Conformal Methods in General Relativity

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8th International Meeting on Lorentzian Geometry,
Málaga, September 22, 2016

Outline

1. Penrose's conformal boundary & Friedrich's conformal field equations
2. Asymptotically de Sitter-like spacetimes
3. Asymptotically Minkowski-like spacetimes
4. Spatial infinity

Penrose's conformal boundary

Einstein's vacuum field equations with cosmological constant Λ (in this talk $\Lambda \geq 0$)

$$\mathrm{Ric}[\widetilde{g}] = \Lambda \widetilde{g} \, .$$

- A main aspect of research: construction of physically relevant solutions
- Important notion: "asymptotic flatness" and "asymptotic de Sitterness"
- Delicate issue due to the absence of non-dynamical background fields

Elegant geometric approach due to Penrose [Penrose '63, '65]:
Assume that, after an appropriate conformal rescaling, one can attach a conformal boundary $\mathscr{I}$ to the spacetime through which the rescaled metric admits a smooth extension ("smooth conformal compactification at infinity")

Physical picture
This is possible whenever the gravitational field has an asymptotically Minkowski-(or de Sitter)-like fall-off behavior.

Asymptotic Simplicity

Definition

A smooth spacetime $(\widetilde{\mathscr{M}}, \widetilde{g})$ is called asymptotically simple if there exists a smooth spacetime $(\mathscr{M}, g)$ and a smooth function $\Theta : \mathscr{M} \to \mathbb{R}$ such that

(i) $\mathscr{M}$ is a manifold with boundary $\mathscr{I} = \partial\mathscr{M}$,

(ii) $\Theta > 0$ on $\mathscr{M} \setminus \mathscr{I}$ and $\Theta = 0$ with $\mathrm{d}\Theta \neq 0$ on $\mathscr{I}$,

(iii) $\exists$ an embedding $\phi : \widetilde{\mathscr{M}} \to \mathscr{M}$ such that $\phi(\widetilde{\mathscr{M}}) = \mathscr{M} \setminus \mathscr{I}$ and $\phi^*(\Theta^{-2} g) = \widetilde{g}$,

(iv) each inextendable null geodesic in $(\widetilde{\mathscr{M}}, \widetilde{g})$ acquires two distinct endpoints on $\mathscr{I}$.

- $\mathscr{I}$ provides a representation of (null) infinity.
- $\mathscr{I}$ consists of two disjoint components $\mathscr{I}^+$ and $\mathscr{I}^-$, future and past (null) infinity.
- To model e.g. an isolated body with sufficiently weak gravitational field so that no collapse to a black hole etc. occurs.

Let $(\widetilde{\mathscr{M}}, \widetilde{g})$ be an asymptotically simple vacuum spacetime with $\Lambda = 0$. Then $\mathscr{I}^+$ and $\mathscr{I}^-$ are both topologically $\mathbb{R} \times S^2$, while $\widetilde{\mathscr{M}}$ is topologically $\mathbb{R}^4$ [Geroch 1971], [Hawking & Ellis 1973].

Asymptotic Simplicity

One may think of various variations of this definition:

- weaken smoothness requirement
- weaken completeness condition

Definition

$(\widetilde{\mathscr{M}}, \widetilde{g})$ is called weakly asymptotically simple if its asymptotic region is diffeomorphic to an asymptotically simple spacetime.

According to Penrose, isolated systems, should be described by weakly asymptotically simple spacetimes.

Properties of a smooth conformal boundary $\mathscr{I}$ (in vacuum)

- null hypersurface if $\Lambda = 0$
- spacelike hypersurface if $\Lambda > 0$
- conformal Weyl tensor vanishes at $\mathscr{I}$ (supposing that, for $\Lambda = 0$, $\mathscr{I} \cong \mathbb{R} \times S^2$)
- Killing vector fields are tangential to $\mathscr{I}$

Asymptotic Simplicity

Advantages

- Problems on unbounded domains can be reformulated in terms of bounded domains $\hookrightarrow$ easier to analyze from a PDE point of view.
- Asymptotic behavior of the gravitational field can be analyzed in terms of a local problem ($\hookrightarrow$ asymptotic Cauchy problem)

Issue: "Size" of the class of (weakly) asymptotically simple spacetimes

- Compatibility of Penrose's geometric concept and Einstein's field equations
- Smooth $\mathscr{I}$ versus polyhomogeneous $\mathscr{I}$
- How can such spacetimes be constructed?

Friedrich's conformal field equations

Need equations in $(\mathscr{M}, g)$:
Introduce Θ as a conformal gauge freedom and solve $\mathrm{Ric}[\Theta^{-2}g] = \Lambda\Theta^{-2}g$.

Issue: Equations are singular at $\mathscr{I}$.

Use the conformal field equations (CFE) [Friedrich '81]:

- Instead of Θ regard the curvature scalar $R[g]$ as a conformal gauge freedom and treat Θ as an unknown.
- Regard the Schouten tensor L, the rescaled Weyl tensor $W := \Theta^{-1}\mathrm{Weyl}$ and the scalar $s := \frac{1}{4}\Box_g\Theta + \frac{1}{24}R\Theta$ as independent of g and Θ.
- The conformal field equations for (g, Θ, L, W, s) read

$$
\begin{aligned}
&\nabla_\rho W_{\mu\nu\sigma}{}^\rho = 0 \,, \\
&\nabla_\mu L_{\nu\sigma} - \nabla_\nu L_{\mu\sigma} = \nabla_\rho\Theta\, W_{\nu\mu\sigma}{}^\rho \,, \\
&\nabla_\mu\nabla_\nu\Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu} \,, \\
&\nabla_\mu s = -L_{\mu\nu}\nabla^\nu\Theta \,, \\
&2\Theta s - \nabla_\mu\Theta\nabla^\mu\Theta = \Lambda/3 \,, \\
&R_{\mu\nu\sigma}{}^\kappa[g] = \Theta W_{\mu\nu\sigma}{}^\kappa + 2(g_{\sigma[\mu}L_{\nu]}{}^\kappa - \delta_{[\mu}{}^\kappa L_{\nu]\sigma}) \,.
\end{aligned}
$$

- There are alternative versions which introduce additional gauge degrees of freedom.

Friedrich's conformal field equations

Properties:

- Remain regular at $\{\Theta = 0\}$.
- Equivalent to the vacuum equations where $\Theta \neq 0$.
- Split into a symmetric hyperbolic system of evolution equations and a set of constraint equations.

In this talk:
Construction of $\Lambda \geq 0$-vacuum spacetimes which admit a smooth conformal compactification at infinity by constructing $(\mathscr{M}, g, \Theta)$ directly via the CFE.

Some crucial results can be established just by using standard results on symmetric hyperbolic systems (such as local existence or Cauchy stability).

Conformal compactification of de Sitter

Assumption: $\Lambda > 0$.

De Sitter spacetime is asymptotically simple.

In global coordinates the de Sitter line element reads (on $\mathbb{R} \times S^3$)

$$\widetilde{g}_{\mathrm{dS}} = -\mathrm{d}\tau^2 + \frac{3}{\Lambda}\cosh^2\Big(\sqrt{\frac{\Lambda}{3}}\,\tau\Big)\mathrm{d}\Omega_3 \,, \quad \tau \in \mathbb{R} \,.$$

Application of the coordinate transformation $\tau \mapsto t := 2\arctan\Big[\tanh\Big(\frac{1}{2}\sqrt{\frac{\Lambda}{3}}\tau\Big)\Big]$ yields

$$\widetilde{g}_{\mathrm{dS}} = \frac{3}{\Lambda}\frac{1}{\cos^2 t}\Big(-\mathrm{d}t^2 + \mathrm{d}\Omega_3\Big) \,, \quad t \in (-\pi/2, \pi/2) \,.$$

Set $\Theta = \sqrt{\frac{\Lambda}{3}}\cos t$. Then

$$g_{\mathrm{dS}} = \Theta^2 \widetilde{g}_{\mathrm{dS}} = -\mathrm{d}t^2 + \mathrm{d}\Omega_3 \quad \text{on } [-\pi/2, \pi/2] \times S^3 \,,$$

with attached conformal boundary $\mathscr{I}^{\pm} = \{\pm\pi/2\} \times S^3$.

Asymptotic Cauchy problem

Construction of $\Lambda > 0$-vacuum spacetimes which admit a smooth $\mathscr{I}^-$:

- Regard $\mathscr{I}^-$ as initial surface $\rightsquigarrow$ Cauchy problem in $(\mathscr{M}, g)$ for the CFEs
- Asymptotic Cauchy data: Riemannian 3-manifold (Σ, h) and a symmetric 2-tensor D subject to the following constraints

$$\mathrm{tr}\, D = 0\,, \quad \mathrm{div}\, D = 0\,.$$

Theorem (Friedrich, 1986)

Consider asymptotic Cauchy data (Σ, h, D). Then there exists a unique solution $(\mathscr{M}, g)$ of the CFE such that in the emerging spacetime: $\mathscr{I}^- \cong \Sigma$, the induced metric on $\mathscr{I}^-$ can be identified with h and $W(n, \cdot, n, \cdot) = D$, where n is a unit future normal to $\mathscr{I}^-$.

- Permits the construction of semi-global solutions to Einsteins field equations with positive Λ.
- Asymptotic de Sitter data: $(\Sigma, h) = (S^3, \mathrm{d}\Omega_3)$ and $D = 0$.

Stability

Theorem (Friedrich, 1986, 1991)

Let $(\widetilde{\mathscr{M}}, \widetilde{g})$ be an asymptotically simple $\Lambda > 0$-vacuum spacetime. Then $(\widetilde{\mathscr{M}}, \widetilde{g})$ is stable.

- As Cauchy surface one may take either $\mathscr{I}^-$ or any non-asymptotic Cauchy surface.
- The perturbed spacetimes are asymptotically simple.
- Situation much more involved in the case of e.g. Schwarzschild- or Kerr-de Sitter.

Conclusion

Asymptotic simplicity seems to be a reasonable concept for $\Lambda > 0$.

Conformal compactification of Minkowski

Assumption: $\Lambda = 0$.

Minkowski spacetime is asymptotically simple.

Its line element can be written as

$$\widetilde{g}_{\mathrm{mink}} = -\mathrm{d}\tau^2 + \mathrm{d}r^2 + r^2\mathrm{d}\Omega_2 \,.$$

Introduce new coordinates $u := \arctan\left(\frac{\tau - r}{\sqrt{2}}\right)$ and $v := \arctan\left(\frac{\tau + r}{\sqrt{2}}\right)$,

$$\widetilde{g}_{\mathrm{mink}} = \frac{1}{\cos^2 u \cos^2 v}\Big(-2\mathrm{d}u\mathrm{d}v + \frac{1}{2}\sin^2(v-u)\mathrm{d}\Omega_2\Big) \,.$$

Set $\Theta = \cos u \cos v$. Then

$$g_{\mathrm{mink}} = \Theta^2 \widetilde{g}_{\mathrm{mink}} = -2\mathrm{d}u\mathrm{d}v + \frac{1}{2}\sin^2(v-u)\mathrm{d}\Omega_2 \,, \quad u, v \in [-\pi/2, \pi/2] \,, \ \ v \geq u \,,$$

with attached conformal boundary $\mathscr{I}^- \cup \mathscr{I}^+ \cup \{i^-\} \cup \{i^+\} \cup \{i^0\}$

- $\mathscr{I}^- = \{u = -\pi/2, v \in (-\pi/2, \pi/2)\}$
- $\mathscr{I}^+ = \{v = +\pi/2, u \in (-\pi/2, \pi/2)\}$
- past and future timelike infinity $i^\pm = \{u = v = \pm\pi/2\}$
- spatial infinity $i^0 = \{v = -u = \pi/2\}$

Asymptotic characteristic initial value problem

Construction of $\Lambda = 0$-vacuum spacetimes with smooth $\mathscr{I}^-$ (from data prescribed on $\mathscr{I}^-$):

- Consider an incoming null hypersurface $\mathscr{H}$ which intersects $\mathscr{I}^-$ in a smooth spherical cross-section S.
- CFE split into constraint and evolution equations.
- Given appropriate free "seed" data all the remaining data can be computed by solving algebraic equations and ODEs.
- In adapted null coordinates (u, r, x^A): $[g_{AB}\mathrm{d}x^A\mathrm{d}x^B]$ on $\mathscr{H}$ and the radiation field W_{rArB} on $\mathscr{I}^-$ (supplemented by certain data on S).

Theorem (Kánnár, 1996)

Local existence holds to the future of $\mathscr{H} \cup \mathscr{I}^-$ near S.

Remark: Use local existence result for symmetric hyperbolic systems for such a characteristic initial value problem [Rendall '90].

Purely radiative spacetimes

Consider a spacetime which is generated solely by gravitational radiation coming in from $\mathscr{I}^-$ and interacting with itself, with no information coming in from i^-:

- One stipulates the spacetime to be smoothly extendable through both $\mathscr{I}^-$ *and* i^-, and that $\mathscr{I}^-$ is the future light-cone C_{i^-} of i^-.
- $\rightsquigarrow$ Consider an asymptotic initial value problem with data on C_{i^-}.
- As free data one identifies the radiation field $W_{AB} := W_{rArB}|_{\mathscr{I}^-}$ (subject to certain regularity conditions at i^-).

Theorem (Chruściel & P., 2013)

Local existence holds to the future of C_{i^-} near i^- supposing that the radiation field admits an extension to a smooth spacetime tensor field.

(i) Use a representation of the CFE as a system of wave equations [P. '13].

(ii) Make sure that the solutions of the constraint equations are restrictions to C_{i^-} of smooth spacetime fields (via approximate solutions [Friedrich '13]).

(iii) Apply local existence result for wave equations for such an initial value problem [Dossa '03].

Hyperboloidal Cauchy problem

- The results discussed so far give no insight how generic spacetimes with a smooth $\mathscr{I}$ are.
- $\rightsquigarrow$ Need to study Cauchy problems with data prescribed on "ordinary" hypersurfaces.

To avoid difficulties at i^0 consider the hyperboloidal Cauchy problem, where Cauchy data are given on a spacelike hypersurface which intersects $\mathscr{I}^+$ in a 2-sphere S.

- Cauchy data $(\widetilde{\Sigma}, \widetilde{h}, \widetilde{K})$ satisfy the vacuum constraints

$$R[\widetilde{h}] - |\widetilde{K}|^2 + \widetilde{K}^2 = 0\,, \quad \nabla^{(\widetilde{h})}_j \widetilde{K}_i{}^j - \nabla^{(\widetilde{h})}_i \widetilde{K} = 0\,,$$

supplemented by certain topological and asymptotic conditions.

Theorem (Friedrich, 1983)

For such hyperboloidal Cauchy data local existence holds and the emerging spacetime admits a piece of a smooth $\mathscr{I}^+$, supposing that the relevant data are smoothly extendable through S.

Hyperboloidal Cauchy problem

Theorem (Friedrich, 1986)

Consider hyperboloidal Cauchy data which are smoothly extendable through S and sufficiently close to hyperboloidal Minkowskian data. Then the emerging spacetime is future asymptotically simple and admits a smooth i^+.

Remarks:

- The CFE force the null geodesic generator of $\mathscr{I}^+$ to meet in one regular point.
- One shows $\exists\ p \in \gamma$ on a null geodesic γ on the Cauchy horizon where $\mathrm{d}\Theta = 0$ with $s \neq 0$.
- Then it follows from the CFE

$$\nabla_\mu \nabla_\nu \Theta = -\Theta L_{\mu\nu} + s g_{\mu\nu}$$

  that p is an isolated critical point for Θ.
- Generalization to permit larger class of reference spacetimes [Lübbe & Kroon, 2011]

Friedrich's result permits the construction of a large class of asymptotically simple spacetimes from Cauchy data which are stationary (e.g. Schwarzschild) near spatial infinity [Chruściel & Delay '02, '03].

Remark: Stationary, asympt. Euclidean spacetimes are weakly asymptotically simple [Dain, '01].

Characteristic Cauchy problem

Consider a characteristic Cauchy problem on a future light-cone C_O of some point $O \in \widetilde{\mathscr{M}}$ (alternatively, on two transversally intersecting null hypersurfaces):

- **Seed data** (in $(\widetilde{M}, \widetilde{g})$) in adapted null coordinates (u, r, x^A): Conformal class $[\widetilde{g}_{AB}\mathrm{d}x^A\mathrm{d}x^B]$, which is a one-parameter family of Riemannian metrics on S^2, sufficiently well-behaved at the vertex [Chruściel, 2014].
- Local existence holds to the future of C_O near O [Choquet-Bruhat, Chruściel & Martin-Garcia, 2011].
- Local existence holds to the future of C_O (supposing that the characteristic constraint equations admit a solution) [Luk, 2012].
- Obstruction provided by the Raychaudhuri equation.

Theorem (Cabet, Chruściel & Tagne Wafo, 2016)

Consider smooth data for the CFE in $(\mathscr{M}, g)$ on a light-cone C_0 which admit a smooth expansion through the 2-sphere S where $\Theta|_{C_O}$ vanishes. Then local existence holds and the emerging spacetime admits a piece of a smooth $\mathscr{I}^+$.

Question for both types of initial value problems:
Do seed data which are smooth at S produce data for the CFE which are smooth at S?

Characteristic Cauchy problem

- Consider smooth data $\widetilde{\gamma}_{AB} \in [\widetilde{g}_{AB}]$ on C_O which admit an expansion of the form

$$\widetilde{\gamma}_{AB} \sim r^2\Big(s_{AB} + \sum_{n=1}^{\infty} h^{(n)}_{AB} r^{-n}\Big).$$

- Necessary in spacetimes which admit a smooth $\mathscr{I}$.
- Solve Raychaudhuri equation for the expansion $\widetilde{\tau}$ ($\widetilde{g}_{AB} = e^{\int \widetilde{\tau}}\widetilde{\gamma}_{AB}$)

$$(\partial_r - \widetilde{\kappa})\widetilde{\tau} + \frac{1}{2}\widetilde{\tau}^2 + |\widetilde{\sigma}|^2(\widetilde{\gamma}) = 0\,, \quad \widetilde{\kappa} = O(r^{-3})\,.$$

- Solution satisfies

$$\widetilde{\tau} = 2r^{-1} + \widetilde{\tau}_2 r^{-2} + O(r^{-3})\,.$$

- No-logs-condition [Chruściel & P. '15] [P. '15]

$$(h^{(2)}_{AB})_{\mathrm{tf}} - \frac{1}{2}\Big(\mathrm{tr} h^{(1)} + \widetilde{\tau}_2\Big)(h^{(1)}_{AB})_{\mathrm{tf}} = 0\,.$$

- The no-logs conditions holds if and only if the Weyl tensor vanishes at S.

Smoothness of $\mathscr{I}$

Theorem (Andersson, Chruściel & Friedrich '92, Andersson & Chruściel '93; Chruściel & P. '15, P. '15)

(a) *Generically, solutions of the constraint equations constructed from smooth "seed" data for*
   (i) *the hyperboloidal, and*
   (ii) *the characteristic Cauchy problem*

   are polyhomogeneous rather than smooth at S.

(b) *There exists a large class of "non-generic" data which admit a smooth conformal completion.*

Consequence:

Mild regularity conditions on the asymptotic behavior of the seed data are sufficient to guarantee at least a piece of a smooth $\mathscr{I}$.

Spatial infinity as a point

- So far spatial infinity i^0 has been excluded from the considerations.
- Ultimately one would like to construct asymptotically simple spacetimes from asymptotically Euclidean Cauchy data sets.
- **Issue:** If $m_{ADM} \neq 0$ spatial infinity i^0 cannot be a regular point.

$\rightarrow$ Representation of spatial infinity as a point too compressed.

Alternative conformal transformation of Minkowski

Consider again the Minkowski line element

$$\widetilde{g}_{\mathrm{mink}} = -(\mathrm{d}y^0)^2 + (\mathrm{d}y^1)^2 + (\mathrm{d}y^2)^2 + (\mathrm{d}y^3)^2 \,.$$

Introduce new coordinates: First, set

$$x^\mu := \frac{y^\mu}{y^\nu y_\nu}\,, \quad y^\nu y_\nu > 0\,,$$

and then

$$R := \sqrt{x^i x_i}\,, \qquad \tau := x^0/\sqrt{x^i x_i}\,, \quad i = 1,2,3\,.$$

The Minkowski metric can be written in the form $\widetilde{g}_{\mathrm{mink}} = \Theta^{-2} g_{\mathrm{mink}}$, where

$$\Theta = R(1-\tau^2)\,,$$

and

$$g_{\mathrm{mink}} = -\mathrm{d}\tau^2 - 2\frac{\tau}{R}\mathrm{d}\tau\mathrm{d}R + \frac{1-\tau^2}{R^2}\mathrm{d}R^2 + \mathrm{d}\Omega_2\,, \quad |\tau| < 1 \quad , \quad R > 0\,.$$

- $\mathscr{I}^\pm = \{\tau = \pm 1, R > 0\}$,
- spatial infinity $I = \{|\tau| < 1, R = 0\} \cong (-1,1) \times S^2$,
- the "critical sets" $I^\pm = \{\tau = \pm 1, R = 0\} \cong S^2$.

$\Rightarrow$ Cylinder-representation of spatial infinity [Friedrich, 1998]

General conformal field equations

Introduce additional gauge degrees of freedom:

- orthonormal frame field e_a, i.e. $g(e_a, e_b) = \delta_{ab}$
- Weyl connection $\widehat{\nabla}$ (satisfies $\widehat{\nabla}_\alpha g_{\mu\nu} = -2b_\alpha g_{\mu\nu}$ for a 1-form b)

As unknowns one regards $e^\mu{}_a$, $\widehat{\Gamma}_a{}^b{}_c$, $\widehat{L}_{ab}$, and $W^a{}_{bcd}$.

The general conformal field equations (GCFE) read

$$
\begin{aligned}
[e_a, e_b] &= 2\widehat{\Gamma}_{[a}{}^c{}_{b]} e_c \,, \\
e_{[a}(\widehat{\Gamma}_{b]}{}^i{}_j) - \widehat{\Gamma}_k{}^i{}_j \widehat{\Gamma}_{[a}{}^k{}_{b]} + \widehat{\Gamma}_{[a}{}^i{}_{|k|} \widehat{\Gamma}_{b]}{}^k{}_j &= \delta_{[a}{}^i \widehat{L}_{b]j} - \delta_j{}^i \widehat{L}_{[ab]} - \delta_{j[a} \widehat{L}_{b]}{}^i + \frac{\Theta}{2} W^i{}_{jab} \,, \\
2\widehat{\nabla}_{[a} \widehat{L}_{b]c} &= d_e W^e{}_{cab} \,, \\
\widehat{\nabla}_e W^e{}_{cab} &= \frac{1}{4} \widehat{\Gamma}_e{}^f{}_f W^e{}_{cab} \,.
\end{aligned}
$$

where $d_a := \Theta b_a + \widehat{\nabla}_a \Theta$.

Conformal Gauß gauge

A conformal geodesic is a curve in $(\mathscr{M}, g)$ such that $\exists$ 1-form b such that

$$
\begin{aligned}
\nabla_{\dot{x}}\dot{x} + \langle \dot{x}, b\rangle \dot{x} - g(\dot{x},\dot{x}) g^{\flat}(b,\cdot) &= 0\,, \\
\nabla_{\dot{x}} b - \langle \dot{x}, b\rangle b + \frac{1}{2} g^{\flat}(b,b) g(\dot{x},\cdot) &= L(\dot{x},\cdot)\,.
\end{aligned}
$$

- Given data $(x_*, \dot{x}_*, b_*)$ there exists a unique conformal geodesic $(x(\tau), b(\tau))$ near x_*.
- The sign of $g(\dot{x},\dot{x})$ is preserved along a conformal geodesic.

Construct a gauge adapted to a congruence of (timelike) conformal geodesics intersecting an initial surface Σ transversally.

Gauge conditions:

$$
\widehat{\nabla}_{\dot{x}} g = 0\,, \quad \widehat{\nabla}_{\dot{x}} e_a = 0\,,
$$

supplemented by certain gauge data on Σ, in particular $\Theta|_{\Sigma}$.

Lemma (Friedrich, 1995)

$\widehat{\nabla}_{\dot{x}}\widehat{\nabla}_{\dot{x}}\widehat{\nabla}_{\dot{x}}\Theta = 0.$

- In the conformal Gauß gauge the location of the conformal boundary is a priori known.

Regular initial value problem at spatial infinity

This setting allows the formulation of a regular initial value problem at spatial infinity for appropriate Euclidean data $(\widetilde{\Sigma}, \widetilde{h}, \widetilde{K})$ [Friedrich, 1998], [Dain & Friedrich, 2001].

GCFE imply a symmetric hyperbolic system of evolution equations

Main issue: Hyperbolicity breaks down at $I^{\pm}$.

GCFE imply inner equations on the cylinder I which can be analyzed.

- Generically the solutions are polyhomogenous at $I^{\pm}$ [Friedrich, 1998] [Kroon, 2008].
- It is expected that appropriate regularity conditions at I^0 ensure smoothness at $I^{\pm}$.
- For time symmetric data which are conformally flat near I^0 the emerging spacetime is smooth at $I^{\pm}$ if and only if the data are Schwarzschildean near I^0 [Kroon, 2009].

Open issue: Does smoothness at $I^{\pm}$ imply smoothness of $\mathscr{I}^{\pm}$?

Conclusions

1. CFE permit the construction of $\Lambda \geq 0$-vacuum spacetimes which admit at least a piece of a smooth $\mathscr{I}$ (and in some situations even more).
2. For $\Lambda > 0$ asymptotic simplicity seems to be a very reasonable notion.
3. For $\Lambda = 0$ a smooth $\mathscr{I}$ arises from hyperboloidal or characteristic Cauchy problem only if certain regularity conditions are satisfied.
4. To get a full understanding of the compatibility of asymptotic simplicity with Einstein's field equations a better understanding of spatial infinity is essential, in particular of the critical sets $I^{\pm}$.

Thank you!