Some variational problems involving Lancret curves

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This is a survey on a series of joint papers with
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1. Introduction

Helical configurations are structures commonly found in Nature. They appear in microscopic systems (biomolecules, bacterial fibers, nanosprings, protein chains in particular DNA,...) as well as in macroscopic phenomena (strings, ropes, climbing plants,...) (see for example [1, 9, 11, 12, 13, 14, 21, 25] and references therein). In particular, they are very important and ubiquitous in biology as a consequence of the following known, in the biological community since the work of Pauling, theorem: *Identical objects, regularly assembled, form a helix* (see [10] and references therein).

As far as we know, helical structures are usually identified with the simplest idea of circular helices. However, nothing could be further from the truth. Nobody can believe that squirrels chasing one another up and around tree trunks follow a circular helix path. First, because the cross section of a tree trunk is not circular, but also because its axis is not exactly a straight line. On the other hand, many types of bacteria, such as certain strains of *Escherichia coli* or *Salmonella typhimorium* swim by rotating flagellar filaments. These are polymers which are flexible enough to switch among different helical forms quite different from circular helices. Then we will deal with generalized helices to get answers to those questions.

2. Lancret curves in 3-dimensional space forms

- M. Barros, General helices and a theorem of Lancret. *Proc. AMS*, **125** (1997), 1503-1509 ([2])
- M. Barros, —, P. Lucas and M. A. Meroño, General helices in the three-dimensional Lorentzian space forms. *Rocky Mountain J. Math.*, **31** (2) (2001), 373-388 ([5])

The seminal paper was that of M. Barros [2].

Let $M(0)$ be $\mathbb{R}^3$ or $\mathbb{L}^3$. A Lancret curve (or general helix) in $M(0)$ is a Frenet curve whose tangent indicatrix is contained in some plane $\Pi \subset M(0)$. It will be called degenerate or nondegenerate according to the causal character of such a plane. As in the Euclidean setting, the Lancret curves in $\mathbb{L}^3$ are those for which the ratio of curvature to torsion is constant.

In [2, 5] the notion of Lancret curve was extended to real space forms and spacetimes $M(C)$, with $C \neq 0$, respectively, where the notion of Killing vector field along a curve played an important role. We will consider the class of Lancret curves including not only those curves with torsion vanishing identically, but also the ordinary helices (or simply helices), whose curvature and torsion are both nonzero constants. We will refer to these two cases as trivial Lancret curves. As a resume from [2] and [5] we have:

Lancret curves in $\mathbb{R}^3$. The only ones are geodesics of a right cylinders.

Lancret curves in $\mathbb{L}^3$. The only ones are geodesics in either right cylinders (nondegenerate case) or flat B-scrolls (degenerate case) [see Graves [20] for details on scrolls].

Lancret curves in the hyperbolic space $\mathbb{H}^3(C)$, $C < 0$. A curve is Lancret if and only if either its torsion vanishes identically and it is lying in some hyperbolic plane $\mathbb{H}^2(C)$ or it is an ordinary helix.

Lancret curves in the de Sitter space $\mathbf{dS}_3(C)$, $C > 0$. A curve is Lancret if and only if either its torsion vanishes identically or it is an ordinary helix.

Lancret curves in the 3-sphere $\mathbb{S}^3(C)$, $C > 0$. A curve is Lancret if and only if either its torsion vanishes identically and it is lying in some 2-sphere $\mathbb{S}^2(C)$ or there exists a constant b such that curvature κ and the torsion τ are related by $\tau = q\kappa \pm \sqrt{C}$, where q will be viewed as a sort of slope.

Lancret curves in the anti de Sitter space $\mathbf{AdS}_3(C)$, $C < 0$. A curve γ is Lancret if and only if either its torsion vanishes identically or the curvature κ and the torsion τ are related by $\tau = q\kappa \pm \sqrt{-C}$, where q will be viewed as a sort of slope.

(i) **Nondegenerate case:** γ is Lancret if and only if it is a geodesic of either a Hopf tube or a hyperbolic Hopf tube.

(ii) **Degenerate case:** γ is Lancret if and only if it is a geodesic of a flat scroll over a null curve.

3. Variational problem & Euler-Lagrange equations in 3-dimensional Lorentzian space forms

- M. Barros, —, M. A. Javaloyes and P. Lucas, Relativistic particles with rigidity and torsion in $D = 3$ spacetimes, *Class. Quantum Grav.* 22 (2005) 489-513 ([7])

Let $M(C)$ be a 3-dimensional Lorentzian space with constant curvature C. In a suitable space Λ of Frenet curves in $M(C)$ (for example, the space of closed curves or curves satisfying certain second order boundary data, such as clamped curves), we have a three-parameter family $\{\mathcal{F}_{mnp} : \Lambda \to \mathbb{R} \,|\, m, n, p \in \mathbb{R}\}$ of lagrangians defined by

$$\boxed{\mathcal{F}_{mnp}(\gamma) = \int_\gamma (m + n\kappa + p\tau)ds,} \tag{1}$$

where s, κ and τ stand for the arclength parameter, curvature and torsion of γ, respectively, and the parameters m, n and p are not allowed to be zero simultaneously.

We have found out the moduli space of trajectories regarding the model $[M(C), \mathcal{F}_{mnp}]$, as well as the corresponding algorithms to obtain the trajectories of a given model.

The closed trajectories, when there exist, are also obtained from an interesting quantization principle.

We have used standard arguments, involving integrations by parts, to get the variation of $\mathcal{F}_{mnp}$ along γ in the direction of W

$$\delta\mathcal{F}_{mnp}(\gamma)[W] = \int_\gamma \langle \Omega(\gamma), W\rangle \, ds + [\mathcal{B}(\gamma, W)]_0^L \, , \tag{2}$$

where $\Omega(\gamma)$ and $\mathcal{B}(\gamma, W)$ stand for the Euler-Lagrange and boundary operators, respectively, which are given by

$$\Omega(\gamma) = (-\varepsilon_1\varepsilon_2 m\kappa + \varepsilon_1\varepsilon_2 p\, \kappa\tau - \varepsilon_2\varepsilon_3 n\tau^2 + \varepsilon_1 n\, C)N + (-\varepsilon_1 p\, \kappa_s + \varepsilon_3 n\tau_s)B,$$

$$\begin{aligned} \mathcal{B}(\gamma, W) &= \varepsilon_2 \frac{p}{\kappa} \left\langle \nabla_T^2 W, B \right\rangle + n \left\langle \nabla_T W, N\right\rangle \\ &+ \varepsilon_1 m \left\langle W, T\right\rangle + \left(-\varepsilon_3 n\tau + \varepsilon_1\varepsilon_2 \frac{p\, C}{\kappa} + \varepsilon_1 p\, \kappa \right) \left\langle W, B\right\rangle . \end{aligned}$$

Second order boundary conditions *Given $q_1, q_2 \in M$ and $\{x_1, y_1\}$, $\{x_2, y_2\}$ orthonormal vectors in $T_{q_1}M$ and $T_{q_2}M$, respectively, define the space of curves*

$$\Lambda = \{\gamma : [t_1, t_2] \to M \,|\, \gamma(t_i) = q_i, T(t_i) = x_i, N(t_i) = y_i, 1 \le i \le 2\}.$$

Then the critical points of the variational problem $\mathcal{F}_{mnp} : \Lambda \to \mathbb{R}$ are characterized by the following Euler-Lagrange equations

$$\varepsilon_3 m\kappa - \varepsilon_3 p\, \kappa\tau + \varepsilon_1 n\tau^2 + \varepsilon_1 n\, C = 0, \tag{3}$$

$$-\varepsilon_1 p\, \kappa_s + \varepsilon_3 n\tau_s = 0. \tag{4}$$

The moduli spaces of trajectories

Equations (3) and (4) yield

$$\begin{aligned} \varepsilon_1 p\kappa - \varepsilon_3 n\tau &= a, & (5)\\ -\varepsilon_1 m\kappa + a\tau &= \varepsilon_3 nC, & (6)\end{aligned}$$

a being an integration constant.

The moduli space of trajectories is summarized in the following tables.

m	n	p	**Solutions in** $\mathbb{L}^3$**,** $C = 0$
$\neq 0$	$= 0$	$= 0$	Geodesics ($\kappa = 0$)
$= 0$	$= 0$	$\neq 0$	Circles (κ constant and $\tau = 0$)
$= 0$	$\neq 0$	$= 0$	Plane curves ($\tau = 0$)
$\neq 0$	$\neq 0$	$= 0$	Helices with arbitrary τ and $\kappa = \varepsilon_2 \frac{n\tau^2}{m}$
$\neq 0$	$= 0$	$\neq 0$	Helices with arbitrary κ and $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	Circles and **Lancret curves** with $\tau = -\varepsilon_2 \frac{p}{n}\kappa$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{\varepsilon_1 a^2}{ap - \varepsilon_3 nm}$ and $\tau = \frac{ma}{ap - \varepsilon_3 nm}$, $a \in \mathbb{R} - \{\frac{\varepsilon_3 nm}{p}\}$

m	n	p	**Solutions in dS$_3$,** $C = c^2$
$\neq 0$	$= 0$	$= 0$	Geodesics ($\kappa = 0$)
$= 0$	$= 0$	$\neq 0$	Circles (κ constant and $\tau = 0$)
$= 0$	$\neq 0$	$= 0$	Do not exist
$\neq 0$	$\neq 0$	$= 0$	Helices with arbitrary τ and $\kappa = \varepsilon_2 \frac{n(c^2+\tau^2)}{m}$
$\neq 0$	$= 0$	$\neq 0$	Helices with arbitrary κ and $\tau = \frac{m}{p}$
$= 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \varepsilon_1 \frac{n^2 c^2 + a^2}{ap}$ and $\tau = \varepsilon_3 \frac{n\, c^2}{a}$, $a \in \mathbb{R} - \{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa = \frac{n^2 c^2 + a^2}{\varepsilon_1 p\, a + \varepsilon_2 mn}$ and $\tau = \frac{ma + \varepsilon_3 np\, c^2}{p\, a - \varepsilon_3 mn}$, $a \in \mathbb{R} - \{\varepsilon_3 \frac{mn}{p}\}$

m	n	p	**Solutions in $\mathbf{AdS}_3$**, $C=-c^2$
$\neq 0$	$=0$	$=0$	Geodesics ($\kappa=0$)
$=0$	$=0$	$\neq 0$	Circles (κ constant and $\tau=0$)
$=0$	$\neq 0$	$=0$	Horizontal lifts, via a Hopf map π_- or λ, of curves in either $\mathbb{H}^2(-4c^2)$ or $\mathbf{AdS}_2(-4c^2)$
$\neq 0$	$\neq 0$	$=0$	Helices with arbitrary τ and $\kappa=\varepsilon_2\frac{n(\tau^2-c^2)}{m}$
$\neq 0$	$=0$	$\neq 0$	Helices with arbitrary κ and $\tau=\frac{m}{p}$
$=0$	$\neq 0$	$\neq 0$	Helices with $\kappa=\varepsilon_1\frac{a^2-n^2c^2}{ap}$ and $\tau=-\varepsilon_3\frac{n\,c^2}{a}$, $a\in\mathbb{R}-\{0\}$
$\neq 0$	$\neq 0$	$\neq 0$	Helices with $\kappa=\frac{a^2-n^2c^2}{\varepsilon_1 p\,a+\varepsilon_2 mn}$ and $\tau=\frac{ma-\varepsilon_3 np\,c^2}{p\,a-\varepsilon_3 mn}$, $a\in\mathbb{R}-\{\varepsilon_3\frac{mn}{p}\}$
$\neq 0$	$\neq 0$	$\neq 0$	**Lancret curves** with $\tau=-\varepsilon_2\frac{p}{n}\kappa\pm\frac{m}{p}$ and $c=\pm\frac{m}{p}$

4. Going deeply into non flat backgrounds: creating algorithms

- M. Barros, —, M. A. Javaloyes and P. Lucas, Relativistic particles with rigidity and torsion in $D = 3$ spacetimes, *Class. Quantum Grav.* 22 (2005) 489-513 ([7])

The most interesting models appear in the anti de Sitter space $\mathbf{AdS}_3$ and they are $[\mathbf{AdS}_3(C), \mathcal{F}_{0n0}]$ and $[\mathbf{AdS}_3(C), \mathcal{F}_{mnp}]$ with $mnp \neq 0$.

The former corresponds again with the action giving the total curvature, that we have called the Plyushchay model describing a massless relativistic particle.

In [4] it is shown that the trajectories of this model are horizontal lifts, via either the usual Hopf map or the Lorentzian Hopf map, of arbitrary curves in either the hyperbolic plane or the anti de Sitter plane, respectively. It should be noticed that those horizontal curves are Lancret ones, where the curvature is an arbitrary function and the torsion is determined by the radius of the anti de Sitter space (for instance, $\tau = \pm 1$ if $C = -1$).

The model admits a one-parameter class $\mathcal{T}$ of trajectories which are ordinary helices (see Tables above). They can be geometrically obtained as geodesics of either a Hopf tube over a curve with constant curvature in the corresponding hyperbolic plane or a hyperbolic Hopf tube over a curve with constant curvature in the anti de Sitter plane (see [3]).

The dynamics are completed with classes of non trivial Lancret paths whose existence is related to the values of the parameters defining the action.

First of all, notice that the ratio $\frac{m}{p}$ and the curvature C of $\mathbf{AdS}_3(C)$ should satisfy $\frac{m}{p} = \pm\sqrt{-C}$. Therefore, without loss of generality, we may assume that $C = -1$ and $m = \pm p$, so we will put $m = p$ in the discussion. On the other hand, the non trivial Lancret curves in the anti de Sitter space are characterized by the following constraint between curvature and torsion

$$\tau = q\kappa \pm 1, \quad \text{for a certain constant } b \in \mathbb{R}.$$

Furthermore, as in the flat case, degenerate Lancret curves correspond with $q = \pm 1$ and spacelike acceleration ($\varepsilon_2 = 1$) (see [5]). Then, we have to distinguish two cases.

4.1. The dynamics in $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 \neq p^2$

This model has a second class, $\mathcal{T}_{n^2 \neq p^2}$, of trajectories which, according to the above Table, are nondegenerate Lancret curves (because $n^2 \neq p^2$) satisfying

$$\tau = -\varepsilon_2 \frac{p}{n}\kappa \pm 1.$$

- **Geodesics of Hopf tubes**

  The algorithm to get the solutions of this subfamily runs as follows.

1. Take a unit speed curve $\gamma(s)$ in the hyperbolic plane $\mathbb{H}^2(-4)$ and consider its Hopf tube $\pi_-^{-1}(\gamma)$ in $\mathbf{AdS}_3$.
2. This is a Lorentzian flat surface that can be parametrized with coordinate curves being, respectively, the fibers and the horizontal lifts of γ, in the following way
$$\Phi(s,t) = \cos(t)\,\bar{\gamma}(s) + \sin(t)\,i\bar{\gamma}(s).$$
3. Choose now the arclength parametrized geodesic of $\pi_-^{-1}(\gamma)$ defined by
$$\gamma_{np}(u) = \Phi(au, bu), \qquad a^2 - b^2 = \varepsilon_1, \qquad \frac{b^2}{a^2} = \frac{p^2}{n^2}.$$
4. Let ρ be the curvature function of γ into $\mathbb{H}^2(-4)$. Then a direct computation gives the curvature κ and the torsion τ of γ_{np} in $\mathbf{AdS}_3$
$$\begin{aligned} \kappa &= a^2\rho + 2ab, \\ \tau^2 &= \kappa^2 - \varepsilon_1\kappa\rho + 1. \end{aligned}$$
From these equations, we obtain that $\tau = -\varepsilon_2\frac{p}{n}\kappa \pm 1$. Therefore, γ_{np} is a path in $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 \neq p^2$.
5. Finally, notice that all of solutions γ_{np} of this kind are either spacelike or timelike, according to $n^2 > p^2$ or $n^2 < p^2$, respectively.

- **Geodesics of hyperbolic Hopf tubes**

  A similar algorithm holds.

4.2. The dynamics in $[\mathbf{AdS}_3, \mathcal{F}_{mnp}]$ with $n^2 = p^2$

The model presents the following families of trajectories:

- **Geodesics of Riemannian hyperbolic Hopf tubes**
- **Geodesics of scrolls over null curves**

  with their corresponding algorithms.

5. A general Lagrangian density in 3-dimensional space forms

- —, J. Guerrero, M. A. Javaloyes and P. Lucas, Particles with curvature and torsion in 3-dimensional pseudo-riemannian space forms. *J. Geom. Phys.* **56** (2006), 1666-1687 ([19])

Let $M^3_\nu(C)$ be a 3-dimensional pseudo-Riemannian space form of curvature C and index ν. Let $\gamma : I \to M^3_\nu(C)$ be an immersed curve with speed $v(t) = |\gamma'(t)|$, curvature k, torsion τ and Frenet frame $\{T, N, B\}$. The Frenet equations write down as follows

$$\begin{cases} \nabla_T T = \varepsilon_2 k N, \\ \nabla_T N = -\varepsilon_1 k T + \varepsilon_3 \tau B, \\ \nabla_T B = -\varepsilon_2 \tau N, \end{cases}$$

where $\varepsilon_1 = \langle T, T\rangle$, $\varepsilon_2 = \langle N, N\rangle$ and $\varepsilon_3 = \langle B, B\rangle$. Let

$$\boxed{\mathcal{L}(\gamma) = \int_\gamma f(k,\tau)ds} \tag{7}$$

Then first variation of $\mathcal{L}(\gamma)$ along γ in the direction of W is given by

$$\mathcal{L}'(0) = [\mathcal{B}(\gamma, W)]_0^L - \int_0^L \left\langle \nabla_T P - \varepsilon_1 C f_k N + \varepsilon_1\varepsilon_2 C \frac{f'_\tau}{k} B, W \right\rangle ds, \tag{8}$$

where the vector P is given by

$$P = \varepsilon_1(f - (2kf_k + \tau f_\tau))T + \varepsilon_1 k f_\tau B - \nabla_T (f_k N) + \varepsilon_2 \nabla_T \left(\frac{f_\tau'}{k} B \right)$$

and the boundary term is

$$\mathcal{B}(\gamma, W) = \left\langle \nabla_T^2 W, \varepsilon_2 \frac{f_\tau}{k} B \right\rangle + \left\langle \nabla_T W, f_k N - \frac{\varepsilon_2}{k} f_\tau' B \right\rangle + \left\langle W, P + \frac{\varepsilon_1 \varepsilon_2 C f_\tau}{k} B \right\rangle .$$

The critical curves are characterized by the vanishing of the Euler-Lagrange operator

$$\boxed{\mathcal{E} := -\left(\nabla_T P - \varepsilon_1 C f_k N + \varepsilon_1 \varepsilon_2 C \frac{f_\tau'}{k} B \right) = 0} \tag{9}$$

which is equivalent to the Euler-Lagrange equations

$$-\varepsilon_1 \varepsilon_2 k f - \varepsilon_2(\varepsilon_3 \tau^2 - \varepsilon_1 k^2) f_k + 2\varepsilon_1 \varepsilon_2 k \tau f_\tau + f_k'' + \left(\tau \frac{f_\tau'}{k} \right)' + \tau \left(\frac{f_\tau'}{k} \right)' + \varepsilon_1 C f_k = 0, \tag{10}$$

$$\varepsilon_3 \tau f_k' + \varepsilon_3 \frac{\tau^2}{k} f_\tau' + \varepsilon_3 (\tau f_k)' - \varepsilon_1 (k f_\tau)' - \varepsilon_2 \left(\frac{f_\tau'}{k} \right)'' - \varepsilon_1 \varepsilon_2 C \frac{f_\tau'}{k} = 0. \tag{11}$$

The critical curves of the Lagrangian (7) admit two Killing vector fields P and J given by

$$P = \varepsilon_1 \left(f - (k f_k + \tau f_\tau)\right) T - \left(f_k' + \frac{\tau}{k} f_\tau'\right) N + \left(-\varepsilon_3 \tau f_k + \varepsilon_1 k f_\tau + \varepsilon_2 \left(\frac{f_\tau'}{k}\right)'\right) B, \quad (12)$$

$$J = -\varepsilon_1 f_\tau T - \frac{f_\tau'}{k} N - \varepsilon_3 f_k B, \quad (13)$$

satisfying that

i) $\mathcal{E} = -(\nabla_T P + \varepsilon C J \wedge T)$

ii) $\nabla_T J = -P \wedge T$

where $\varepsilon = \varepsilon_1 \varepsilon_2 \varepsilon_3$.

Furthermore they satisfy the integral equations

$$\begin{cases} \langle P, P \rangle + \varepsilon C \langle J, J \rangle = d, \\ \langle P, J \rangle = e, \end{cases} \quad (14)$$

for suitable constants d and e.

The integral equations are reduced to a system involving P and J, which is equivalent to the Euler-Lagrange equations if, and only if, $\langle J, J \rangle$ is not constant. We note that when the Lagrangian density is $m + nk + p\tau$, then $\langle J, J \rangle$ is constant.

Then we have solved the motion equations and found out solutions which, as a pretty interesting fact, **are not Lancret curves**.

To obtain explicitly the critical curves of the Lagrangian, we have chosen suitable coordinate frames where the Frenet equations have been integrated. With the help of the corresponding Lie algebras, a complete system of solutions is given in the de Sitter $\mathbb{S}^3_1$ and anti de Sitter $\mathbb{H}^3_1$ worlds as well as in the non-flat Riemannian space forms $\mathbb{S}^3$ and $\mathbb{H}^3$.

6. Variational problems with torsion in greater dimensions

- M. Barros, —, M. A. Javaloyes and P. Lucas, Geometry of relativistic particles with torsion, *Int. J. Mod. Phys. A* Vol. 19, No. 11 (2004), 1737-1745 ([6])

We consider the action $\mathcal{L} : \Lambda \longrightarrow \mathbb{R}$ given by

$$\boxed{\mathcal{L}(\gamma) = \int_\gamma (pk_2 + q)ds}$$

Then by using the Frenet equations we have

$$\boxed{\mathcal{L}'(0) = [\mathcal{B}(\gamma, W)]_0^L - \int_0^L \langle \nabla_T P, W \rangle \, vdt} \tag{15}$$

where

$$P = \varepsilon_1 \varepsilon_3 p \nabla_T \left(\frac{k_3}{k_1} N_3 \right) + \varepsilon_0 p k_1 N_2 + \varepsilon_0 q T$$

and the boundary term reads

$$\mathcal{B}(\gamma, W) \;\; = \;\; \left\langle \nabla_T^2 W, \varepsilon_1 \frac{p}{k_1} N_2 \right\rangle + \left\langle \nabla_T W, -\varepsilon_1 \varepsilon_3 p \frac{k_3}{k_1} N_3 \right\rangle + \langle W, P \rangle \,,$$

W standing for a generic variational vector field along γ.

We take curves with the same endpoints and having there the same Frenet frame, so that $[\mathcal{B}(\gamma, W)]_0^L$ vanishes. Then

The trajectory $\boldsymbol{\gamma} \in \boldsymbol{\Lambda}$ is the worldline of a relativistic particle in the $\boldsymbol{d}$-dimensional background $\mathbb{R}^{\boldsymbol{d}}_{\boldsymbol{\nu}}$ if and only if the vector field $\boldsymbol{P}$ is constant along $\boldsymbol{\gamma}$.

A straightforward computation shows that P is constant if and only if the following equations hold:

$$\begin{aligned} pk_2(1-\varepsilon\varphi^2) - q &= 0, & (16)\\ k_1'(1-\varepsilon\varphi^2) - 3\varepsilon k_1\varphi\varphi' &= 0, & (17)\\ -\varepsilon_2\varepsilon_3\varphi'' + \varepsilon_2\varepsilon_4\varphi k_4^2 - \varepsilon k_1^2\varphi(1-\varepsilon\varphi^2) &= 0, & (18)\\ 2k_4\varphi' + \varphi k_4' &= 0, & (19)\\ k_3k_4k_5 &= 0, & (20) \end{aligned}$$

where $\varepsilon = \varepsilon_0\varepsilon_1\varepsilon_2\varepsilon_3$ and $\varphi = k_3/k_1$. The last equation yields $k_5 = 0$, so that the motion will be restricted to (at most) a 5-dimensional subspace. On the other hand, from Eq. (19) we easily find that $\varphi^2 k_4$ is a constant B, which determines k_4 in terms of the lower curvatures.

As for the solutions we have

- **The 4-dimensional case.** The curvatures are given by

$$k_1^2 = \frac{A}{\left(1-\varepsilon\varphi^2\right)^3}, \qquad k_2 = \frac{q}{p\left(1-\varepsilon\varphi^2\right)},$$

$$k_3^2 = \frac{A\varphi^2}{\left(1-\varepsilon\varphi^2\right)^3}.$$

If $A = 0$ then we easily obtain $\varphi^2 = \varepsilon$, so that $k_3 = \pm k_1$, and from equation (16) we deduce $q = 0$.

Note that this case can not appear in the Lorentzian background.

- **The 5-dimensional case.** A similar reasoning yields

$$k_1^2 = \frac{A}{\left(1-\varepsilon\varphi^2\right)^3}, \qquad k_2 = \frac{q}{p\left(1-\varepsilon\varphi^2\right)},$$

$$k_3^2 = \frac{A\varphi^2}{\left(1-\varepsilon\varphi^2\right)^3}, \qquad k_4 = \frac{B}{\varphi^2}.$$

From now on, we will deal with lightlike curves

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7. Variational problems concerning light-like curves

7.1. Lagrangian density linear in the curvature in $M_1^3(C)$

- —, A. Giménez and P. Lucas, Geometrical particles models on 3D null curves. *Phys. Let. B*, **543** (2002), 311-317 ([16])

Let $(M_1^3, \langle , \rangle)$ denote a 3-dimensional space-time, with constant curvature G and Levi-Civita connection ∇.

We consider mechanical systems with Lagrangians which linearly depend on the curvature of a light-like curve.

Consider the set Λ of null Cartan curves, [15], satisfying given first order boundary data to drop out the boundary terms which appear when computing the first order variation of the action.

Let $\gamma : I = [a, b] \to M_1^3$ be a null Cartan curve with Cartan frame $\{L = \gamma', W, N\}$, where $\langle L, L\rangle = \langle N, N\rangle = 0$ and $\langle L, N\rangle = -1$. The Cartan equations are given by (see [15] for details):

$$\begin{aligned} L' &= W, \\ W' &= -kL + N, \\ N' &= -kW, \end{aligned} \tag{21}$$

where the prime $()'$ denotes covariant derivative.

We consider the action $S : \Lambda \to \mathbb{R}$ given by

$$\boxed{S(\gamma) = 2c \int_\gamma (\lambda + \mu k(s)) ds}$$

When $\lambda = 1$ and $\mu = 0$ it leads to the action studied by Nersessian and Ramos in [22, 23]. The case $\mu = 1$ has been considered by Nersessian in [24].

As for the first variation we have

$$S'(0) = [\Omega]_a^b - c \int_a^b \langle V, (\mu k''' + 3\mu k k' - \lambda k')L \rangle \, ds \tag{22}$$

where

$$\begin{aligned} \Omega = &-c\mu L(h) + 2c\mu \left\langle \nabla_L^2 V, N \right\rangle + c(\mu k + \lambda) \left\langle \nabla_L V, W \right\rangle \\ &- c \left\langle V, \nabla_L((\mu k + \lambda)W) \right\rangle + 2c(\frac{1}{2}\mu k'' - \lambda k + 2\mu G) \left\langle V, L \right\rangle + 2c\mu k \left\langle V, N \right\rangle , \end{aligned} \tag{23}$$

V standing for a generic variational vector field along γ and $h = -\left\langle \nabla_L^2 V, W \right\rangle$.

We take curves with the same endpoints and having the same Cartan frame in them, so that $[\Omega]_a^b$ vanishes. Under these conditions, the first-order variation is

$$\boxed{S'(0) = -c \int_a^b \langle V, (\mu k''' + 3\mu k k' - \lambda k')L \rangle \, ds}$$

The trajectory $\gamma \in \Lambda$ is the null worldline of a relativistic particle in the (2+1)-dimensional spacetime if and only if the following differential equation is satisfied

$$\boxed{\mu k''' + 3\mu k k' - \lambda k' = 0} \tag{24}$$

To get the explicit solution of the motion equation, we write $(k')^2 = P(k)$, where P is a polynomial of degree 3.

I. P has a real root of multiplicity 3: $\alpha = \alpha_1 = \alpha_2 = \alpha_3$

The curvature function is given by

$$k(s) = \frac{\lambda}{3} - \frac{4}{(s+E)^2}, \qquad s \in (-\infty, \lambda/3)$$

where E is a constant of integration depending on the initial condition (see Fig. 1 (i)).

II. P has two real roots, the lowest with multiplicity 2: $\alpha = \alpha_1 = \alpha_2 < \alpha_3$

The root α_3 is given by $\lambda - 2\alpha$. There are two possibilities:

$$k(s) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(s+E)\right), \quad s \in (-\infty, \alpha)$$
$$k(s) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(s+E)\right), \quad s \in (\alpha, \lambda - 2\alpha]$$

where E is a constant (see Fig. 1 (ii)-(iii)).

III. P has two real roots, the greatest with multiplicity 2: $\alpha = \alpha_1 < \alpha_2 = \alpha_3$

We obtain that $\alpha_2 = \alpha_3 = (\lambda - \alpha)/2$, and the solution is given by

$$k(s) = \alpha + \frac{3\alpha - \lambda}{2}\tan^2\left(\frac{1}{2}\sqrt{\frac{\lambda - 3\alpha}{2}}(s+E)\right), \qquad s \in (-\infty, \alpha]$$

where E is a constant (see Fig. 2 (iv)).

IV. P has three distinct real roots: $\alpha_1 < \alpha_2 < \alpha_3$

Let us denote $\alpha = \alpha_1$ and $\beta = \alpha_2$, then $\alpha_3 = \lambda - \alpha - \beta$. There are two possibilities for the curvature:

$$k(s) = \alpha - (\beta - \alpha)\,\mathrm{tn}^2\left(\frac{1}{2}\sqrt{\lambda - 2\alpha - \beta}(s+E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}}\right),$$

$$k(s) = \lambda - \alpha - \beta + (\alpha + 2\beta - \lambda)\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\lambda - 2\alpha - \beta}(s+E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}}\right),$$

defined on the intervals $(-\infty, \alpha]$ or $[\beta, \lambda - \alpha - \beta]$, respectively (see Fig. 2 (v)-(vi)).

V. P has complex roots

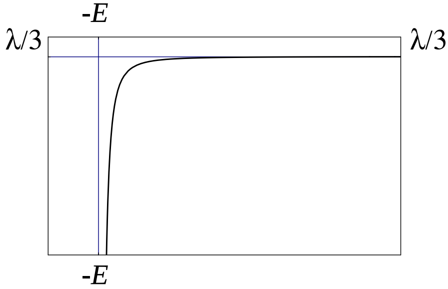
Let us suppose that α_1 and α_2 are complex (so α_3 is real). Then the curvature is given by

$$k(s) = \alpha_3 - (\alpha_3 - \alpha_2)\,\mathrm{sn}^2\left(\frac{1}{2}\sqrt{\alpha_3 - \alpha_1}(s+E), \sqrt{\frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}}\right), \qquad s \in (-\infty, \alpha_3].$$

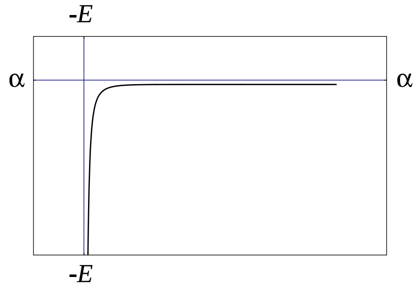
(see Fig. 2 (vii)).

7.2. Sketching worldlines

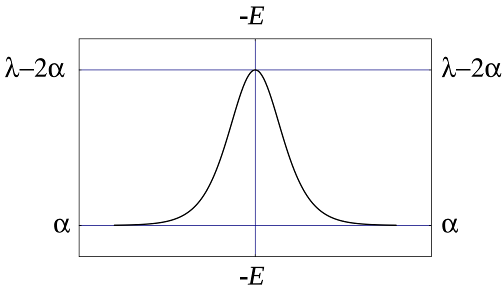
Once we know the curvature functions, the worldlines of the relativistic particles can be obtained by integrating the Cartan equations. The explicit integration of these equations is a difficult task, sometimes impossible (even when the curvature is a nice function). In our case, the goal of finding the exact worldlines can be reached by numeric integration. We will sketch, with the help of MATHEMATICA, the particle worldlines in all discussed cases in the preceding section.



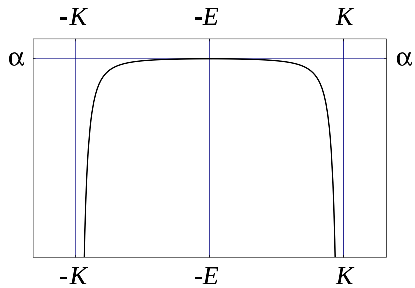
(i) $\alpha_1 = \alpha_2 = \alpha_3$



(ii) $\alpha_1 = \alpha_2 < \alpha_3$

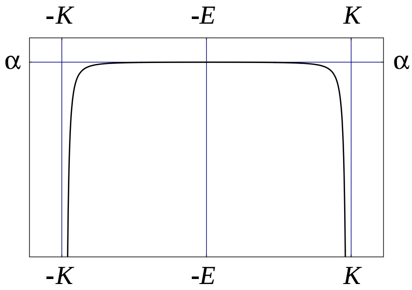


(iii) $\alpha_1 = \alpha_2 < \alpha_3$

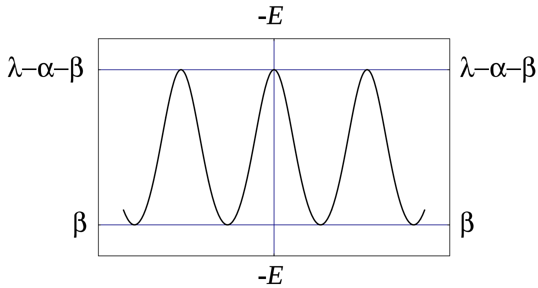


(iv) $\alpha_1 < \alpha_2 = \alpha_3$

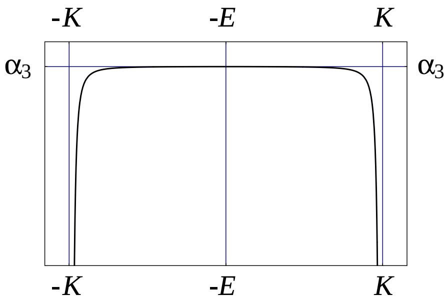
Fig.1: *Curvature function for the different possibilities of the roots of the polynomial P. It is quite interesting to remark that in cases (i), (ii) and (iii), as s increases, $k(s)$ approaches to a constant, said otherwise, the trajectory looks like a helix.*



(v) $\alpha_1 < \alpha_2 < \alpha_3$

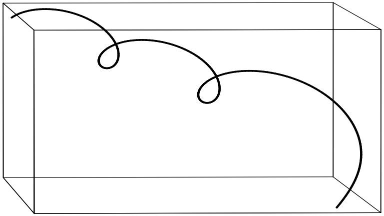


(vi) $\alpha_1 < \alpha_2 < \alpha_3$

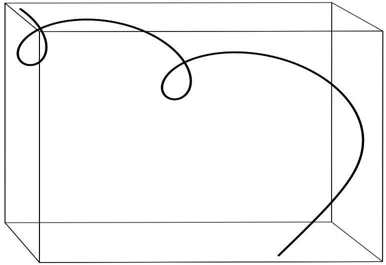


(vii) Complex roots

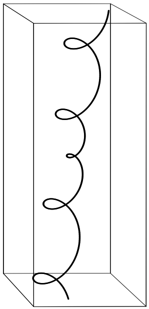
Fig.2: *Curvature function for the different possibilities of the roots of the polynomial* P.



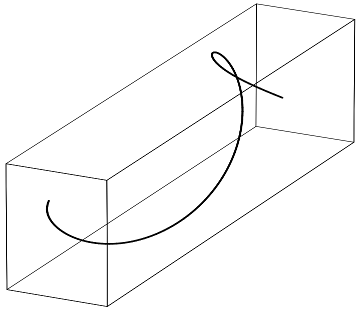
$\alpha_1 = \alpha_2 = \alpha_3,$
$k(s) \in (-\infty, \alpha_1]$



$\alpha_1 = \alpha_2 < \alpha_3,$
$k(s) \in (-\infty, \alpha_1]$

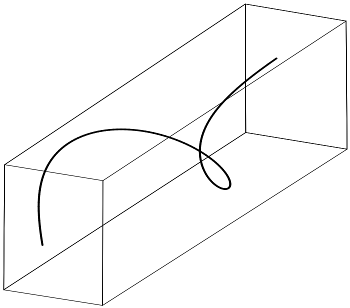


$\alpha_1 = \alpha_2 < \alpha_3,$
$k(s) \in (\alpha_2, \alpha_3]$

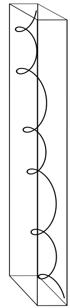


$\alpha_1 < \alpha_2 = \alpha_3,$
$k(s) \in (-\infty, \alpha_1]$

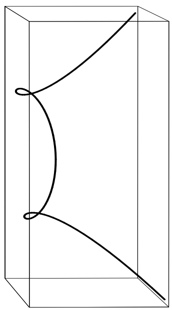
Fig.3: *Worldlines for corresponding curvature functions.*



$\alpha_1 < \alpha_2 < \alpha_3$,
$k(s) \in (-\infty, \alpha_1]$



$\alpha_1 < \alpha_2 < \alpha_3$,
$k(s) \in [\alpha_2, \alpha_3]$



$\alpha_1 = \alpha + \beta i$,
$\alpha_2 = \alpha - \beta i$, α_3 real,
$k(s) \in (-\infty, \alpha_3]$

Fig.4: *Worldlines for corresponding curvature functions.*

7.3. Lagrangian density linear in the curvature in $\mathbb{L}^n$

- —, A. Giménez and P. Lucas, Relativistic particles and the geometry of 4D null curves. *J. Geom. Phys.*, **57** (2007), 2124-2135 ([18])

Let $\mathbb{L}^n$ be an n-dimensional Lorentz-Minkowski space with background gravitational field $\langle , \rangle$ and Levi-Civita connection ∇. First of all, we will describe the geometry of null curves in $\mathbb{L}^n$ in terms of the Cartan frame of the curve (see [15] for details).

Let $\gamma : [a, b] \to \mathbb{L}^n$ be a null curve with Cartan frame $\{L = \gamma', N, W_1, W_2, \ldots, W_{n-2}\}$, such that

$$\begin{aligned} \langle L, L\rangle = \langle N, N\rangle = 0, \qquad & \langle L, N\rangle = -1, \\ \langle W_i, L\rangle = \langle W_i, N\rangle = 0, \qquad & \langle W_i, W_j\rangle = \pm 1. \end{aligned}$$

The Cartan equations read

$$\begin{aligned} L' &= W_1, \\ W_1' &= -k_1 L + N, \\ N' &= -k_1 W_1 + k_2 W_2, \\ W_2' &= k_2 L + k_3 W_3, \\ W_i' &= -k_i W_{i-1} + k_{i+1} W_{i+1} \quad i \in \{3, \ldots, n-3\}, \\ W_{n-2}' &= -k_{n-2} W_{n-3}, \end{aligned} \tag{25}$$

where $()'$ means covariant derivative and k_i are the *Cartan curvatures* of the curve.

We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\boxed{\mathcal{L}(\gamma) = \int_\gamma (\mu k_1 + \lambda) d\sigma} \tag{26}$$

μ and λ both being constant.

The simplest action describing the motion of a particle is achieved when it is proportional to the pseudo-arc length parameter (i.e. $\mu = 0$), which has been studied by Nersessian and Ramos in [22, 23] when $n = 2, 3$. When the action is linear on the curvature of the particle path, some advances have been achieved in [16, 24].

The first variation is

$$\mathcal{L}'(0) = \frac{1}{2}[\Omega(\gamma, V)]_a^b - \frac{1}{2}\int_a^b \langle V, \mathcal{E}_1(\gamma)L + \mathcal{E}_2(\gamma)W_2 + \mathcal{E}_3(\gamma)W_3 + \mathcal{E}_4(\gamma)W_4 \rangle \, d\sigma, \tag{27}$$

where

$$\begin{aligned} \mathcal{E}_1(\gamma) &= \mu k_1''' + 2\mu k_2 k_2' + 3\mu k_1 k_1' - \lambda k_1', \\ \mathcal{E}_2(\gamma) &= 2\mu k_2'' - k_2(2\mu k_3 - \mu k_1 + \lambda), \\ \mathcal{E}_3(\gamma) &= 2\mu(k_2 k_3' - 2k_2' k_3), \\ \mathcal{E}_4(\gamma) &= k_2 k_3 k_4, \end{aligned} \tag{28}$$

and the boundary term reads

$$\begin{aligned}\Omega(\gamma, V) = &\left\langle \nabla_L^3 V, \mu W_1 \right\rangle + \left\langle \nabla_L^2 V, -\mu k_1 L + 3\mu N \right\rangle \\ &+ \langle \nabla_L V, (\mu k_1 + \lambda) W_1 - \mu k_2 W_2 \rangle + \langle V, P_1 \rangle ,\end{aligned} \tag{29}$$

where P_1 is the vector field given by

$$P_1 = \left(\mu k_1'' + \mu k_1^2 - \lambda k_1 \right) L - \mu k_1' W_1 + (\mu k_1 - \lambda) N + 2\mu k_2' W_2 + 2\mu k_2 k_3 W_3, \tag{30}$$

and V stands for a generic variational vector field along γ.

To drop $[\Omega(\gamma, V)]_a^b$ we have to consider curves with the same endpoints and having the same Cartan frame there. Under these conditions, the first-order variation reads

$$\mathcal{L}'(0) = -\frac{1}{2} \int_a^b \langle V, \mathcal{E}_1(\gamma) L + \mathcal{E}_2(\gamma) W_2 + \mathcal{E}_3(\gamma) W_3 + \mathcal{E}_4(\gamma) W_4 \rangle \, d\sigma.$$

As a consequence we have

A null curve $\gamma \in \Lambda$ is critical for the action $\mathcal{L}(\gamma)$ in $\mathbb{L}^n$ if and only if the *Euler-Lagrange equations* **hold**

$$\boxed{\mathcal{E}_1(\gamma) = 0, \quad \mathcal{E}_2(\gamma) = 0, \quad \mathcal{E}_3(\gamma) = 0, \quad \mathcal{E}_4(\gamma) = 0}$$

The critical points for the linear action $\mathcal{L}(\gamma)$ in $\mathbb{L}^n$ lie in a Lorentzian subspace of dimension not greater than five.

By considering the special case where the action is constant ($\mu = 0$), the Euler-Lagrange equations are reduced to

$$-\lambda k_1' = 0, \qquad -\lambda k_2 = 0, \qquad k_2 k_3 k_4 = 0.$$

As a consequence we have

The critical points for the constant action in $\mathbb{L}^n$ are just null helices in 3-dimensional Lorentzian linear subspaces.

7.4. Lagrangian density as a general function on the curvature in $\mathbb{L}^3$

- —, A. Giménez and P. Lucas, Relativistic particles with rigidity along light-like curves. *Horizons in World Physics*, **245** (2004), 135-150 ([17])

We consider the action $\mathcal{L} : \Lambda \to \mathbb{R}$ given by

$$\boxed{\mathcal{L}(\gamma) = \int_\gamma f(k)dt}$$

where f is a differentiable function.

The first variation is

$$\mathcal{L}'(0) = \frac{1}{2}\left[\Omega(\gamma, V)\right]_a^b - \frac{1}{2}\int_a^b \langle V, \mathcal{E}(\gamma)L\rangle\, dt, \tag{31}$$

where

$$\mathcal{E}(\gamma) = \varphi''' + (k\varphi)' + k\varphi', \qquad \varphi = -f(k) + 2kf'(k) + k''f''(k) + (k')^2 f^{(3)}(k) \tag{32}$$

and the boundary term reads

$$\begin{aligned}\Omega(\gamma, V) \;&=\; \left\langle \nabla_L^3 V, f'(k)W\right\rangle + \left\langle \nabla_L^2 V, f'(k)(3N - kL) - f''(k)k'W\right\rangle \\ &\quad + \left\langle \nabla_L V, (f(k) + f''(k)k'' + f^{(3)}(k)(k')^2)W - 2f''(k)k'N\right\rangle + \langle V, P\rangle\,.\end{aligned}$$

Here the vector field P is given by

$$P = (\varphi'' + k\varphi)L - \varphi' W + \varphi N, \tag{33}$$

V standing for a generic variational vector field along γ.

Then f and k have to satisfy the following ordinary differential equations

$$\begin{aligned} (\varphi')^2 - 2\varphi(\varphi'' + k\varphi) &= \varepsilon p^2, \\ -2f'(k)\varphi'' + 2k' f''(k)\varphi' - 2f(k)\varphi - \varphi^2 &= \omega. \end{aligned}$$

Then, according to P is either non-null or null, the curve γ is described in cylindrical coordinates around P.

Special case: $\boldsymbol{f}$ being a quadratic function

$$\boxed{f(k) = \rho k^2 + \mu k + \lambda}$$

Case 1: $\boldsymbol{\rho = \mu = 0}$, $\boldsymbol{\lambda \neq 0}$ (the constant case)

Then γ is a Cartan helix, [8, 15], with axis given by the vector P.

Case 2: $\boldsymbol{\rho = 0}$, $\boldsymbol{\mu \neq 0}$ (the linear case)

From that we obtain

$$(k')^2 + k^3 - \lambda k^2 + (\omega - \lambda^2)k + \lambda^3 - \omega\lambda - \varepsilon p^2 = 0, \tag{34}$$

which can be written as $(k')^2 + Q(k) = 0$, Q being the polynomial $Q(X) = X^3 - \lambda X^2 + (\omega - \lambda^2)X + \lambda^3 - \omega\lambda - \varepsilon p^2$. Observe that since $Q(k) = -(k')^2$ then k takes values only where Q is negative. Now we are going to analyze all possible cases.

- I. $\boldsymbol{Q}$ **has a real root of multiplicity 3:** $\boldsymbol{\alpha = \alpha_1 = \alpha_2 = \alpha_3}$

  The curvature function is given by

$$k(t) = \frac{\lambda}{3} - \frac{4}{(t+E)^2}, \quad s \in (-\infty, \lambda/3)$$

  where E is a constant of integration, depending on the initial conditions.

- II. $\boldsymbol{Q}$ **has two real roots, the lowest with multiplicity 2:** $\boldsymbol{\alpha = \alpha_1 = \alpha_2 < \alpha_3}$

$$k(t) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(t+E)\right), \quad s \in (-\infty, \alpha)$$
$$k(t) = \lambda - 2\alpha + (3\alpha - \lambda)\coth^2\left(\frac{1}{2}\sqrt{\lambda - 3\alpha}(t+E)\right), \quad s \in (\alpha, \lambda - 2\alpha]$$

- III. $\boldsymbol{Q}$ has two real roots, the greatest with multiplicity 2: $\boldsymbol{\alpha = \alpha_1 < \alpha_2 = \alpha_3}$

$$k(t) = \alpha + \frac{3\alpha - \lambda}{2} \tan^2 \left(\frac{1}{2} \sqrt{\frac{\lambda - 3\alpha}{2}} (t + E) \right), \quad s \in (-\infty, \alpha]$$

- IV. $\boldsymbol{Q}$ has three distinct real roots: $\boldsymbol{\alpha_1 < \alpha_2 < \alpha_3}$

$$k(t) = \alpha - (\beta - \alpha) \operatorname{tn}^2 \left(\frac{1}{2} \sqrt{\lambda - 2\alpha - \beta} (t + E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}} \right),$$

$$k(t) = \lambda - \alpha - \beta + (\alpha + 2\beta - \lambda) \operatorname{sn}^2 \left(\frac{1}{2} \sqrt{\lambda - 2\alpha - \beta} (t + E), \sqrt{\frac{\lambda - \alpha - 2\beta}{\lambda - 2\alpha - \beta}} \right),$$

- V. $\boldsymbol{Q}$ has complex roots

$$k(t) = \alpha_3 - (\alpha_3 - \alpha_2) \operatorname{sn}^2 \left(\frac{1}{2} \sqrt{\alpha_3 - \alpha_1} (t + E), \sqrt{\frac{\alpha_2 - \alpha_3}{\alpha_1 - \alpha_3}} \right), \quad s \in (-\infty, \alpha_3].$$

We use cylindrical coordinates to integrate the Cartan equations of the curves.

Case 3: $\boldsymbol{\rho \neq 0}$ (the quadratic case)

The Euler-Lagrange equation is given by

$$2k^{(5)} + (10k + \mu)\, k^{(3)} + 20k''k' + k' \left(15k^2 + 3\mu k - \lambda\right) = 0, \tag{35}$$

which can be reduced to

$$\begin{aligned} \frac{1}{16}(k')^2 \left(-4(k'')^2 - 2(2k+\mu)(k')^2 + 5k^4 + 2\mu k^3 - 2\lambda k^2 + 4ck + 2c\mu + \lambda^2 + \omega\right)^2 & \\ +(2k'' + 3k^2 + \mu k - \lambda)(4kk'' - 2(k')^2 + 4k^3 + \mu k^2 + 2c) - \varepsilon p^2 &= 0. \end{aligned}$$

The integration of this equation is very complicated, but we can use computing methods to make us an idea of their solutions (see the following pictures).

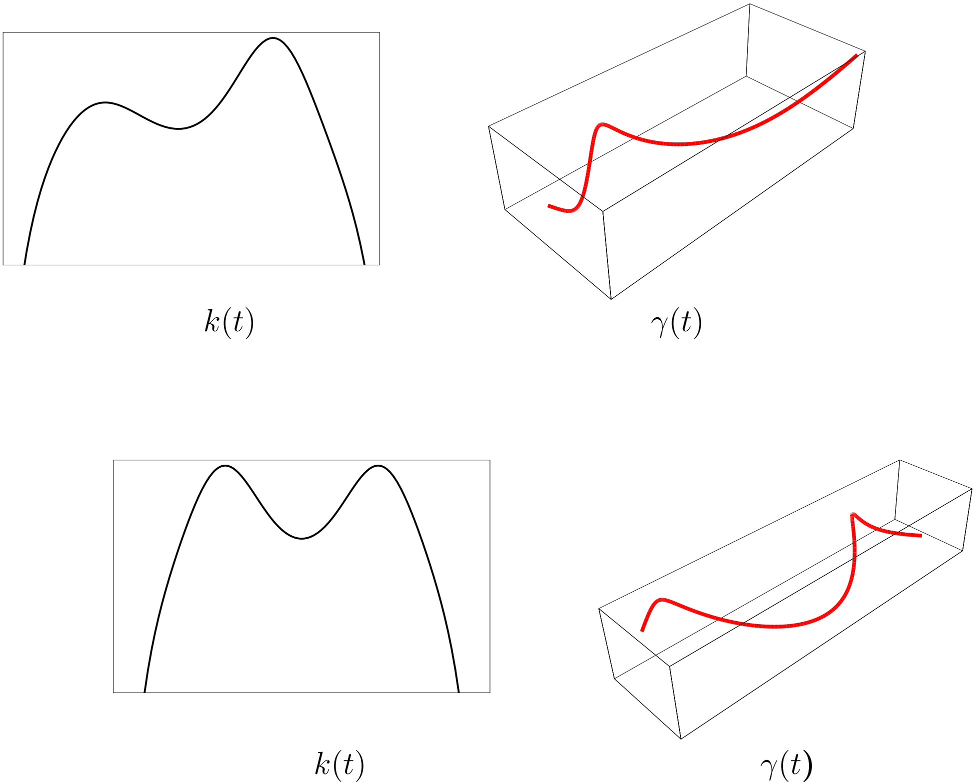
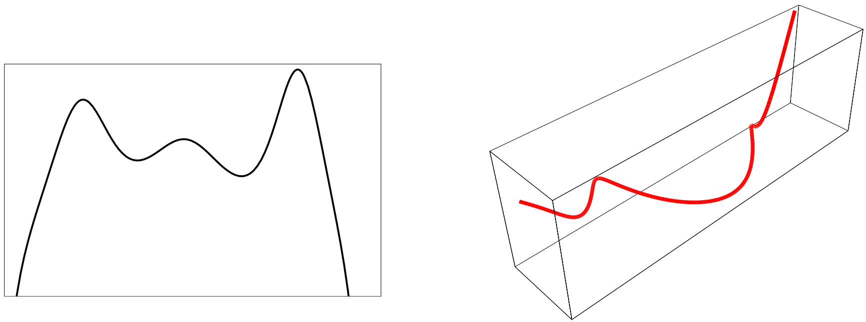


Fig. 5 (*Quadratic case: spacelike J and timelike J*)



$k(t)$ $\gamma(t)$

Fig. 6 (*Quadratic case and lightlike* J)

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Appendix: **What's a B-scroll?**

Let $x(s)$ be a null (*lightlike*) curve in $\mathbb{L}^3$ with a Cartan frame $\{A = \dot{x}, B, C\}$, i. e., B and C are vector fields along $x(s)$ such that

$$\langle A, A\rangle = \langle B, B\rangle = 0, \qquad \langle C, C\rangle = 1$$
$$\langle A, C\rangle = \langle B, C\rangle = 0, \qquad \langle A, B\rangle = -1$$

and

$$\dot{A} = f(s)C$$
$$\dot{B} = g(s)C$$
$$\dot{C} = g(s)A + f(s)B.$$

Then the immersion $\varphi : (s, u) \longrightarrow x(s) + uB(s)$ defines a Lorentz surface which L. K. Graves called B-scroll.

Its shape operator, in the usual frame $\{\varphi_s, \varphi_u\}$, is given by

$$\begin{pmatrix} g(s) & 0 \\ f(s) & g(s) \end{pmatrix}$$

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