosh\theta + \lambda)^2 g_u \tag{22}$$

on $F$.

The completeness of the metric (21) assures that $g_u^*$ is also complete.

Moreover, it has the advantage over $g_u'$ that we can control its Gaussian curvature under reasonable assumptions.

In order to concrete this assertion, denote by $K_u^*$ and $K_u$ the Gaussian curvatures of the Riemannian metrics $g_u^*$ and $g_u$, respectively. From (22) and using the relation between Gaussian curvatures for conformal changes[33] , we have

$$K_u - (f(u)\cosh\theta + \lambda)^2 K_u^* = \Delta_u \log(f(u)\cosh\theta + \lambda). \tag{23}$$

[33]See for instance, A.L. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.

**Lemma** *Suppose that* $(F, g)$ *is complete, with finite total curvature. If* $\inf f > 0$, $\sup f < \infty$ *and the inequality* $\frac{K^F}{f(u)^2} - (\log f)''(u) \geq 0$ *holds on* $F$*, then the complete Riemannian surface* $(F, g_u^*)$ *has finite total curvature.*

***Proof.*** From the previous expresions (19) and (18) we get,

$$\Delta_u \log(f(u)\cosh\theta + \lambda) \leq \frac{1}{f(u)\cosh\theta + \lambda}\left\{\Big(K_u - \frac{K^F}{f(u)^2}\Big)f(u)\cosh\theta + \Big(K_u - \frac{K^F}{f(u)^2}\Big)\lambda\right\}$$

$$\leq K_u - \frac{K^F}{f(u)^2}.$$

Since the Riemannian area elements of the metrics $g$ and $g_u^*$ satisfy

$$dA_u^* = \frac{(f(u)\cosh\theta + \lambda)^2 f(u)^2}{\cosh\theta}\, dA,$$

make use of (23), we obtain

$$\int_F \text{máx}(-K_u^*, 0)\, dA_u^* \leq \int_F \text{máx}(-K^F, 0) \frac{1}{\cosh\theta}\, dA < \int_F \text{máx}(-K^F, 0)\, dA < \infty.$$

□

Now, we can state one of main results of Rubio-Salamanca's work.

**Theorem** *Let $M = I \times_f F$ a Lorentzian warped product, with fiber $(F, g)$ a complete Riemannian surface, which has finite total curvature and whose warping function satisfies* $\text{ínf}\, f > 0$ *and* $\sup f < \infty$. *If $M$ obeys the NCC, then any entire maximal graph $(\Sigma_u, \langle\, ,\, \rangle)$ must be totally geodesic. Moreover, if there exists a point $p \in F$ such that $\frac{K^F(p)}{f(u(p))^2} - (\log f)''(u(p)) > 0$, then $u$ is constant.*

***Proof.*** From previous Lemma, we have that $(F, g_u^*)$ is complete with finite total curvature. Consider the function $\frac{1}{f(u)\cosh\theta}$ on $(F, g_u)$. Then, some computations allow to show that the laplacian

$$\Delta_u \left( \frac{1}{f(u)\cosh\theta} \right) = -\frac{1}{f(u)^2 \cosh^2\theta} \Delta_u (f(u)\cosh\theta) + 2 \frac{|\nabla_u (f(u)\cosh\theta)|^2}{(f(u)^3 \cosh^3\theta)}$$

is non-positive

Taking into account the invariance of superharmonic functions by conformal changes of metric, we get a positive superharmonic function on the complete parabolic Riemannian surface $(F, g_u^*)$ and as a consequence the function $f(u)\cosh\theta$ must be constant.

Thus, from the second term of (18), whose expresion we will recall,

$$\begin{aligned}\Delta_u(f(u)\cosh\theta) &= \left\{\frac{K^F}{f(u)^2} - (\log f)''(u)\right\}|\nabla_u u|^2 f(u)\cosh\theta \\ &+ \frac{1}{2}\mathrm{trace}(A^2)f(u)\cosh\theta,\end{aligned}$$

we obtain that the graph $(\Sigma_u, \langle\, , \rangle)$ is totally geodesic.

On the other hand, if moreover there exists a point $p \in F$ such that $\frac{K^F(p)}{f(u(p))^2} - (\log f)''(u(p)) > 0$, taking into account the first addend of (18), then there exists an open neighborhood of $(p, u(p))$ in $\Sigma_u$ which is contained in the complete maximal graph $u = u_0$, with $f'(u_0) = 0$.

As $(\Sigma_u, \langle\ ,\ \rangle)$ is entire and totally geodesic, it must to coincide with the totally geodesic spacelike slice $t = u_0$.

$\square$

## Uniqueness of complete maximal hypersurfaces in spacetimes

The importance in General Relativity of maximal and constant mean curvature spacelike hypersurfaces in spacetimes is well-known; a summary of several reasons justifying it can be found in the paper of Marsden and Tipler.[34]

Each maximal hypersurface can describe, in some relevant cases, the transition between the expanding and contracting phases of a relativistic universe.

On the one hand, they can constitute an initial set for the Cauchy problem.[35] Specifically, Lichnerowicz proved that a Cauchy problem with initial conditions on a maximal hypersurface is reduced to a second-order non-linear elliptic differential equation and a first-order linear differential system.[36]

Also, the deep understanding of this kind of hypersurfaces is essential to prove the positivity of the gravitational mass.

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[34] J.E. Marsden and F.J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in General Relativity, *Phys. Rep.*, **66** (1980), 109–139.

[35] H. Ringström, *The Cauchy problem in General Relativity*, ESI Lectures in Mathematics and Physics, 2009.

[36] A. Lichnerowicz, L' integration des équations de la gravitation relativiste et le problème des n corps, *J. Math. Pures et Appl.*, **23** (1944), 37–63.

They are also interesting for Numerical Relativity, where maximal hypersurfaces are used for integrating forward in time.[37]

From a mathematical point of view, it is necessary to study the maximal hypersurfaces of a spacetime in order to understand its structure. Especially, for some asymptotically flat spacetimes, the existence of a foliation by maximal hypersurfaces is established (see, for instance [38] and references therein).

The existence results and, consequently, uniqueness appear as kernel topics.

Let us remark that the completeness of a spacelike hypersurface is required whenever we study its global properties, and also, from a physical viewpoint, completeness implies that the whole physical space is take into consideration.

[37] J.L. Jaramillo, J.A.V. Kroon and E. Gourgoulhon, From geometry to numerics: interdisciplinary aspects in mathematical and numerical relativity, *Classical Quant. Grav.*, **25** (2008), 093001.

[38] R. Bartnik, Existence of maximal surfaces in asymptotically flat spacetimes, *Comm. Math. Phys.*, **94** (1984), 155–175.

A maximal hypersurface is (locally) a critical point for a natural variational problem, namely of the volume functional (see, for instance.[39]).

After the relevant result of the Bernstein-Calabi conjecture[40] for the $n$-dimensional Lorentz-Minkowski spacetime given by Cheng and Yau,[41] classical papers dealing with uniqueness results for constant mean curvature (CMC) hypersurfaces are [42], and [43].

[39] A. Brasil, A.G. Colares, On constant mean curvature spacelike hypersurfaces in Lorentz manifolds, *Mat. Contemp.*, **17** (1999) 99–136.

[40] E. Calabi, Examples of Bernstein problems for some nonlinear equations, *P. Symp. Pure Math.*, **15** (1970), 223–230.

[41] Cheng and S.T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces, *Ann. of Math.*, **104** (1976), 407–419.

[42] D. Brill and F. Flaherty, Isolated maximal surfaces in spacetime, *Commun. Math. Phys.* **50** (1984), 157–165.

[43] J.E. Marsden and F.J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in General Relativity, *Phys. Rep.*, **66** (1980), 109–139.

In their work[44], Brill and Flaherty replaced the Lorentz-Minkowski spacetime by a spatially closed universe, and proved uniqueness results for CMC hypersurfaces in the large by assuming $\overline{\mathrm{Ric}}(z, z) > 0$ for every timelike vector $z$. This assumption may be interpreted as the fact that there is real present matter at every point of the spacetime. It is known as the Ubiquitous Energy Condition.

This energy condition was relaxed by Marsden and Tipler[45] to include, for instance, non-flat vacuum spacetimes.

More recently, Bartnik[46] proved very general existence theorems and consequently, he claimed that it would be useful to find new satisfactory uniqueness results.

[44] D. Brill and F. Flaherty, Isolated maximal surfaces in spacetime, *Commun. Math. Phys.* **50** (1984), 157–165.

[45] J.E. Marsden and F.J. Tipler, Maximal hypersurfaces and foliations of constant mean curvature in General Relativity, *Phys. Rep.*, **66** (1980), 109–139.

[46] R. Bartnik, Existence of maximal surfaces in asymptotically flat spacetimes, *Commun. Math. Phys.*, **94** (1984), 155–175.

Later, Alías, Romero and Sánchez[47] proved new uniqueness results in the class of spacetimes that they called spatially closed Generalized Robertson-Walker (GRW) spacetimes (which includes the spatially closed Robertson-Walker spacetimes) under the Timelike Convergence Condition.

This GRW spacetimes differ from the classical Robertson-Walker spacetimes due to the fact that, despite being both defined as the warped product of an open interval endowed with negative definite metric and a Riemannian manifold as a fiber, this fiber does not necessarily has constant sectional curvature in the case of GRW spacetimes.

Also, Alías and Montiel[48] proved that in a GRW spacetime whose warping function satisfies the convexity condition $(\log f)'' \leq 0$, the spacelike slices are the only compact constant mean curvature spacelike hypersurfaces.

[47] L.J. Alías, A. Romero and M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson-Walker spacetimes, *Gen. Relat. Gravit.*, **27** (1995), 71–84.

[48] L.J. Alías and S. Montiel, Uniqueness of spacelike hypersurfaces with constant mean curvature in generalized Robertson-Walker spacetimes, *Differential geometry*, Valencia 2001, 59–69. World Science Publication, River Edge (2002).

Furthermore in 2011, this result was generalized by Caballero, Romero and Rubio[49] for a larger class of spatially closed spacetimes.

Up to this point, except the Cheng-Yau theorem, all the uniqueness results aforementioned are shown in spatially closed spacetimes.

In spite of the historical importance of spatially closed GRW spacetimes, a number of observational and theoretical arguments about the total mass balance of the universe[50] suggest the convenience of taking into consideration open cosmological models.

Even more, a spatially closed GRW spacetime violates the holographic principle[51] whereas a GRW spacetime with non-compact fiber could be a suitable model that follows that principle.[52]

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[49] M. Caballero, A. Romero and R.M. Rubio, Constant mean curvature spacelike hypersurfaces in Lorentzian manifolds with a timelike gradient conformal vector field, *Class. Quantum Grav.*, **28** (2011), 145009–145022.

[50] H.Y. Chiu, A cosmological model for our universe, *Annals of Physics*, **43** (1967), 1–41.

[51] R. Bousso, The holographic principle, *Rev. Mod. Phys.*, **74** (2002), 825–874

[52] D. Bak and S-J Rey, Cosmic holography, *Class. Quantum Grav.*, **17** (2000), L83–L89.

Romero, Rubio and Salamanca[53] introduce a new class of spatially open GRW spacetimes, which is called Spatially parabolic GRW spacetimes.

This new notion of spatially parabolic GRW spacetime is a natural counterpart of the spatially closed GRW spacetime. So, a GRW spacetime is spatially parabolic if its fiber is a parabolic Riemannian manifold.

Recall that a complete (non-compact) Riemannian manifold is said to be parabolic if the only positive superharmonic functions are the constants

The parabolicity of the fiber of a GRW spacetime provides wealth in a geometric-analytic point of view. So, the authors obtain several uniqueness and non-existence results on complete maximal hypersurfaces immersed in certain families of these spacetimes, whose hyperbolic angle is bounded (see also[54]).

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[53] A. Romero, R.M. Rubio and Juan J. Salamanca, Uniqueness of complete maximal hypersurfaces in spatially parabolic Generalized Robertson-Walker spacetimes, *Class. Quantum Grav.*, **30** (2013).

[54] A. Romero, R.M. Rubio and Juan J. Salamanca, A new approach for uniqueness of complete maximal hypersurfaces in spatially parabolic GRW spacetimes, *J. Math. Anal. Appl.*, **419** (2014), 355–372.

For arbitrary dimension, parabolicity has no clear relationship with sectional curvature. Indeed, the Euclidean space $\mathbb{R}^n$ is parabolic if and only if $n \leq 2$. Moreover, there exist parabolic Riemannian manifolds whose sectional curvature is not bounded from below.

The family of spatially parabolic GRW spacetimes is very large, although some other interesting GRW spacetimes do not belong to this family. For instance, those Robertson-Walker spacetimes whose fiber is the hyperbolic space $\mathbb{H}^n$ are excluded.

Making use of two maximun principle: the strong Liouville property and the Omori-Yau generalized maximum principle, Romero, Rubio and Salamanca[55] obtain new uniqueness results in other relevant spatially open GRW spacetimes for complete maximal hypersurfaces which are between two spacelike slices (time bounded) and/or have a bounded hyperbolic angle.

In contrast to parabolicity, some curvature assumptions should be imposed here.

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[55] A. Romero, R.M. Rubio and J.J. Salamanca, Complete maximal hypersurfaces in certain spatially open generalized Robertson-Walker spacetimes, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Math. RACSAM*, (2014), 1–10.

On the other hand, in the case of the Einstein-de Sitter spacetime, which is a spatially open model, a new uniqueness result for complete constant mean curvature hypersurfaces is given[56].

Finally, focusing on the problems of uniqueness and non-existence of complete maximal hypersurfaces immersed in a spatially open Robertson-Walker spacetime with flat fiber, Pelegrín, Romero and Rubio[57] give new non-existence and uniqueness results on complete maximal hypersurfaces.

Note that these models have aroused a great deal of interest, since recent observations have shown that the current universe is very close to a spatially flat geometry.[58]

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[56] R.M. Rubio, Complete constant mean curvature spacelike hypersurfaces in the Einstein-de Sitter spacetime, *Rep. Math. Phys.*, **74** (2014), 127–133.

[57] J.A. Pelegrín, A. Romero and R.M. Rubio, Uniqueness of complete maximal hypersurfaces in spatially open $(n+1)$-dimensional Robertson-Walker spacetimes with flat fiber, *Gen. Relativity Gravitation* **48** (2016),

[58] E.J. Copeland, M. Sami and S. Tsujikawa, Dynamics of dark energy, *Int. J. Mod. Phys. D*, **15** (2006), 1753–1935.

It is important to say that the authors do not need the hyperbolic angle of the hypersurface to be bounded, which was an assumption used in some previous works studying the spatially open case.

Thus, they are able to deal with spacelike hypersurfaces approaching the null boundary at infinity, such as hyperboloids in Minkowski spacetime.

**Thank you very much**

**for your kind attention!**

**¡Muchas gracias**

**por vuestra amable atención!**