Sergio Daín (1969-2016)
Understanding isolated system dynamics in General Relativity
A perspective on Sergio Dain’s contribution to General Relativity

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1. Gravitational collapse in General Relativity: the general framework

2. Aspects of the Cauchy problem in General Relativity
   - Elliptic problems in General Relativity
   - Einstein equations: physical content

3. Geometric inequalities: the role of angular momentum
   - Global inequalities: $|J| \leq m^2$
   - Local inequalities: $A \geq 8\pi|J|$

4. Perspective
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Gravitational collapse in General Relativity: the general framework

A general framework of research

Understanding Einstein equations: interplay of Geometry, Analysis and Physics

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]
A general framework of research

Understanding Einstein equations: interplay of Geometry, Analysis and Physics

\[ R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 8\pi \, T_{\mu\nu} \]
A general framework of research

Understanding Einstein equations: interplay of Geometry, Analysis and Physics

\[ R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 8\pi \, T_{\mu\nu} \]

Classical gravitational collapse picture

1. **Singularity Theorems**: incomplete inextendible causal geodesic, given “strong gravitational field” data on \( \tilde{S} \)
   - Trapped surfaces [Penrose, Hawking, 65, 67, 70, 73...].
   - Non-simply connected data [Gannon, Lee 75, 76...].

2. **(Weak) Cosmic Censorship conjecture** [Penrose 69]:
   Complete \( \mathcal{I}^+ \) and Black Hole region and Horizon.

3. **Spacetime settles down to a stationary final state**:
   Positivity mass theorems [Schoen & Yau 79, 80, Witten 81].

4. **BH uniqueness “theorems”** [e.g. Chruściel et al. 12]:
   Final state given by Kerr spacetime, \((m, J)\).

   **Initial value problem**: 3+1 approach.
Aspects of the Cauchy problem in General Relativity

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A physical formulation

“Suppose we want to describe an isolated self-gravitating system. For example a star, a binary system, a black hole or colliding black holes. Typically these astrophysical systems are located far away from the Earth, so that we can receive from them only electromagnetic and gravitational radiation. How is this radiation? For example one can ask how much energy is radiated, or which are the typical frequencies for some systems. This is the general problem we want to study.”

[S. Dain, Lecture Notes in Physics 604, 161-182 (2002)]

A first step in the mathematical study: elliptic systems

‘Elliptic problems appear naturally in physics mainly in two situations: as equations which describe equilibrium (for example, stationary solutions in General Relativity) and as constraints for the evolution equations (for example, constraint equations in Electromagnetism and General Relativity). In addition, in General Relativity they appear often as gauge conditions for the evolution equations.’

[S. Dain, Lecture Notes in Physics 692, 117-139 (2006)]
Initial data problem in General Relativity: Cauchy problem

Initial data set for Einstein equations

\((\tilde{S}, \tilde{h}_{ij}, \tilde{K}_{ij}, \mu, j^i)\)

- \(\tilde{S}\): connected three-dimensional manifold.
- \(\tilde{h}_{ij}\): Riemannian metric.
- \(\tilde{K}_{ij}\): symmetric tensor field.
- \(\mu\): scalar field.
- \(j^i\): vector field on \(\tilde{S}\).

Constraint equations

\[
\begin{align*}
\tilde{R} - \tilde{K}_{ij} \tilde{K}^{ij} + \tilde{K}^2 &= 16\pi \mu \quad \text{(Hamiltonian constraint)} \\
\tilde{D}_j \tilde{K}^{ij} - \tilde{D}^i \tilde{K} &= -8\pi j^i \quad \text{(Momentum constraint)}
\end{align*}
\]
Asymptotic flatness

Asymptotically Euclidean initial data

Data on $\tilde{S}$ are asymptotically flat with $N$ asymptotic ends, if for some compact set $\Omega$ we have $\tilde{S} \setminus \Omega = \bigcup_{k=1}^{N} \tilde{S}_{(k)}$ where $\tilde{S}_{(k)}$ are open sets that can be mapped by a coordinate system $\tilde{x}^j$ diffeomorphically onto the complement of a closed ball in $\mathbb{R}^3$ such that

$$\tilde{h}_{ij} = \left(1 + \frac{2m}{\tilde{r}}\right) \delta_{ij} + O(\tilde{r}^{-2}), \quad \partial_k \tilde{h}_{ij} = O(\tilde{r}^{-2}), \quad \partial_l \partial_k \tilde{h}_{ij} = O(\tilde{r}^{-3})$$

$$\tilde{K}_{ij} = O(\tilde{r}^{-2}), \quad \partial_k \tilde{K}_{ij} = O(\tilde{r}^{-3})$$

as $\tilde{r} = \left(\sum_{j=1}^{3} (\tilde{x})^2\right)^{1/2} \to \infty$ in each “asymptotic end” $\tilde{S}_{(k)}$. 
Mass $m$ and angular momentum $J$

Conserved quantities at spatial infinity

- ADM mass $m$:

$$m = \frac{1}{16\pi} \lim_{\tilde{r} \to \infty} \int_{S_r} \left( \partial_j \tilde{h}_{ij} - \partial_i \tilde{h}_{jj} \right) \tilde{\nu}^i dA$$

- ADM momentum $P_i$:

$$P_i = \frac{1}{8\pi} \lim_{\tilde{r} \to \infty} \int_{S_r} \left( \tilde{K}_{jk} - \tilde{K} \tilde{h}_{ij} \right) (\partial_i)^j \tilde{\nu}^k dA$$

- Angular momentum $J_i$ at spatial infinity:

$$J_i = \frac{1}{8\pi} \lim_{\tilde{r} \to \infty} \int_{S_r} \left( \tilde{K}_{jk} - \tilde{K} \tilde{h}_{jk} \right) (\phi_i)^j \tilde{\nu}^k dA$$

, with $\phi_i = \epsilon_{ijk} \tilde{x}^j \partial_k$
Conformal compactification of the data

Focus on vacuum data: \( \mu = 0, j^i = 0 \).

Conformally compactified data:

\[
(S, h_{ij}, K_{ij})
\]

with \( \tilde{S} = S \setminus \{i\} \) (\( i \) point at infinity) and

\[
\tilde{h}_{ij} = \psi^4 h_{ij}, \quad \tilde{K}_{ij} = \psi^{-2} K_{ij} + \frac{1}{3} \tilde{h}_{ij} \tilde{K}
\]

Constraint equations

\[
\left(D_i D^i - \frac{1}{8} R\right) \psi = -\frac{1}{8} K_{ij} K^{ij} \psi^{-7} + \frac{1}{12} \tilde{K}^2 \psi^5 \quad \text{(Lichnerowicz equation)}
\]

\[
D_j K^{ij} = \frac{2}{3} \psi^6 D^i \tilde{K}
\]
Conformal compactification of the data

Focus on vacuum data: $\mu = 0, j^i = 0$.

Conformally compactified data:

$$(S, h_{ij}, K_{ij})$$

with $\tilde{S} = S \setminus \{i\}$ (i point at infinity) and

$$\tilde{h}_{ij} = \psi^4 h_{ij}, \quad \tilde{K}_{ij} = \psi^{-2} K_{ij} + \frac{1}{3} \tilde{h}_{ij} \tilde{K}$$

Constraint equations: maximal slicing $\tilde{K} = 0$

$$\left(D_i D^i - \frac{1}{8} R\right) \psi = -\frac{1}{8} K_{ij} K^{ij} \psi^{-7}$$ (Lichnerowicz equation)

$$D_j K^{ij} = 0$$
The PDE problem: constrained and free initial data

Resolution on the compact \( S \): technical advantages

- Simpler to prove existence of solutions for elliptic equations.
- Simpler to analyze fields in terms of local differentiability in a neighborhood of \( i \).

With [York 73]: \( K_{ij} = Q_{ij} - (\mathbb{L}w)^{ij} \), with \( (\mathbb{L}w)^{ij} = D^i w^j + D^j w^i - \frac{2}{3} h_{ij} D_k w^k \)

\[
\left( D_i D^i - \frac{1}{8} R \right) \psi = -\frac{1}{8} K_{ij} K^{ij} \psi^{-7}, \quad \psi > 0
\]

\[
D^i (\mathbb{L}w)_{ij} \equiv \Delta w_i + \frac{1}{3} D_i D^j w_j + R^i_j w^j = D^j Q_{ji}
\]

Boundary conditions (\( x^j \): \( h \)-normal coordinates centered at \( i \), i.e. \( r = (\sum_{i=1}^{3} (x^j)^2)^{\frac{1}{2}} \))

\[
K_{ij} = O(r^{-4}), \quad (\text{as } r \to 0)
\]

\[
\lim_{r \to 0} r \psi = 1 \quad (\text{at compactified } i)
\]

Free data: \( (h_{ij}, Q_{ij}, \tilde{K}) \). Constrained data: \( (\psi, w^i) \)
Examples of asymptotically flat Initial Data for Black Holes

Two asymptotic ends data

Misner wormhole data

Brill-Lindquist data

Misner data ("images" method)

Non-time symmetric generalizations

Bowen-York, (generalized) punctures...
Asymptotically Flat ID with prescribed Regularity at Infinity


(Initial) Objectives

Construction of data satisfying “multipole expansion”:

\[ \tilde{h}_{ij} \sim \left( 1 + \frac{2m}{\tilde{r}} \right) \delta_{ij} + \sum_{k \geq 2} \frac{\tilde{h}_{ij}^k}{\tilde{r}^k}, \quad \tilde{K}_{ij} \sim \sum_{k \geq 2} \frac{\tilde{K}_{ij}^k}{\tilde{r}^k} \]

- Problem of evolving asymptotically flat Initial Data near space-like and null infinity with Friedrich’s conformal field equations [Friedrich 98; cf. T-T. Paetz’s talk].
- Implications of “regular finite initial value problem near space-like infinity” in numerical relativity.

Results

A series of (hard) theorems giving sufficient conditions for the existence of solutions with prescribed regularity to the elliptic system. Explicit constructions.

Keywords/Elements: Sobolev imbedding, Rellich-Kondrakov, Schauder fixed point, $L^p$ regularity, Schauder elliptic regularity, Fredholm alternative, Weak and Strong Maximum Principle...
Aspects of the Cauchy problem in General Relativity

Elliptic problems in General Relativity

Asymptotically Flat ID with prescribed Regularity at Infinity


Theorem 1 (Hamiltonian constraint). Let $h_{ij}$ be a smooth metric on $S$ with positive Ricci scalar $R$. Assume that $K_{ij}$ is smooth in $\tilde{S}$ and satisfies in a convex normal neighborhood $B_a$

$$r^8 K_{ij} K^{ij} \in \mathbf{E}^\infty(B_a).$$

Then there exists on $\tilde{S}$ a unique solution $\psi$ of the Hamiltonian constraint, which is positive, satisfies the boundary conditions, and has in $B_a$ the form

$$\psi = \frac{\hat{\psi}}{r}, \quad \hat{\psi} \in \mathbf{E}^\infty(B_a), \quad \hat{\psi}(i) = 1.

(f \in C^\infty(\tilde{S}) is in $\mathbf{E}^m(B_a)$ if on $B_a$ we can write $f = f_1 + rf_2$ with $f_1, f_2 \in C^m(B_a)$).

Theorem 2 (Momentum constraint). Let $h_{ij}$ be a smooth metric in $S$. There exist trace-free tensor fields $K_{ij} \in C^\infty(S \setminus i)$ satisfying $K_{ij} \sim \sum_{k \geq -4} K_{ij}^k r^k$ (with $K_{ij}^k \in C^\infty(S^2)$) with the following properties:

i) $K_{ij} = K_{ij}^{AJ} + \hat{K}_{ij}$, with $\hat{K}_{ij} = O(r^{-2})$ and $K_{ij}^{AJ} = \frac{A}{r^3} (3n_i n_j - \delta_{ij}) + \frac{3}{r^3} (n_j \varepsilon_{kil} J^l n^k + n_i \varepsilon_{ljk} J^k n)$

ii) $D_j K^{ji} = 0$

iii) $r^8 K_{ij} K^{ij} \in \mathbf{E}^\infty(B_a)$
Aspects of the Cauchy problem in General Relativity

Elliptic problems in General Relativity

Comments

General Comments

- **Rigorous construction** of a large class of solutions to the constraint equations, providing regular initial data aiming at smooth asymptotic structure at null infinity from the evolution of Friedrich’s conformal equations.
- **Regularity condition kills linear momentum** $P^i$: generic logarithmic terms in the expansion if $P^i \neq 0$ (no multipole expansion of the prescribed form).
- Explicit solutions of the momentum constraint (with **angular momentum**).
  - Euclidean space.
  - Axisymmetric solutions. (!)

Foundational article in Sergio’s career: “*Train in the right trail*” [H. Friedrich]

- Very solid piece of work, technically and conceptually, but smoothness $I^+$ turned out to be more complicated [Valiente-Kroon 04,05; cf. T-T. Paetz’s talk].
- Difficult to overestimate the formative character of this article (“*this elliptic problem played a chord in the mathematical sensitivity of Sergio*”, [H. Friedrich])
- (A plausible interpretation as a “Wittgenstein ladder”).
Aspects of the Cauchy problem in General Relativity

Trapped surfaces as boundaries for the constraint equations

**Article:** [S. Dain, Class. Quantum Grav. 21, 555-573 (2004)]

**Not yet BBHs!!!** [Pretorius 05]

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**BH initial data: excision technique**

Given $C$ closed surface, expansion null (geodesics), with $\tilde{H} = \tilde{D}_i\tilde{v}^i$:

$$\theta_\pm = \nabla_i (t^i \mp \tilde{v}^i) = \tilde{K} \mp \tilde{H} - \tilde{K}_{ij} \tilde{v}^i \tilde{v}^j$$

Under the proper global and energy conditions future (marginally) trapped surfaces [cf. Galloway’s course]

$$\theta_+ \leq 0 \ , \ \theta_- \leq 0$$

lay in the Black Hole region.

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**Elliptic reduction, conformal method:**

$$\theta_\mp = \psi^{-3} \left( \pm 4\nu^i D_i \psi \pm H \psi - \psi^{-3} K_{ij} \nu^i \nu^j \right) \quad \text{[cf. López’s talk]}

\[
\left( D_i D^i - \frac{1}{8} R \right) \psi = -\frac{1}{8} K_{ij} K^{ij} \psi^{-7} \quad , \quad \psi > 0 \quad , \quad \text{on } \tilde{\Omega}
\]

\[
\left( 4\nu^i D_i \psi + H \right) \psi = \mp \left( \psi^3 \theta_\pm + \psi^{-3} K_{ij} \nu^i \nu^j \right) \quad , \quad \text{on } \partial \Omega
\]

\[
\lim_{r \to 0} r \psi = 1 \quad , \quad \text{(at compactified } i)\]
Theorem. Assume a conformal metric \( h_{ij} \in C^\infty(\tilde{\Omega}) \cap C^2(\tilde{\Omega}) \) and \( \partial \Omega \) smooth. Assume \( R \geq 0 \) in \( \tilde{\Omega} \) and \( H \geq 0 \) on \( \partial \Omega \) (either \( R \) or \( H \) not identically zero). Let us prescribe at \( \partial \Omega \)
\[
\theta_- \leq 0, \quad |\theta_-| \leq \psi_1 K^{ij} \nu_i \nu_j
\]
Then there exists a unique, positive, solution \( \psi \) and \( \theta_+ \leq \theta_- \leq 0 \).

Complementary results and extensions

**Marginally trapped surfaces:** \( \theta_+ = 0 \) [D. Maxwell, Commun. Math. Phys. 253, 561 (2004)]:
Assumptions: \( R = 0, H < 0, H \leq -K^{ij} \nu_i \nu_j \leq 0 \) and \( \lambda_{h,\Omega} > 0 \), where \( \lambda_{h,\Omega} \) generalizes the Yamabe invariant (note \( \theta_- \leq 0 \)):
\[
\lambda_{h,\Omega} = \inf_{\varphi \in C^\infty_c(M), \varphi \neq 0} \frac{\int_{\Omega} (8 D_i \varphi D^i \varphi + R \varphi^2) + 2 \int_{\partial \Omega} H \varphi^2}{\|\varphi\|_{L^6}^2}
\]

**Stationary apparent horizons (isolated horizons):** \( \theta_+ = \sigma_+^{ij} = 0 \) [S. Dain, J. L. Jaramillo, B. Krishnan, PRD, 71 064003 (2005)]. Well-posed problem with mixed Neumann-Dirichlet conditions for the momentum constraint (Lopatinski-Schapiro ellipticity conditions). ID for black hole in instantaneous equilibrium.
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Understanding regularity of stationary solutions

Understanding regularity of stationary initial data

[S. Dain, Class. Quantum Grav. 18, 4329-4338 (2001)]
Characterization of the fall-off behaviour of the intrinsic metric and the extrinsic curvature of Cauchy initial data for asymptotically flat, stationary vacuum spacetimes near spacelike infinity.

- **It fills gap in the proof** of analytic compactification at null infinity of asymptotically flat, vacuum stationary spacetimes [Beig & Simon 81; Damour & Schmidt 90].
- **Understanding** of regularity conditions to be imposed at infinity for stationary data: **key for constructing data containing Kerr**.

Understanding content of initial data: Initial Data for Two Kerr-like Black Holes

**Known families of binary data do not contained Kerr (stationary solutions):** difficulty to assess their physical meaning, namely the gravitational waves content.

  Family of data containing two Kerr as individual limit. Key property:
  \[ h_{ij} \in W^{4,p}(S^3), \quad p < 3 \]

- [S. Dain, Phys. Rev. D64, 124002 (2001)]:
  Improved family of binary Kerr in head-collision and in the close limit.
Binary black holes: energy and Gravitational Waves

Characterization of absence of “gravitational wave content”

**Geometric invariant characterizing staticity** [S. Dain, Phys.Rev.Lett 93, 231101 (2004)]

- 2nd-order operator $\mathcal{P}$ from “constrained map”.
- Obtain $\alpha$ by solving 4-order elliptic equation $\mathcal{PP}^{*} \eta = 0$ on $(S, h_{ij})$.
- $\lambda_{(k)} = -\frac{1}{8\pi} \oint_{\partial S_{(k)}} n^{i} D_{i} \alpha_{(k)}$. Thm.: $(S, h_{ij})$ is static iff $\lambda_{(k)} = 0$ at come end $i_{k}$.

Conserved quantities

Brill-Lindquist-type data, with two “internal” asymptotic ends $i_{1}, i_{2}$ and a third “external” one, with respective masses $m_{1}, m_{2}, m_{3}$.

- **Black hole interaction energy** [S. Dain, Phys. Rev. D 66, 084019 (2002)].

\[ E = m_{3} - m_{1} - m_{2} = \frac{-m_{1}m_{2}}{r_{12}} + \frac{-J_{1} \cdot J_{2} + 3(J_{1} \cdot \hat{n})(J_{2} \cdot \hat{n})}{r_{12}^{3}} + \text{higher order terms} \]

- **Conserved quantities in a black hole collision** [S. Dain, J.A. Valiente-Kroon, Class. Quantum Grav. 19, 811-815 (2002)].

\[ G_{0} = -\frac{127\sqrt{5}\pi}{4} r_{12}^{2} m_{1} m_{2} \]

Constant along $\mathcal{J}^{+}$: information about the late time evolution of the collision.
Axisymmetry

Axisymmetric (maximal) initial data parametrized by two functions: \((q, \omega)\)

Axial Killing vector \(\eta^i\) of \((S, \tilde{h}_{ij}, \tilde{K}_{ij})\) (with \(\tilde{K} = 0\)): \(\mathcal{L}_\eta \tilde{h}_{ij} = 0\), \(\mathcal{L}_\eta \tilde{K}_{ij} = 0\).

Conformal decomposition: \(\tilde{h}_{ij} = \psi^4 h_{ij}\), \(\tilde{K}_{ij} = \psi^{-2} K_{ij}\), and \(\eta = \eta^i \eta^j h_{ij}\). Then, with

\[
\begin{align*}
  h_{ij} &= e^{-2q} \left( d\rho^2 + dz^2 \right) + \frac{1}{\rho^2} \eta_i \eta_j \quad \text{[D. Brill, Ann. Phys. 7 466-83 (1959)]} \\
  K^{ij} &= \frac{2S^{(i} \eta^{j)}}{\eta} , \quad S^i = \frac{1}{2\eta} \epsilon^{ijk} \eta_j D_k \omega \quad , \quad \mathcal{L}_\eta \omega = 0
\end{align*}
\]

so that constraints become

\[
D_j K^{ji} = 0 \ , \ \left( \Delta - \frac{1}{8} R \right) \psi = -\frac{1}{8} \frac{D_i \omega D^i \omega}{2\eta^2} \psi^{-7}
\]

Komar angular momentum of a closed surface \(S\)

\[
J(S) = \frac{1}{16\pi} \int_S \epsilon_{\mu\nu\lambda\gamma} \nabla^\lambda \eta^\gamma = \frac{1}{8\pi} \int_S K_{ij} \eta^i \nu^j dA =
\]

Remark: in vacuum, if \(S'\) contains \(S\), it holds: \(J(S') = J(S)\).
Aspects of the Cauchy problem in General Relativity

Einstein equations: physical content

Geometry, Analysis, Physics... and Numerics

New axisymmetric data: exploring extremality

Axisymmetric data with: \( q = 0 \) and \( \omega = \omega_{\text{Kerr}} \).


\[
\varepsilon_J \equiv \frac{J}{m^2}, \quad \varepsilon_J \leq 1 \\
\varepsilon_A \equiv \frac{A}{8\pi \left( m^2 + \sqrt{m^4 - J^2} \right)}, \quad \varepsilon_A \leq 1
\]

Constructing binary black hole with maximum “kick” velocity: \textbf{cylindrical ends in extremal data}

\( v_{\text{recoil}} = 3290 \pm 47 \text{ km} \cdot \text{s}^{-1} \): maximum with numerics!


Based on existence theorem for extremal Bowen-York data:


Numerical exploration of geometric inequalities [Jaramillo, Vasset, Ansorg, Novak 08, 09]

\( \varepsilon_A \equiv \frac{A}{8\pi \left( m^2 + \sqrt{m^4 - J^2} \right)} \)

coined as \textbf{Dain’s number: RIGIDITY!!!}.
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coined as Dain’s number: RIGIDITY!!!
To retain:

- Mastery of elliptic theory.
- **Axisymmetry**.
- Sharp geometric bounds: angular momentum.
- Equality: **rigidity** (and cylindrical ends).
Geometric inequalities: the role of angular momentum

Scheme

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4. Perspective
Prototype: Isoperimetric inequality

\[ L^2 \geq 4\pi A \quad (= \text{circle}) \]

**Geometry**

“General Relativity is a geometric theory, hence it is not surprising that geometric inequalities appear naturally in it. Many of these inequalities are similar in spirit as the isoperimetric inequality. […] the inequality applies for a rich class of objects and the equality only applies for an object of “optimal shape”. This object, like the circle, can be described by few parameters and it has also a variational characterization.”

**Physics**

“[…] General Relativity is also a physical theory. It is often the case that the quantities involved have a clear physical interpretation and the expected behavior of the gravitational and matter fields often suggests geometric inequalities which can be highly non-trivial from the mathematical point of view. The interplay between physics and geometry gives to geometric inequalities in General Relativity their distinguished character.”
BH uniqueness “theorem” [e.g. Chruściel et al. 12]

- Stationary black holes in vacuum are characterized by sub-extremal Kerr solution: $\sqrt{|J|} \leq m$ [cf. F. Finster’s talk].
- Kerr family parametrized by $(m, J)$: solutions for all values of the parameters.

Relation $(A, m, J)$ in BH case

Horizon Area $(A)$, mass $(m)$, angular momentum $(J)$:

$$A = 8\pi \left( m^2 + \sqrt{m^4 - J^2} \right)$$

Related inequalities

$$A \leq 16\pi m^2$$  \hspace{1cm} (= Schwarzschild)

$$J \leq m^2$$ \hspace{1cm} (= Extreme Kerr)

$$8\pi |J| \leq A$$ \hspace{1cm} (= Extreme Kerr)
Geometric inequalities: the role of angular momentum

Kerr spacetime: stationary BHs (and naked singularities)

Relation \((A, m, J)\) in BH case

Horizon Area \((A)\), mass \((m)\), angular momentum \((J)\):

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A = 8\pi \left( m^2 + \sqrt{m^4 - J^2} \right)
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Related inequalities

\[
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A & \leq 16\pi m^2 \\
J & \leq m^2 \\
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\(=\) Schwarzschild
\(=\) Extreme Kerr
\(=\) Extreme Kerr
**Kerr spacetime: stationary BHs (and naked singularities)**

Geometric inequalities: the role of angular momentum

Relation \((A, m, J)\) in BH case

- Horizon Area \((A)\), mass \((m)\), angular momentum \((J)\):
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  \]

Related inequalities

- \(A \leq 16\pi m^2\) (\(=\) Schwarzschild)
- \(J \leq m^2\) (\(=\) Extreme Kerr)
- \(8\pi |J| \leq A\) (\(=\) Extreme Kerr)

**Question**: \(J \leq m^2\) and \(8\pi |J| \leq A\) in the **Dynamical** case?
Plan

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4. Perspective
Heuristic motivation of $|J| \leq m^2$: weak cosmic censorship

Penrose inequality: $A \leq 16\pi m^2$ [Penrose 69]

Global version:

- BH horizon section $\mathcal{H}_S$, with area $A$, in a slice $S$ with ADM mass $m$.
- Area grows along BH horizon [Hawking 73].
- **Spacetime settles to a stationary state.**
- BH uniqueness: final state is Kerr $(m_o, A_o)$ with $A \leq A_o \leq 16\pi m_o^2$.
- GWs take energy away, Trautman-Bondi mass at $\mathcal{I}^+$ is decreasing: $m_o \leq m$ [cf. Nurowski's talk].
- On $S$ we have: $A \leq 16\pi m^2$.

Local in time:

- Initial Data $(S)$: apparent horizon $\Sigma$.
- **Weak cosmic censorship**: BH event horizon $\mathcal{H}$.
- Minimal surface enclosing $\Sigma$: $A_{\text{min}}(\Sigma) \leq A(\mathcal{H}_S)$.
- $A_{\text{min}}(\Sigma) \leq 16\pi m^2$

Remark: refinement of **mass possitivity** theorem $m \geq 0$. 
Heuristic motivation of $|J| \leq m^2$: weak cosmic censorship

**Dain’s inequality:** $|J| \leq m^2$ [Friedman & Mayer 82]

- BH Initial Data on $S$ with $(m, J)$.
- **Weak Cosmic censorship:** Gravitational collapse results in a Black Hole.
- Axial symmetry: angular momentum cannot be radiated away, $J$ constant along the evolution.
- **Spacetime settles to a stationary state.**
- BH uniqueness, final state is Kerr $(m_o, J_o)$ with $J = J_o \leq m_o^2$.
- GWs take energy away, Trautman-Bondi mass at $\mathcal{I}^+$ is decreasing: $m_o \leq m$ [cf. Nurowski’s talk].
- On $S$ we have: $J \leq m^2$.

\[ m \leq \frac{J^2}{m} \]

\[ (m_o, J_o) \leq \frac{J_s}{m_s} \]

\[ A \leq A_0 \]

\[ m_0 \leq m \]

\[ J_0 = J \]

\[ (J, m) \]

Singularity

Horizon

Radiation

$I^+ +$
Heuristic motivation of $|J| \leq m^2$: weak cosmic censorship

Dain's inequality: $|J| \leq m^2$ [Friedman & Mayer 82]

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- On $S$ we have: $J \leq m^2$.

“Weak Cosmic Censorship” and “settlement to stationarity”

“[...] a counter example will imply that the standard picture of the gravitational collapse is not true. Conversely, a proof of gives indirect evidence of its validity, since it is very hard to understand why this highly nontrivial inequality should hold unless [WCC and final stationary] can be thought of as providing the underlying physical reason behind it.”

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Inequality $J \leq m^2$ (Dain’s inequality)

Theorem. Consider an axially symmetric, vacuum, asymptotically flat and maximal initial data set with two asymptotics ends. Let $m$ and $J$ denote the total mass and angular momentum at one of the ends. Then, the following inequality holds

$$\sqrt{|J|} \leq m$$

(= Extreme Kerr)

Steps of the proof: lower bound + variational analysis

- **Variational principle** [S. Dain, Class. Quantum. Grav., 23, 6857-6871 (2006)]
  Brill’s expression for the total mass, bounded below by mass functional $M(\sigma, \omega)$:
  $$m \geq M(\sigma, \omega) = \frac{1}{32} \int_{\mathbb{R}^3} \left( |\partial \sigma|^2 + \rho^{-4} e^{-2\sigma} |\partial \omega|^2 \right) dV$$
  Positivity of mass “as a tool”.

- **Local version** [S. Dain, CQG 23, 6845-6855 (2006); Phys.Rev.Lett. 96, 101101(2006)]
  Proof that extreme Kerr is a local minimum of $M(\sigma, \omega)$, with $M(\sigma, \omega) \geq \sqrt{|J|}$.

- **Global version** [S. Dain, J. Differential Geometry, 79 (1) 33-67 (2008)]
  $$M(\sigma, \omega) \sim M'_\Omega(\eta, \omega) = \frac{1}{32\pi} \int_{\Omega} \left( \frac{|\partial \eta|^2 + |\partial \omega|^2}{\eta^2} \right) dV$$
  Unique absolute minimum through harmonic maps argument [Hildebrandt et al. 77]

Generalizations and improvements

- Simplification, charge, improvement of rigidity [Chruściel, Weinstein, Costa, Schoen, Zhou...]
- Numerical exploration of multiple $i_k$ [S. Dain, O. Ortiz, Phys. Rev. D, 80, 024045 (2009)]
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4. Perspective
Heuristic motivation of local Black Hole extremality

Assessing Christodolou mass as BH quasi-local mass candidate

\[ m_{\text{Chris}} = \sqrt{\frac{A}{16\pi} + \frac{4\pi J^2}{A}} \]

- Mass not growing in the generic case: “Penrose process/superradiance”, extraction of \( J \) [cf. F. Finster talk].

- In vacuum and axisymmetry, no loss of \( J \) is possible \( \delta J = 0 \).

**Mass expected to grow on physical grounds:**

\[ \delta m_{\text{Chris}} = \frac{\kappa}{8\pi} \delta A + \Omega_H \delta J = \frac{\kappa}{8\pi} \delta A \geq 0 \]

Then, since \( \delta A \geq 0 \) [Hawking 73], we have \( \delta m_{\text{Chris}} \geq 0 \) iff:

\[ \kappa = \frac{1}{4m_{\text{Chris}}} \left( 1 - \frac{(8\pi J)^2}{A^2} \right) \geq 0 \iff A \geq 8\pi |J| \]

Research context

- Conjectured in the stationary case with surrounding matter [Ansorg, Petroff 05,06]. **Proof** (including equality case and charge) [Ansorg, Pfister, Hennig, Cederbaum 08,11]

- Extremality conditions [Booth & Fairhurst 08].

- 1st law of Thermodynamics in Dynamical Horizons: [Ashtekar & Krishnan 02,03]
Inequality $A \geq 8\pi J$

**Theorem.** Given an axisymmetric closed marginally trapped and stable surface $S$, in a non-vacuum spacetime with $\Lambda \geq 0$ and with matter fulfilling the dominant energy condition, it holds the inequality (with $A$ and $J$ the area and angular momentum of $S$)

$$A \geq 8\pi |J| \quad (= \text{Extreme Kerr throat})$$

**Steps of the proof:** stability condition + variational analysis

- **Variational functional** [S. Dain, Phys. Rev. D 8, 104010 (2010)]
  
  Definition of an extreme throat geometry, with $J$. Identification of $\mathcal{M}_S$
  
  $$\mathcal{M}_S = \frac{1}{2\pi} \int_S \left( |D\sigma|^2 + 4\sigma + \left| \frac{D\omega}{\eta} \right|^2 \right) dS_o$$

- **Proof of** $e^{(\mathcal{M}_S - 8)/8} \geq 2|J|$ [A. Aceña, S. Dain, M.E. Gabach-Clément, CQG 28, 105014 (2011)]

- **Proof for vacuum, maximal ($\tilde{K} = 0$), globally axisym. data** [S. Dain, M. Reiris, PRL 107, 051101 (2011)]: Stability minimal surfaces: $A \geq 4\pi e^{(\mathcal{M}_S - 8)/8}$. Rigidity proof.

- **Quasilocal spacetime proof** [J.L. Jaramillo, M. Reiris, S. Dain, PRD 84, 121503(R) (2011)]
  
  Essential ingredients: Lorentzian spacetime structure + stability of marginally outer trapped surfaces [Andersson, Mars, Simon 05; Galloway, Schoen 06] [cf. Galloway’s course].

**Generalizations**

- Charges, higher dimensions, Einstein-Maxwell dilaton, $\Lambda$, mass [Dain, Gabach-Clément, Jaramillo, Reiris, Hollands, Paetz, Simon, Yazadjiev, Fajman, Khuri, Weinstein, Yamada...]

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**Geometric inequalities for bodies**

**Conjecture:** $R^2(U) \gtrsim \frac{G}{c^3} |J(U)|$

For a rotating body $U$ with $J(U)$ and $R(U)$ “size measure”:
- The speed of light $c$ is the maximum speed (Lorentzian str.).
- *Trapped surface conjecture* [Seifert 79]: $R(U) \gtrsim \frac{G}{c^2} m(U)$.
- It holds for black holes: $A \geq 8\pi \frac{G}{c^3} |J|$.

**Theorem** [S. Dain, Phys. Rev. Lett. 112, 041101 (2014)]. Consider a maximal, axially symmetric, initial data set that satisfy the dominant energy condition. Let $U$ be an open set on the data. Assume that the energy density is constant on $U$. Then it holds

$$R^2(U) \geq \frac{24}{\pi^3} \frac{G}{c^3} |J(U)|$$

where, for a Killing vector $\eta^i$ with norm $\lambda$, we define $R(U)$ as $R(U) \equiv \frac{2}{\pi} \left( \int_U \lambda \right)^{1/2}.$

**Further (ongoing...) steps in inequalities for bodies**

- **Charge, angular momentum, energy, size** [S. Dain, PRL 112, 041101 (2014)]:

$$\frac{Q^4}{4R^2} + \frac{c^2 J^2}{R^2} \leq E^2$$

(motivated by Bekenstein bounds)

- **Sharp lower bounds charge-radius** [P. Anglada, S. Dain, O. Ortiz, PRD 93, 044055 (2016)].

Theorem, physical discussion, numerical exploration (published: 23 February 2016)
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Back to the classical gravitational collapse picture

1. Singularity Theorems.
2. (Weak) Cosmic Censorship conjecture.
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Gravitational radiation
Back to the classical gravitational collapse picture

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An ongoing research program...

- **1st stage, elliptic systems**: initial data and stationarity.
- **2nd stage, variational analysis**: geometric inequalities.
- **3rd stage, hyperbolic problems**: well-posedness of (axisymmetric) evolutions systems, wave equations/linear stability and extremality...

Final words

Research: Mathematical answers for Physical questions, Physical intuitions for Mathematical queries

Richness, soundness, astonishing inner consistency. Results he highlighted:

- A stability result in black holes: $|J| \leq m^2$.
- Relation between size and angular momentum: $A \geq 8\pi |J|$.

Teaching and mentoring: a source of satisfaction ("bringing ideas…")

- Crucial role in the GR community: a gift for distilling the finest ideas of the previous generation and transmitting them (enriched!) to the younger one.
- A gift for pedagogy: his articles as optimal places to get started in a domain.
- A gift to light up enthusiasm in his many "students".
- A gift to express important ideas in simple, elegant… and few words.
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Gracias Sergio