Methods of Quantization

Elisa Ercolessi

Department of Physics, Università di Bologna, and INFN-Sezione di Bologna, via Irnerio 46, 40123, Bologna, Italy

E-mail: `elisa.ercolessi@unibo.it`

Abstract. These notes collect the lectures for the mini course given by the author to the "XXIII International Workshop on Geometry and Physics" held in Granada (Spain) in September 2014, which followed a brief course on fundamentals of Quantum Mechanics.
The aim of these lectures is to give an introduction to the Methods of Quantization. In doing so, for the richness of the problem, we had necessarily to make a choice of topics to be covered. Thus, we give only some details about the so-called Geometric Quantization, about which there are many good reviews, and concentrate on two approaches, Coherent states and Weyl map, which are deeply rooted in the geometric concepts we have developed in the pre-course. Finally, we introduce a very different approach to quantization, the path-integral one, due to Feynman.
(Preliminary version: some additional content and references will be added subsequently.)

Introduction

The conceptual framework of Quantum Mechanics (QM) is both physically and mathematically very well understood. The success of this theory is demonstrated both by the innumerable technological applications and by the sophisticated theories about quantum field theory, that have been elaborated in the last decades and confirmed with an unbelievable degree of precision by experiments*. Similarly to classical theories, dynamical equations can be written also for a quantum mechanical system and the possibility of solving a problem then relies on our ability to solve such equations.
However, one can argue that there is still quite an important open problem: indeed, despite the inner coherence of QM, there is not a clear and unique way to "quantize" a classical system, i.e. to give a precise set of rules to construct the quantum version of a given system of which we know the classical description. The opposite limit, in which we want to extract the classical version of a given quantum system, is to some extent much simpler to obtain♯. This is because QM contains a new fundamental constant, the Planck

* This holds true both for in particle physics and in condensed matter systems, the only difference being that the former has to be treated relativistically.

♯ We have to stress, however, that not all quantum systems admit a classical limit and that more than one quantum systems might correspond to the same classical one.

constant h (or better $\hbar = h/2\pi$), which plays the role of a quantization parameter, so that the classical limit might be obtained for $\hbar \to 0$, once one has specified in which sense this limit has to be taken*.

A complete overview of the different "quantization techniques" that have been proposed in literature goes well beyond the scope of these lectures. Here we have decided to concentrate only on some of them, both because they are founded on the geometric structures of QM -as we have described them in the pre-course- and because they represent the ones that are more operatively used in modern theoretical physics. We will discuss the so-called "Coherent State Approach" which, mainly complemented by Feynman path-integral technique, is the methods which is most widely used in quantum field theory and the "Weyl Quantization Approach" which is at the origin of modern tomographic techniques, originally used in optics and now in quantum information theory.

The outline of the paper is as follows.
In Sect. 1, we will introduce the problem of "quantization", starting from the first attempts to quantize a system due to the founders of QM and briefly outlining the so-called Geometric Quantization technique, which has been developed in many papers and books (see [1] for a very complete review and a list of references). These attempts essentially rely on the so-called Schroedinger picture, in which one chooses to work with wave functions, i.e. on a Hilbert space given by $L^2(Q)$, Q being an n-dimensional manifold representing half of the degrees of freedom of the system, typically the classical space of coordinates q_i or of momenta p_i.
In Sect. 2 we first present the notion of coherent states, discussing their mathematical structure and their physical properties, and then introduce the so-called Bargmann-Fock representation, in which the Hilbert space is the space of square integrable functions of a suitable complex manifold.
In Sect. 3 we discuss the technique proposed by Weyl and Wigner, which naturally leads to a formulation of QM directly on the space of square integrable functions on the phase space. In this context the limit $\hbar \to 0$ in which one can recover classical mechanics becomes very transparent and will be discussed.
Finally, in Sect. 4 we sketch the approach to quantization that uses path-integrals, due to Feynman. After a brief overview, we will concentrate on path-integrals with coherent states.

1. The problem of quantization

1.1. The origin of quantization rules

The discussion about the procedure from which one has to pass from the classical to the quantum description of given physical system accompanied the birth of a QM theory M and took place mainly in the context of the so-called Schroedinger picture, according

* Regarding this, $\hbar$ is seen to play the same role as the speed of light c in the theory of special relativity.

to which quantum states are represented by wave-functions, i.e. -say in the coordinate representation- by square integrable functions over the classical configuration manifold $Q = \{q \equiv (q_1, \cdots, q_n)\}$, taken usually as $\mathbb{R}^n$: $\mathcal{H} = L^2(Q) = \{\psi(q) : \|\psi(q)\|_2 < \infty\}$. Quantum observables are self-adjoint operators on $\mathcal{H}$, so to have a real spectrum and admitting a spectral decomposition. This is the analogue of what we do in a classical context, in which the space of states is given by the phase space $T^*Q = \{(p, q) \equiv (p_1, \cdots, p_n; q_1, \cdots, q_n)\}$ and observables are real (regular, usually C^∞) functions on it: $f(p, q) \in \mathbb{R}$. Notice that both the space of self-adjoint operators on $\mathcal{H}$ and of real regular functions on T^*Q are vector spaces, actually algebras, on which we might want to assign suitable topologies.
In "quantizing" a classical system we would like to assign a set of rules that allows to univocally allows us to pick up a self-adjoint operators $\hat{O}_f$ for each classical observable f. We want this map to preserve the algebraic structures of these two spaces [1], i.e.:

- the map $f \mapsto \hat{O}_f$ is linear
- if $f = id_{T^*Q}$ then $\hat{O}_{id} = \mathbb{I}_{\mathcal{H}}$
- if $g = \Phi \circ f$, with $\Phi : \mathbb{R} \to \mathbb{R}$ for which both $\hat{O}_f$ and $\hat{O}_g$ are well defined, then $\hat{O}_g = \hat{O}_{\Phi\circ f} = \Phi(\hat{O}_f)$

and to encompass the largest class of functions as possible.
In particular it must be possible to find the operators $\hat{p}_j, \hat{q}_j$ associated to the coordinate functions p_j and q_j $(j = 1, \cdots, n)$ on T^*Q, about which we put an additional requirement. From the simple examples we have developed in the pre-course, such as the free particle and the harmonic oscillator, we have argued that Classical Poisson Brackets (CPB):

$$\{p_i, q_j\} = \delta_{ij} \;\; ; \;\; \{q_i, q_j\} = \{p_i, p_j\} = 0 \;\; ; \{q_i, id\} = \{p_i, id\} = 0 \tag{1}$$

have to be replaced by the following Canonical Commutation Relations (CCR):

$$[\hat{q}_i, \hat{p}_j] = \imath\hbar\delta_{ij} \;\; , \;\; [\hat{q}_i, \hat{q}_j] = [\hat{p}_i, \hat{p}_j] = 0 \;\; ; \;\; [\hat{q}_i, \mathbb{I}] = [\hat{p}_i, \mathbb{I}] = 0 \tag{2}$$

From the first of this commutators with $i = j$, one can see that, for at least (and indeed both) one of the operators $\hat{p}_i, \hat{q}_i$ has to be unbounded [14]. A theorem by Stone and Von-Neumann [14] then states that, up to unitarity equivalence, there is only one irreducible representation of such algebra of observables, called the *Weyl algebra*, which on $\mathcal{H} = L^2(Q)$ reads as:

$$\hat{q}_i\psi(q) = q_i\psi(q) \;\; , \;\; \hat{p}_i\psi(q) = -\imath\hbar\frac{\partial}{\partial q_i}\psi(q) \tag{3}$$

Actually, knowing that in classical mechanics we can reconstruct the Poisson bracket between any two (regular, such as polynomials) functions $f(p, q), g(p, q)$ out of (1), we would like to have more, i.e. we would like to to find a map such that if, $f \mapsto \hat{O}_f$ and $g \mapsto \hat{O}_g$, then:

$$\hat{O}_{\{f,g\}} = \imath\hbar[\hat{O}_f, \hat{O}_g] \tag{4}$$

This is possible if f, g are both functions of only the q- or the p-coordinates or if they are linear in them, but we know that this does not hold already if we consider a quadratic function such as (considering for simplicity only one degree of freedom):

$$H = \frac{\hbar\omega}{2}(p^2 + x^2) = \frac{\hbar\omega}{2}(x + ip)(x - ip) = \frac{\hbar\omega}{2}(x - ip)(x + ip)$$

which represents a 1D harmonic oscillator. This is because, at the quantum level, we have an ordering problem due to the fact that the operators $\hat{p}, \hat{q}$ do not commute, contrary to the functions p, q.
It is therefore evident that we will not be able to find a correspondence $f \mapsto \hat{O}_f$ which fulfils all (five) requirements*, at least if we insist on trying to quantize the whole space of observables.
We also remark that one is interesting to the development of a technique that is not only suited to describe $T^*Q = \mathbb{R}^{2n}$, but the more general case in which the phase space is a symplectic manifold (Γ, ω) .

To solve this question, there have been developed two main approaches, that we briefly outline.

Geometric quantization. The idea behind this approach is to overcome the problems described above by restricting the space of quantizable observables. More precisely, one would like to assign to (Γ, ω) a separable Hilbert space $\mathcal{H}$ and a mapping $Q : f \mapsto \hat{O}_f$ from a *sub-Lie algebra* $\emptyset$ (as large as possible) of real-valued functions on Γ into self-adjoint linear operators on $\mathcal{H}$, satisfying:

(i) $Q : f \mapsto \hat{O}_f$ is linear
(ii) $\hat{O}_{id_\Gamma} = \mathbb{I}_\mathcal{H}$
(iii) $\hat{O}_{\{f,g\}} = \imath\hbar[\hat{O}_f, \hat{O}_g]$, $\forall f, g \in \emptyset$
(iv) for $\Gamma = \mathbb{R}^{2n}$ and $\omega = \omega_0$, the standard symplectic form, we should recover the operators $\hat{q}_i, \hat{p}_i$ as in (3)
(v) the procedure is functorial, in the sense that for any two symplectic manifolds (Γ_1, ω_1), (Γ_2, ω_2) and a symplectic diffeomorphism $\Phi : \Gamma_1 \to \Gamma_2$, the composition with Φ should map $\emptyset_1$ into $\emptyset_2$ and there should exist a unitary operator $U_\Phi : \mathcal{H}_1 \to \mathcal{H}_2$ such that $O^{(1)}_{f \circ \Phi} = U^\dagger_\Phi O^{(2)}_f U_\Phi$, $\forall f \in \emptyset_2$

The solution to this problem was first given by Konstant and Sourieau and goes through two main steps, called *prequantization* and *polarization*. The theory of geometric quantization has become a topic of studies both in physics and in mathematics, but is goes beyond the scope of these lectures. We refer the interested reader to ref. [1] for a review and an exhaustive list of references.

Weyl quantization. In this approach one modifies relations (3) and (4). We will describe it in detail in Sect. 3. Here we make only a couple of observations.

* This is not only because we insist on eq. (4). In [1], the interested reader may found a detailed discussions and many examples showing all existing inconsistencies among the various assumptions we made.

Restricting to $\Gamma = \mathbb{R}^{2n}$ and $\omega = \omega_0$ for simplicity, we notice first of all that, in order to avoid dealing with unbounded operators, one works with the (strongly-continuous) one-parameter groups generated by the operators $\hat{q}_j, \hat{p}_j$:

$$\widehat{U}(q_j) = \exp\{\imath q_j \widehat{p}_j/\hbar\} \quad , \quad \widehat{V}(p_j) = \exp\{\imath p_j \widehat{q}_j/\hbar\} \tag{5}$$

out of which one can construct the Weyl operator:

$$\widehat{W}_j = \widehat{U}(q_j)\widehat{V}(p_j)\exp\left(-\imath q_j p_j/\hbar\right) \tag{6}$$

which is the building blocks of the whole procedure, inducing a bijective map (the Weyl-Wigner map) between classical observables $f(p,q)$ and unitary operators W_f. Second, we will prove that it is possible to endow the space of functions on Γ with an associative but non-commutative product $f \star g$, called the Moyal or star product, which -by construction- satisfies: $W_f W_g = W_{f\star g}$, so that we can construct a "deformed" Poisson bracket

$$\{f, g\}_\star \equiv \frac{1}{\imath\hbar}\left(f \star g - g \star f\right) \tag{7}$$

such that

$$[W_f, W_g] = W_{\{f,g\}_\star} \tag{8}$$

Also, such deformed Poisson bracket satisfies the asymptotic (or semiclassical) expansion:

$$\{f, g\}_\star = \{f, g\} + O(\hbar^2) \tag{9}$$

These are not the only quantization method that have been studied in literature. In Sect. 2 we will present a different technique, which is based on the definition of generalized coherent states, while in Sect. 4 we will present Feynman path integral approach. The explanation of techniques might be found in [1].

In the next subsection, we recall the main properties of the Heisenberg-Weyl algebra (and group), which we have seen to play a key role in the process of quantization and will be extensively used in the following.

1.2. The Heisenberg-Weyl Group

We describe the Heisenberg-Weyl algebra and group in detail just for one degree of freedom, the generalization to an arbitrary number being obvious.

Starting from the canonical degrees of freedom q, p, we can construct the so-called creation/annihilation operators:

$$a^\dagger = \frac{\hat{q} - \imath\hat{p}}{\sqrt{2\hbar}} \ , \ a = \frac{\hat{q} + \imath\hat{p}}{\sqrt{2\hbar}} \tag{10}$$

which, in virtue of (2), satisfy:

$$[a, a^\dagger] = \mathbb{I} \ , \ [a, \mathbb{I}] = [a^\dagger, \mathbb{I}] = 0 \tag{11}$$

We say that $\{q, p, \mathbb{I}\}$ or equivalently $\{a, a^\dagger, \mathbb{I}\}$ generate a Lie algebra, the Heisenberg-Weyl algebra w_1.
Introducing the (imaginary) elements:

$$e_1 = \imath p/\sqrt{\hbar}\ ,\ e_2 = \imath q/\sqrt{\hbar}\ ,\ e_3 = \imath\mathbb{I} \tag{12}$$

we can equivalently say that the Heisenberg-Weyl Algebra w_1 is the real 3-dimensional Lie algebra generated by the set $\{e_1, e_2.e_3\}$ such that:

$$[e_1, e_2] = e_3\ ,\ [e_1, e_3] = [e_2, e_3] = 0 \tag{13}$$

A generic element $x \in w_1$ can therefore be written in one of the following form:

$$x = x_1 e_1 + x_2 e_2 + s e_3 \qquad x_1, x_2, s \in \mathbb{R} \tag{14}$$

$$= \imath s\mathbb{I} + \imath(Pp - Qq)/\hbar \quad s, P, Q \in \mathbb{R} \tag{15}$$

$$= \imath s\mathbb{I} + (\alpha a^\dagger - \alpha^* a) \qquad s \in \mathbb{R}\,,\ \alpha \in \mathbb{C} \tag{16}$$

where the relationship between the coefficients of the three expressions can be easily derived by the reader.
The commutator between any two elements of the algebra is easily found to be given by:

$$[x, y] = B(x, y)e_3 \ \text{ with } B(x, y) = x_1 y_2 - x_2 y_1 \tag{17}$$

Notice that $B(x, y)$ is the standard symplectic form on the plane of coordinates (x_1, x_2), i.e. (p, q).

The Heisenberg-Weyl group is, as usual, obtained by exponentiation. Using for example (16), one has:

$$g = e^x = e^{\imath s\mathbb{I}} D(\alpha)\ ,\ D(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a} \tag{18}$$

with:

$$D(\alpha)D(\beta) = e^{\imath\Im(\alpha\beta^*)} D(\alpha + \beta) = e^{2\imath\Im(\alpha\beta^*)} D(\beta)D(\alpha) \tag{19}$$

which can be proved via the Baker-Campbell-Hausdorff formula*.

The latter expression shows that the operators $e^{\imath s\mathbb{I}} D(\alpha)$ give a representation of the group W_1, fixed by (s, α), with s real and α complex. The group W_1 is nilpotent and its center is given by the subgroup generated by the elements with $(s, 0)$, whose most general representation is fixed by a real number λ via:

$$T^\lambda((s, 0)) = e^{\imath s\lambda}\mathbb{I}$$

Any $\lambda \neq 0$ gives a non-equivalent infinite-dimensional representation of W_1, and any unitary irreducible representation is of this kind [13]♯.

* This formula reads: $e^{A+B} = e^A e^B e^{-[A,B]/2} = e^B e^A e^{[A,B]/2}$ whenever A, B are such that $[A, B] = c\mathbb{I}$ $(c \in \mathbb{C})$.
♯ If $\lambda = 0$ the unitary representation is one-dimensional and fixed by a pair of real numbers μ, ν via: $T(g) = e^{\mu x_1 + \nu x_2}$.

2. Coherent states and Bargmann Fock representation

In this section we will show explicit realisations of the representations of the Weyl group, by means of the so called coherent states, introduced first by Glauber and Sudarshan; the interested reader may found more on this subject and a thorough overview of applications in [10].

2.1. Definition and basic properties

Let us consider a Hilbert $\mathcal{H}$, on which we have defined a couple of bosonic creation/annihilation operators: $[a, a^\dagger] = \mathbb{I}$. We denote with

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle \qquad \text{with} \;\; n = 0, 1, 2, ... \tag{20}$$

the orthonormal states of the Fock basis, which are eigenstates of the number operator $\hat{N} = a^\dagger a$, with eigenvalues n: $\hat{N}|n\rangle = n|n\rangle$. In the following we will make use of the following identities which encode the orthonormality and completeness properties of such a basis:

$$\langle n|n'\rangle = \delta_{nn'} \tag{21}$$

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{I} \tag{22}$$

A *coherent state* is, by definition, an eigenstates $|\alpha\rangle$ of the annihilation operator* with eigenvalue $\alpha \in \mathbb{C}$:

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{23}$$

An explicit expression for $|\alpha\rangle$ may be found by expanding it on the Fock basis: $|\alpha\rangle = \sum_{n=0}^{\infty} a_n|n\rangle$ and imposing (23):

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{n!} \left(a^\dagger\right)^n |0\rangle = e^{\alpha a^\dagger}|0\rangle \;,\; \forall \alpha \in \mathbb{C} \tag{24}$$

These states are not orthogonal, since:

$$\langle \alpha|\beta\rangle = e^{\alpha^* \beta} \tag{25}$$

but form a complete set, since:

$$\mathbb{I} = \int \frac{(d\Re\alpha)\,(d\Im\alpha)}{\pi} e^{-|\alpha|^2} |\alpha\rangle\langle\alpha| \tag{26}$$

Thus, the set:

$$\{|\tilde{\alpha}\rangle = e^{-|\alpha|^2/2} e^{\alpha a^\dagger}|0\rangle\}_{\alpha\in\mathbb{C}}$$

is an overcomplete set of normalized vectors.

* One can easily show that the creation operator does not admits instead eigenvectors with non-zero eigenvalue.

Proof. We leave the details of proof of these identities to the reader, giving only some suggestions.
To show (25), one has to use the Baker-Campbell-Hausdorff formula and use the fact that:

$$e^{\alpha^* a}|0\rangle = \sum_{n=0}^{\infty} \frac{(\alpha a)^n}{n!}|0\rangle = |0\rangle$$

To show (26), one uses the following chain of identities:

$$\begin{aligned}\int \frac{(d\Re\alpha)\,(d\Im\alpha)}{\pi}|\tilde{\alpha}\rangle\langle\tilde{\alpha}| &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}|n\rangle\langle m|\int \frac{(d\Re\alpha)\,(d\Im\alpha)}{\pi}\alpha^n\,(\alpha^*)^m\,e^{-|\alpha|^2} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}|n\rangle\langle m|\frac{1}{\pi}\int_0^{2\pi}d\theta\int_0^{\infty}\rho d\rho e^{i(n-m)\theta}\rho^{n+m}e^{-\rho^2} \\ &= \sum_n\sum_m n!\delta_{nm}\frac{1}{\sqrt{n!}}\frac{1}{\sqrt{m!}}|n\rangle\langle m| = \sum_{n=0}^{\infty}|n\rangle\langle n| = \mathbb{I}\end{aligned}$$

where we have used polar coordinates $\alpha = \rho e^{i\theta}$. ■

Coherent states satisfy several interesting properties that we now list:

(i) The mean value of creation/annihilation operators on a coherent state is non-zero; more explicitly:

$$\langle\tilde{\alpha}|a|\tilde{\alpha}\rangle = \alpha\ ,\ \langle\tilde{\alpha}|a^\dagger|\tilde{\alpha}\rangle = \alpha^*$$

(ii) A coherent state is also denominated "displaced vacuum" since it can be written as:

$$|\tilde{\alpha}\rangle = D(\alpha)|0\rangle \tag{27}$$

where we have introduced the *displacement operator*

$$D(\alpha) \equiv e^{\alpha a^\dagger - \alpha^* a} \tag{28}$$

so called because: $D^\dagger(\alpha)aD(\alpha) = a + \alpha$.

Proof. Point (i) follows immediately from the definition. Point (ii) can be proved by noticing that:

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{\frac{|\alpha|^2}{2}}e^{\alpha a^\dagger}e^{-\alpha^* a}$$

$$D(\alpha)^\dagger = e^{-\alpha a^\dagger + \alpha^* a} = e^{-\frac{|\alpha|^2}{2}}e^{\alpha^* a}e^{-\alpha a^\dagger}$$

■

It is interesting to look at formula (27) in the coordinate representation, i.e. when we choose $\mathcal{H} = L^2\left(\mathbb{R} = \{q\},\ d\mu = dq\right)$ and:

$$\hat{q} = q \ \Rightarrow\ e^{\imath p_0 \hat{q}}\psi(q) = e^{\imath p_0 q}\psi(q)$$

$$\hat{p} = -\imath\hbar\frac{\partial}{\partial q} \ \Rightarrow\ e^{-\imath q_0 \hat{p}}\psi(q) = \psi(q - q_0)$$

If we now recall that the vacuum is represented by: $\psi_0(q) = \frac{1}{\pi^{1/4}} e^{-q^2/2}$, with just some algebra we can show that:

$$\begin{aligned}\psi_\alpha(q) \equiv |\tilde{\alpha}\rangle = D(\alpha)|0\rangle &= e^{\imath \frac{p_0 q_0}{2}} e^{-\imath q_0 \hat{p}} e^{p_0 \hat{q}} \psi_0(q) \\ &= \frac{1}{\pi^{1/4}} e^{-\imath \frac{p_0 q_0}{2}} e^{\imath p_0 q} e^{-(q-q_0)^2/2}\end{aligned} \tag{29}$$

where we have set $\alpha = (q_0 + ip_0)/\sqrt{2}$. This shows that, like the vacuum, a coherent state is represented by a wave function of gaussian type, with a mean value displaced from $q = 0$ to $q = q_0$.

The reader might have recognized in eq. (27) the operators that define the Heisenberg-Weyl group that we have encountered in the previous section. Indeed, the set of coherent states that we have just defined gives an explicit example of an irreducible representation.

2.2. Physical properties

Coherent states plays an important role in physical problems. They were introduced in the context of optics and possess some very interesting properties that we now briefly present.

Number of particle Distribution. Coherent states are an infinite superposition of Fock states, hence they have no definite number of particles. However we may easily see that $\langle\tilde{\alpha}|\hat{N}|\tilde{\alpha}\rangle = |\alpha|^2$. Indeed:

$$\begin{aligned}\langle\tilde{\alpha}|\hat{N}|\tilde{\alpha}\rangle &= e^{-|\alpha|^2} \sum_{nm} \frac{(\alpha^*)^n}{\sqrt{n!}} \frac{(\alpha)^m}{\sqrt{m!}} \langle n|\hat{N}|m\rangle = e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} n \\ &= e^{-|\alpha|^2} |\alpha|^2 \sum_{n=1}^{\infty} \frac{(|\alpha|^2)^{n-1}}{(n-1)!} = e^{-|\alpha|^2} |\alpha|^2 \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{m!} = |\alpha|^2\end{aligned}$$

The last expression before the final result, shows that the probability of finding the value m is given by:

$$p_m = \frac{|\alpha|^{2m}}{m!} e^{-|\alpha|^2}$$

This means that, in a coherent state the number of particles obeys a Poisson distribution, with average $|\alpha|^2$, i.e. a distribution that could be obtained form a classical counting of particles which are randomly distributed (with fixed mean).

Quasi-classical states. Let us consider the equations of motion of a classical 1D oscillator: $\dot{q}(t) = \omega p(t)$, $\dot{p}(t) = -\omega q(t)$. Setting $\alpha(t) \equiv \frac{1}{\sqrt{2}} [q(t) + \imath p(t)]$ we can summarize them in the complex equation:

$$\dot{\alpha}(t) = -\imath\omega\alpha(t)$$

whose solution is easily found to be:

$$\alpha(t) = \alpha(t=0)\, e^{-\imath\omega t} \equiv \alpha_0\, e^{-\imath\omega t} \tag{30}$$

with energy

$$E = \frac{\hbar\omega}{2}\left[x(t)^2 + p(t)^2\right] = \hbar\omega|\alpha(t)|^2 = \hbar\omega|\alpha_0|^2 \tag{31}$$

We notice that eq.ns (10) is the quantum counterpart of what we are doing here at the classical level. To compare the two, let us recall that an operator $\hat{A}$ evolves in time as: $\hat{A}(t) = e^{\frac{\imath tH}{\hbar}}\hat{A}e^{-\frac{\imath tH}{\hbar}}$ so that its mean value on a state, $\langle\hat{A}\rangle_t \equiv \langle\psi|\hat{A}(t)|\psi\rangle$ evolves according to the equation:

$$\imath\hbar\frac{d}{dt}\langle\hat{A}\rangle_t = \langle\left[\hat{A},\hat{H}\right]\rangle \tag{32}$$

In particular, it is immediate to see that for $a, a^\dagger$ and $H = \hbar\omega\left(a^\dagger a + 1/2\right)$ one has:

$$\langle a\rangle_t = \langle a\rangle_0 e^{-\imath\omega t} \quad , \quad \langle a^\dagger\rangle_t = \langle a^\dagger\rangle_0 e^{\imath\omega t} \tag{33}$$

$$\langle H\rangle_t = \hbar\omega\left(\langle a^\dagger a\rangle_t + \frac{1}{2}\right) \tag{34}$$

These expectation values coincide with the classical solution (30) if:

$$\langle a\rangle_0 = \alpha_0 \quad , \quad \langle a^\dagger\rangle_0 = \alpha_0^* \tag{35}$$

and with the classical energy (31) if:

$$\langle a^\dagger a\rangle_0 = |\alpha_0|^2 \tag{36}$$

up to a content term which becomes negligible in the (classical) limit $|\alpha_0|^2 \gg \hbar\omega$.
A state that satisfies these conditions is called quasi-classical.

It is easy to see that a coherent state $|\alpha\rangle$ does indeed satisfies eq.ns (35,36). Moreover, it is not difficult to show that all quasi-classical states are indeed coherent states.

Proof. Suppose $|\psi\rangle$ is a quasi-classical state and consider the operator $b = a - \alpha_0\mathbb{I}$, so that:

$$\||b|\psi\rangle\|^2 = \langle\psi|b^\dagger b|\psi\rangle = \langle\psi|\left(a^\dagger - \alpha_0^*\right)(a - \alpha_0)|\psi\rangle = |\alpha_0|^2 - \alpha_0\alpha_0^* - \alpha_0^*\alpha_0 + |\alpha|^2 = 0$$

Since the scalar product is non-degenerate, one must have

$$0 = b|\psi\rangle = (a - \alpha_0)|\psi\rangle \Leftrightarrow a|\psi\rangle = \alpha_0|\psi\rangle$$

showing that $|\psi\rangle = |\alpha_0\rangle$. ■

Minimal uncertainty states.

Coherent state are also called minimal uncertainty states because they saturate Heisenberg uncertainty principle:

$$\Delta\hat{x}\Delta\hat{p} = \frac{\hbar}{2}$$

This can be easily seen, by calculating:

$$\langle\alpha_0|\hat{x}|\alpha_0\rangle = \frac{\alpha_0 + \alpha_0^*}{\sqrt{2\hbar}} \quad , \quad \langle\alpha_0|\hat{p}|\alpha_0\rangle = \frac{\alpha_0 - \alpha_0^*}{\sqrt{2\hbar}i}$$

and:

$$\langle\alpha_0|\hat{x}^2|\alpha_0\rangle = \frac{1+(\alpha_0+\alpha_0^*)^2}{2\hbar} \quad , \quad \langle\alpha_0|\hat{p}^2|\alpha_0\rangle\frac{1-(\alpha_0-\alpha_0^*)^2}{2\hbar}$$

so that

$$\Delta\hat{x} = \sqrt{\langle\hat{x}^2\rangle - \langle\hat{x}\rangle^2} = \sqrt{\frac{\hbar}{2}}$$

$$\Delta\hat{p} = \sqrt{\langle\hat{p}^2\rangle - \langle\hat{p}\rangle^2} = \sqrt{\frac{\hbar}{2}}$$

Remark. Let us notice that, for a coherent state, both $\Delta\hat{x}$ and $\Delta\hat{p}$ are minimal, and equal to $\sqrt{\hbar/2}$. It is possible to define also the so-called "squeezed" states for which:

$$\Delta x = \frac{1}{\sqrt{2}}e^{-s}$$

$$\Delta p = \frac{1}{\sqrt{2}}e^{+s}$$

with $s \in \mathbf{R}$, which are again of minimal uncertainty. They can be constructed by applying the displacement operator after having applied to the vacuum another operator, called squeezing operator $\delta(s)$:

$$|\psi_s\rangle = D(\alpha)\,\delta(s)\,|0\rangle$$

$$\delta(s) \equiv e^{\frac{s}{2}\left(a^2 - a^{\dagger 2}\right)}$$

In the coordinate representation a squeezed state is give by the wave-function:

$$\psi_s(q) = \pi^{-\frac{1}{4}}e^{\frac{s}{2}}\exp\left(\imath\frac{q_0 p_0}{2}\right)\exp\left[-\frac{(q-q_0)^2}{2e^{2s}}\right]$$

i.e. by a Gaussian distribution centered in $q = q_0$ e with variance e^s. Such states are usually obtained in non-linear interaction problems in optics, when one adds to the harmonic oscillator hamiltonian a term of the kind: $H_{int} = a^2 - (a^\dagger)^2$. □

We can summarize what we have seen in this subsection by saying that coherent states are those states (and the only ones) for which the quantum expectation values of the operators satisfy the same dynamical laws as the corresponding classical functions (position, momentum, energy) on phase space and for which the corresponding variances about such classical values get minimized.

2.3. Bargmann-Fock representation

In this subsection, instead of using greek letters $\alpha, \beta, \cdots$, we will denote variables in $\mathbb{C}$ with $z, z', \cdots$ and therefore a coherent states will be represented by the ket $|z\rangle$ and its corresponding eigenvalue with respect to the operator a is denoted with $z \in \mathbb{C}$.

Given any $|\psi\rangle \in \mathcal{H}$, we can use the resolution of the identity (26) to write:

$$|\psi\rangle = \left(\int \frac{(d\Re z)\,(d\Im z)}{\pi} e^{-|z|^2} |z\rangle\langle z|\right)|\psi\rangle = \int \frac{(d\Re z)\,(d\Im z)}{\pi} e^{-|z|^2} |z\rangle \psi(z^*) \quad (37)$$

where $\psi(z^*) \equiv \langle z|\psi\rangle$ is the wave functions associated to the vector ψ in the coherent state basis.
It is not difficult to verify that, for any $|\psi\rangle, |\phi\rangle \in \mathcal{H}$:

$$\langle\psi|\phi\rangle = \int \frac{(d\Re z)\,(d\Im z)}{\pi} e^{-|z|^2} \psi(z^*)^* \phi(z^*) \quad (38)$$

which shows that we are working in the Hilbert space $\mathcal{H}_{BF}$ of all anti-holomorphic functions in z such that:

$$\|\psi\|^2_{BF} \equiv \int \frac{(d\Re z)\,(d\Im z)}{\pi} e^{-|z|^2} |\psi(z^*)|^2 < \infty \quad (39)$$

which thus defines the so-called Bargmann-Fock representation. Such Hilbert space might be thought as the completion of the linear space of polynomials in the variable z^*: $\mathcal{P} = \{P(z^*) = a_0 + a_1 z^* + \cdots + a_n (z^*)^n\}$ with respect to the scalar product defined by the measure:

$$d\mu(z) \equiv \frac{(d\Re z)\,(d\Im z)}{\pi} e^{-|z|^2} \quad (40)$$

In this representation, the vectors of the Fock basis are given by the monomials in z^*:

$$\Phi_n(z^*) = \langle z|n\rangle = \sum_{m=0}^{\infty} \frac{(z^*)^m}{\sqrt{m!}} \langle m|n\rangle = \frac{(z^*)^n}{\sqrt{n!}} \quad (41)$$

It is interesting to see how the creation/annihilation operators $a^\dagger/a$ are represented on $\mathcal{H}_{BF}$. From the definition of a coherent state, we know that $a|z\rangle = z|z\rangle$, i.e.: $\langle z|a^\dagger = z^*\langle z|$. Therefore we can write:

$$\begin{aligned} a^\dagger|z\rangle &= \sum_{m=0}^{\infty} \frac{(z^*)^m}{\sqrt{m!}} a^\dagger|m\rangle = \sum_{m=0}^{\infty} \frac{(z^*)^m}{\sqrt{m!}} \sqrt{m+1}|m+1\rangle \\ &= \sum_{n=1}^{\infty} \frac{(z^*)^{n-1}}{\sqrt{n!}} n|n\rangle = \frac{\partial}{\partial z} \sum_{n=0}^{\infty} \frac{(z^*)^n}{\sqrt{n!}} |n\rangle \end{aligned}$$

This means that: $a^\dagger|z\rangle = \frac{\partial}{\partial z}|z\rangle$ and $\langle z|a = \frac{\partial}{\partial z^*}\langle z|$. In other words, in $\mathcal{H}_{BF}$:
• a acts as the derivative with respect to z^*:

$$a : f(z^*) = \langle z|f\rangle \mapsto \langle z|a|f\rangle = \frac{\partial}{\partial z^*}\langle z|f\rangle = \frac{\partial}{\partial z^*} f(z^*)$$

• $a^\dagger$ acts as multiplication by z^*:

$$a^\dagger : f(z^*) = \langle z|f\rangle \mapsto \langle z|a^\dagger|f\rangle = z^*\langle z|f\rangle = z^* f(z^*)$$

We leave to the reader the proof that the operators $\frac{\partial}{\partial z^*}, z^*$ are one the adjoint of the other with respect to the measure (40) and that they satisfy:

$$\left[\frac{\partial}{\partial z^*}, z^*\right] = \mathbb{I}$$

Given any operator A on $\mathcal{H}$, we can represent it in $\mathcal{H}_{BF}$ by means of an integral representation:

$$(A\psi)(z^*) = \int d\mu(z')A(z^*, z')\psi(z'^*) \tag{42}$$

with a kernel $A(z^*, z')$ given by:

$$A(z^*, z') = \sum_{m,k=0}^{\infty} A_{mk}\frac{(z')^k}{\sqrt{k!}}\frac{(z^*)^m}{\sqrt{m!}} = \langle z|A|z'\rangle \ , \ A_{mn} \equiv \langle m|A|n\rangle \tag{43}$$

Proof. For any $|\psi\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \in \mathcal{H}$,

$$\psi(z^*) = \langle z|\psi\rangle = \sum_{n=0}^{\infty} c_n\langle z|n\rangle = \sum_{n=0}^{\infty} c_n\frac{(z^*)^n}{\sqrt{n!}}$$

and

$$(Af)(z^*) = \langle z|A|f\rangle = \sum_{n=0}^{\infty} c_n\langle z|A|n\rangle = \sum_{n,m=0}^{\infty} c_n\langle z|m\rangle\langle m|A|n\rangle = \sum_{n,m=0}^{\infty} c_n A_{mn}\frac{(z^*)^m}{\sqrt{m!}}$$

We can now use the identity

$$\delta_{kn} = \int d\mu(z')\frac{(z')^k}{\sqrt{k!}}\frac{(z'^*)^n}{\sqrt{n!}}$$

to write:

$$\begin{aligned}(Af)(z^*) &= \sum_{n,m,k=0}^{\infty} c_n A_{mk}\delta_{kn}\frac{(z^*)^m}{\sqrt{m!}} = \int d\mu(z') \sum_{n,m,k=0}^{\infty} c_n A_{mk}\frac{(z')^k}{\sqrt{k!}}\frac{(z'^*)^n}{\sqrt{n!}}\frac{(z^*)^m}{\sqrt{m!}} \\ &= \int d\mu(z') \left[\sum_{m,k=0}^{\infty} A_{mk}\frac{(z')^k}{\sqrt{k!}}\frac{(z^*)^m}{\sqrt{m!}}\right]\left(\sum_{n=0}^{\infty} c_n\frac{(z'^*)^n}{\sqrt{n!}}\right)\end{aligned}$$

■

Using such integral representation, it is very simple to calculate the trace of an operator, since we can write:

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}[A] &= \sum_{n=0}^{\infty}\langle n|A|n\rangle = \int d\mu(z)\sum_{n=0}^{\infty}\langle n|A|z\rangle\langle z|n\rangle = \int d\mu(z)\sum_{n=0}^{\infty}\langle z|n\rangle\langle n|A|z\rangle \\ &= \int d\mu(z)e^{-|z|^2}\langle z|\left(\sum_{n=0}^{\infty}|n\rangle\langle n|\right)A|z\rangle = \int d\mu(z)\langle z|A|z\rangle\end{aligned}$$

Thus:

$$\mathrm{Tr}_{\mathcal{H}}[A] = \int \mathsf{d}\mu(z)A(z^*, z) \tag{44}$$

This formula is very useful in a series of application we will see in the following.

Examples.

(i) The "delta"-operator:

$$\delta(z^* - z_0^*) : \psi(z^*) \mapsto \psi(z_0^*)$$

can be written as:

$$\psi(z_0^*) = \langle z_0|\psi\rangle = \langle z_0| \left(\int d\mu(z)|z\rangle\langle z| \right) |\psi\rangle = \int d\mu(z)\langle z_0|z\rangle\psi(z^*)$$
$$= \int d\mu(z)e^{z_0^* z}\psi(z^*) \tag{45}$$

showing that its kernel is given by: $e^{z_0^* z}$.

(ii) The kernel of the annihilation operator a is given by:

$$\langle z|a|z'\rangle = z'\langle z|z'\rangle = z' e^{z^* z'}$$

(iii) The kernel of the creation operator $a^\dagger$ is given by:

$$\langle z|a^\dagger|z'\rangle = z^*\langle z|z'\rangle = z^* e^{z^* z'}$$

(iv) The kernel of the number operator $a^\dagger a$ is given by:

$$\langle z|a^\dagger a|z\rangle = z^* z'\langle z|z"\rangle = z^* z' e^{z^* z'}$$

(v) More generally, the kernel of any operator of the form*:

$$K = \sum_{pq} k_{pq}(a^\dagger)^p a^q$$

is simply given by the expression:

$$\langle z|K|z\rangle = \sum_{pq} k_{pq}\langle z|(a^\dagger)^p a^q|z\rangle = \sum_{pq} k_{pq}(z^*)^p (z')^q e^{z^* z'}$$

(vi) The operator of the one-parameter group generated by $\hat{N}$: $A = \exp[-\tau a^\dagger a]$ is not written in the normal form and to calculate its kernel, we may proceed as follows:

$$A_{mk} = \langle m|\exp[-\tau a^\dagger a]|k\rangle = \delta_{mk}\exp[-\tau m]$$

from which one sees that:

$$A(z^*, z') = \sum_{m,k=0}^{\infty} \delta_{mk}\exp[-\tau m]\frac{(z')^k}{\sqrt{k!}}\frac{(z^*)^m}{\sqrt{m!}} = \sum_{m=0}^{\infty} \frac{(e^{-\tau} z^* z')^m}{m!} = \exp[e^{-\tau} z^* z']$$

* A polynomial in $a, a^\dagger$ in which the creation operators are all on the left of the annihilation ones is said to be in its normal form.

(vii) We have all ingredients to calculate the (canonical) partition function for the 1D harmonic oscillator, with Hamiltonian $H = \hbar\omega(a^\dagger a + 1/2)$:

$$Z \equiv \mathrm{Tr}_{\mathcal{H}}[e^{-\beta H}] = e^{-\beta\hbar\omega/2}\mathrm{Tr}_{\mathcal{H}}[e^{-\beta\hbar\omega a^\dagger a}]$$

Setting $\tau = \beta\hbar\omega$ in the last example and using (44), we immediately see that:

$$\begin{aligned} Z &= e^{-\beta\hbar\omega/2}\int \frac{(d\Re z)\,(d\Im z)}{\pi} e^{-|z|^2}\exp[e^{-\beta\hbar\omega}|z|^2] \\ &= e^{-\beta\hbar\omega/2}\int_0^{2\pi}\frac{d\theta}{\pi}\int_0^{\rho}\rho d\rho e^{-\rho^2(1-e^{-\beta\hbar\omega})} = \frac{1}{2\sinh\left(\beta\hbar\omega/2\right)} \end{aligned}$$

2.4. Generalized Coherent States and Comments

We will introduce here the notion of generalized coherent states [13], applied to the Heisenberg-Weyl group W_1.
Let us take any UIR of W_1, $T(g)$, and denote with $|\psi_0\rangle$ any (non-zero) vector in the (necessarily infinite-dimensional) representation space $\mathcal{H}$. The stability group of $|\psi_0\rangle$ is given only by the centre of W_1, i.e. by the elements of the form $T((s,0))$. We define the set of generalized coherent states as:

$$|\alpha\rangle \equiv T(g)\,|\psi_0\rangle = D(\alpha)\,|\psi_0\rangle \tag{46}$$

The set of coherent states we have studied in the previous section corresponds to the choice: $|\psi_0\rangle = |0\rangle$. Similarly to what done before, one can show that $\{|\alpha\rangle\}$ form an overcomplete set of states that generate $\mathcal{H}$. More explicitly we have the following:

law of transformation

$$D(\alpha)\,|\beta\rangle = e^{\imath\Im(\alpha\beta^*)}\,|\alpha+\beta\rangle$$

non-orthogonality condition

$$\langle\alpha|\beta\rangle = e^{\alpha^*\beta}$$

resolution of identity

$$\mathbb{I} = \int \frac{(d\Re\alpha)\,(d\Im\alpha)}{\pi} e^{-|\alpha|^2}|\alpha\rangle\langle\alpha| \equiv \int d\mu_\alpha|\alpha\rangle\langle\alpha|$$

From the latter it is immediate to see that any $|\psi\rangle \in \mathcal{H}$ can be written as:

$$|\psi\rangle = \int dd\mu_\alpha|\alpha\rangle\,\langle\alpha|\psi\rangle = \int d\mu_\alpha\,\psi(\alpha)$$

where $\psi(\alpha) = \langle\alpha|\psi\rangle$ is called the symbol of the state $|\psi\rangle$. Clearly: $\langle\psi|\psi\rangle = \int d\mu_\alpha|\psi(\alpha)|^2$.

Such a construction can be easily extended to finite or infinite number of creation/annihilation operators of bosonic type, i.e. to a set of operators $\{(a_i, a_i^\dagger)\}_i$ satisfying the canonical commutation relations:

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0 \tag{47}$$

$$[a_i, a_j^\dagger] = \delta_{ij}\mathbb{I} \tag{48}$$

acting on the tensor product Hilbert space $\mathcal{H}_F = \otimes_j \mathcal{H}_j$.
Thanks to (47), a coherent state can be immediately defined as the common eigenvector $|\phi\rangle \equiv |\phi_1\,\phi_2\,\cdots\rangle$ of all annihilation operators:

$$a_j|\phi\rangle = \phi_j|\phi\rangle \quad \text{with } \phi_j \in \mathbb{C}$$

The solutions of these equations can be easily found to be given by:

$$|\phi\rangle = \prod_j e^{\phi_j a_j^\dagger}|0\rangle = e^{\sum_j \phi_j a_j^\dagger}|0\rangle \tag{49}$$

where $a_j|0\rangle = 0$, for all j. It is also not difficult to prove the:

non-orthogonality condition

$$\langle\phi|\phi'\rangle = e^{\sum_\alpha \phi^*_\alpha \phi'_\alpha}$$

resolution of identity

$$\mathbb{I} = \int \left(\prod_\alpha \frac{(d\Re\phi_\alpha)\,(d\Im\phi_\alpha)}{\pi} \right) e^{\sum_\alpha |\phi_\alpha|^2}|\phi\rangle\langle\phi|$$

This is the framework in which, for example, one discusses the quantization of the electromagnet field: in the vacuum, the latter satisfies d'Alembert equation and hence can be described by using its Fourier modes, each of which behaves independently as a 1D harmonic oscillator. Thus, each of these modes is described quantum mechanically by a couple of creation/annihilation operators, which in all satisfy eq.ns (47,48).

The situation is more complicated if we would like to discuss a set of finite or infinite number of creation/annihilation operators of fermionic type, i.e. a set of operators $\{(a_i, a_i^\dagger)\}_i$ satisfying the canonical commutation relations:

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0 \tag{50}$$

$$\{a_i, a_j^\dagger\} = \delta_{ij}\mathbb{I} \tag{51}$$

If we insist to define coherent states as common eigenvectors $|\xi\rangle \equiv |\xi_1\,\xi_2\,\cdots\rangle$ of all annihilation operators:

$$a_j|\xi\rangle = \xi_j|\xi\rangle$$

we see that the commutations relations (51) now imply:

$$\xi_i\xi_j + \xi_j\xi_i = 0$$

This condition admits non-trivial solutions only if we allow the "numbers" ξ_j to be not in $\mathbb{C}$ but in a Grassmann algebra $\mathbb{G}$.
If we allow so, then coherent states are given by:

$$|\xi\rangle \equiv |\xi_1\xi_2\cdots\rangle = e^{\sum_j \xi_j a_j^\dagger}|0\rangle = \prod_j (1 - \xi_j a_j^\dagger)|0\rangle \tag{52}$$

which are vectors in the *generalized Fock space*:

$$\tilde{\mathcal{H}}_F = \{|\psi\rangle = \sum_J \chi_J |\phi_J\rangle \ : \ \chi_J \in \mathbb{G}\,,\ |\phi_J\rangle \in \mathcal{H}_F\}$$

The set of states $\{|\xi\rangle\}$ satisfy again relationships of the kind:

$$\langle\xi|\xi'\rangle = e^{\sum_\alpha \xi^*_\alpha \xi'_\alpha} \tag{53}$$

$$\mathbb{I} = \int \left(\prod_\alpha d\xi^*_\alpha\, d\xi_\alpha\right) e^{\sum_\alpha \xi^*_\alpha \xi_\alpha} |\xi\rangle\langle\xi| \tag{54}$$

where the integration over Grassmann variables has to be of course suitably defined. We have not time to explore this subject here and refer the interested reader to the literature [11, 12].

As a final comment, we want to stress that the definition of generalized coherent states can be extended to a more general group G, including all nilpotent and semi-simple Lie-groups, by looking at their UIRR's: $T : g \mapsto T(g)$, $T(g)$ being an unitary operator on some Hilbert space $\mathcal{H}$. As before, taking a reference vector $|\psi_0\rangle$, we construct the set of coherent states as $T(g)\,|\psi_0\rangle$, by taking into account that the fiducial vector might have a non-trivial isotropy group G_0. This means that coherent states are in this case labeled by points in the manifold G/G_0. This gives a hint about the fact that coherent states are related to the geometry of symmetric manifolds and the theory of co-adjoint orbits, a topic we cannot deal with here (see [13]).

3. Weyl quantization

In this section* we will introduce the quantization à la Weyl, which overcomes the problem mentioned in the introduction about the fact that the CCR (2) between p and q implies that at least one of the two operators must be unbounded. We will see how the procedure suggested by Weyl is founded on geometric concepts. This means that it can be generalized to a generic phase space, i.e. a symplectic manifold which is not necessarily a linear space. The Weyl map (and its inverse, the Wigner map) allow for a quantization on the space of functions $f(p, q)$ over the entire phase space, and not on the space of functions on the configuration space or on a suitable maximal Lagrangian subspace, as it is required by geometric quantization. In this setting, it is also transparent to discuss the limit $\hbar \to 0$, which describes the quantum to classical transition.

3.1. The Weyl map

Let $\mathcal{S}$ be a (real) linear vector space endowed with a constant symplectic structure* ω. A Weyl map is a map W from $\mathcal{S}$ to the set of unitary operators on an abstract Hilbert

* The material presented here is taken from [5], to which we refer the interested reader for more details.
* Hence necessarily: $\mathcal{S} \approx \mathbb{R}^{2n}$ for some n.

space $\mathcal{H}$:

$$W : \mathcal{S} \to \mathcal{U}(\mathcal{H}) \tag{55}$$

$$z \mapsto \widehat{W}(z) \ , \ \widehat{W}(z)\widehat{W}^{\dagger}(z) = \widehat{W}^{\dagger}(z)\widehat{W}(z) = \widehat{\mathbb{I}} \tag{56}$$

such that:

- W is a strongly continuous map;
- for any $z, z' \in \mathcal{S}$:

$$\widehat{W}(z + z') = \widehat{W}(z)\widehat{W}(z') \exp\{-\imath\omega(z, z')/2\hbar\} \tag{57}$$

It follows then that:

$$\widehat{W}(z)\widehat{W}(z') = \widehat{W}(z')\widehat{W}(z) \exp\{\imath\omega(z, z')/\hbar\} \ , \ \forall z, z' \tag{58}$$

Notice that setting $z' = 0$ or $z' = -z$ in (57) we obtain respectively:

$$\widehat{W}(0) = \widehat{\mathbb{I}} \quad \text{and} \quad \widehat{W}^{\dagger}(z) = \widehat{W}(-z)$$

Thus a *Weyl system* is a projective unitary representation of the linear vector space $\mathcal{S}$, thought of as the group manifold of the translation group, in the Hilbert space $\mathcal{H}$.

Let us now decompose $\mathcal{S}$ into the direct sum of two Lagrangian subspaces: $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$, so that any vector z can be written as $z = (z_1, 0) + (0, z_2)$, $z_1 \in \mathcal{S}_1$, $z_2 \in \mathcal{S}_2$. We can consider the restrictions of W to the Lagrangian subspaces, i.e.:

$$U = W|_{\mathcal{S}_1} : \mathcal{S}_1 \to \mathcal{H} \ , \ V = W|_{\mathcal{S}_2} : \mathcal{S}_2 \to \mathcal{H}$$

As $\omega|_{\mathcal{S}_1} = \omega|_{\mathcal{S}_2} = 0$, U and V are faithful representations of the corresponding Lagrangian subspaces:

$$\widehat{U}(z_1 + z_1') = \widehat{U}(z_1)\widehat{U}(z_1') \ , \ z_1, z_1' \in \mathcal{S}_1 \tag{59}$$

$$\widehat{V}(z_2 + z_2') = \widehat{V}(z_2)\widehat{V}(z_2') \ , \ z_2, z_2' \in \mathcal{S}_2 \tag{60}$$

Moreover:

$$\widehat{U}(z_1)\widehat{V}(z_2) = \widehat{V}(z_2)\widehat{U}(z_1) \exp\{\imath\omega((z_1, 0), (0, z_2))/\hbar\} \tag{61}$$

Viceversa, it is simple to show that given two faithful representations U and V of two transversal Lagrangian subspaces of a symplectic vector space $\mathcal{S}$ satisfying (61) the map:

$$z \mapsto \widehat{W}(z) = \widehat{U}(z_1)\widehat{V}(z_2) \exp\{-\imath\omega((z_1, 0), (0, z_2))/2\hbar\} \tag{62}$$

is a Weyl system.

Consider now a one-dimensional subspace of $\mathcal{H}$ spanned by a fixed vector z. From (57) we have, with α, β real numbers:

$$\widehat{W}(\alpha z)\widehat{W}(\beta z) = \widehat{W}((\alpha + \beta) z) \tag{63}$$

Therefore, $\left\{\widehat{W}(\alpha z)\right\}_{\alpha \in \mathbb{R}}$ is a strongly continuous one-parameter group of unitaries and, by Stone's theorem [14]:

$$\widehat{W}(\alpha z) = \exp\left\{\imath\alpha\widehat{G}(z)/\hbar\right\} \tag{64}$$

with an (essentially) self-adjoint infinitesimal generator $\widehat{G}(z)$. Furthermore, $\left\{\widehat{W}(\alpha\beta z)\right\}_{\beta\in\mathbb{R}}$ is also a strongly continuous one-parameter group, and therefore:

$$\widehat{W}(\alpha\beta z) = \exp\left\{\imath\beta\widehat{G}(\alpha z)/\hbar\right\} \tag{65}$$

and, setting $\beta = 1$, we find:

$$\widehat{G}(\alpha z) = \alpha\widehat{G}(z) \tag{66}$$

In terms of infinitesimal generators and setting: $z \to \alpha z, z' \to \beta z'$, Eq. (58) reads:

$$e^{\imath\alpha\widehat{G}(z)/\hbar}e^{\imath\beta\widehat{G}(z')/\hbar} = e^{\imath\alpha\beta\omega(z,z')/\hbar}e^{\imath\alpha\widehat{G}(z)/\hbar}e^{\imath\beta\widehat{G}(z')/\hbar} \tag{67}$$

and, for α and β infinitesimal, this yields, to the lowest nontrivial order:

$$\left[\widehat{G}(z), \widehat{G}(z')\right] = -\imath\hbar\omega(z,z') \tag{68}$$

Example: free particle. The simplest case we can consider is given by $\mathcal{S} = \mathbb{R}^2$ with coordinates (q,p) and the standard symplectic form $\omega = dq \wedge dp$ so that $\omega((q,p),(q',p')) = qp' - q'p$. Thus:

$$\widehat{W}((q,p)+(q',p')) = \widehat{W}(q,p)\widehat{W}(q',p')\exp\left\{-\frac{\imath}{2\hbar}(qp'-q'p)\right\} \tag{69}$$

Writing $z = (q,p)$ as $z = z_1 + z_2$ with $z_1 \equiv (q,0)$ and $z_2 \equiv (0,p)$, the Weyl system becomes :

$$\widehat{W}(q,p) = \widehat{W}((q,0)+(0,p)) = \widehat{W}(q,0)\widehat{W}(0,p)\exp\{-\imath qp/2\hbar\} \tag{70}$$

while:

$$\widehat{W}(q+q',0) = \widehat{W}(q,0)\widehat{W}(q',0)\ ,\ \widehat{W}(0,p+p') = \widehat{W}(0,p)\widehat{W}(0,p') \tag{71}$$

Define then

$$\begin{aligned}\widehat{W}(q,0) = \exp\left\{\imath q\widehat{P}/\hbar\right\} \quad &\Rightarrow \quad \widehat{G}(0,1) = \widehat{Q} \\ \widehat{W}(0,p) = \exp\left\{\imath p\widehat{Q}/\hbar\right\} \quad &\Rightarrow \quad \widehat{G}(1,0) = \widehat{P}\end{aligned} \tag{72}$$

with (cfr. Eq. (68)):

$$\left[\widehat{Q},\widehat{P}\right] = \imath\hbar\mathbb{I} \tag{73}$$

so that (using the already mentioned Baker-Campbell-Hausdorff formula), one finds easily:

$$\widehat{W}(q,p) = \exp\left\{\imath\left(q\widehat{P}+p\widehat{Q}\right)/\hbar\right\} \tag{74}$$

To construct a a concrete realization of the Weyl system, we consider now $L^2(\mathbb{R},dx)$ and define the families of operators $\left\{\widehat{U}(q)\right\}_{q\in\mathbb{R}}$ and $\left\{\widehat{V}(p)\right\}_{p\in\mathbb{R}}$ via:

$$\left(\widehat{U}(q)\psi\right)(x) = \psi(x+q) \tag{75}$$

$$\left(\widehat{V}(p)\psi\right)(x) = \exp\{\imath px/\hbar\}\psi(x) \tag{76}$$

for $\psi \in L^2(\mathbb{R}, dx)$. It is easy to show that both families are actually one-parameter, strongly continuous groups of unitaries, and that:

$$\left(\widehat{U}(q)\widehat{V}(p)\psi\right)(x) = \exp\{\imath qp/\hbar\}\left(\widehat{V}(p)\widehat{U}(q)\psi\right)(x) \tag{77}$$

Then:

$$\widehat{W}(q,p) = \widehat{U}(q)\widehat{V}(p)\exp\{-\imath qp/\hbar\} \tag{78}$$

At the infinitesimal level, in the appropriate domains, we have:

$$\left(\widehat{Q}\psi\right)(x) = x\psi(x)\,,\ \left(\widehat{P}\psi\right)(x) = -\imath\hbar\frac{d\psi}{dx} \tag{79}$$

Moreover:

$$\left(\widehat{W}(q,p)\psi\right)(x) = \exp\{\imath p[x+q/2]/\hbar\}\psi(x+q) \tag{80}$$

A generic matrix element of $\widehat{W}(q,p)$ will be given then by:

$$\left\langle\phi, \widehat{W}(q,p)\psi\right\rangle = \exp\{\imath qp/2\hbar\}\int_{-\infty}^{+\infty} dx\phi(x)^* \exp\{\imath px/\hbar\}\psi(x+q) \tag{81}$$

Viewed as a function on T^*Q, $\left\langle\phi, \widehat{W}(q,p)\psi\right\rangle$ is square-integrable for all $\phi, \psi \in L^2(\mathbb{R})$, as it can be easily seen from:

$$\left\|\left\langle\phi, \widehat{W}(q,p)\psi\right\rangle\right\|^2 = \int \frac{dqdp}{2\pi\hbar}\left|\left\langle\phi, \widehat{W}(q,p)\psi\right\rangle\right|^2 = \|\phi\|^2\|\psi\|^2 \tag{82}$$

Instead, for the plane-wave states

$$|k\rangle = \frac{1}{\sqrt{2\pi}}e^{\imath kx}\ ,\ |k'\rangle = \frac{1}{\sqrt{2\pi}}e^{\imath k'x} \tag{83}$$

we obtain:

$$\left\langle k'|\widehat{W}(q,p)|k\right\rangle = \delta(k-k'+p/\hbar)\,e^{\imath q(k+k')/2} \tag{84}$$

□

We look now for concrete realizations of a Weyl system in the general case of a symplectic vector space $(\mathcal{S}, \omega)$ and a decomposition of $\mathcal{S}$ as the direct sum: $\mathcal{S} = \mathcal{S}_1 \oplus \mathcal{S}_2$ with $\mathcal{S}_1$ and $\mathcal{S}_2$ Lagrangian subspaces. Every vector $z \in \mathcal{S}$ can then be decomposed in a unique way as: $z = (z_1, 0) + (0, z_2)\,, z_i \in \mathcal{S}_i, i = 1, 2$. Let us remark first of all that the symplectic structure allows each one of the two subspaces to be identified with the dual of the other: $\mathcal{S}_2 \approx \mathcal{S}_1^*$ (and viceversa), thanks to the pairing :

$$\langle\cdot,\cdot\rangle : \mathcal{S}_2 \times \mathcal{S}_1 \to \mathbb{R} \tag{85}$$

$$\langle z_2, z_1\rangle \equiv \omega((z_1, 0), (0, z_2)) \tag{86}$$

Assume now $\mathcal{H}$ to be a separable Hilbert space and let:

$$U : \mathcal{S}_1 \to \mathcal{H}\ \ ,\ \ V : \mathcal{S}_2 \to \mathcal{H} \tag{87}$$

be unitary, irreducible and strongly continuous representations of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively on $\mathcal{H}$, satisfying the additional condition that defines the "Weyl form" of the commutation relations:

$$\widehat{U}(z_1)\widehat{V}(z_2) = \widehat{V}(z_2)\widehat{U}(z_1)\exp\{\imath\omega((z_1,0),(0,z_2))/\hbar\} \tag{88}$$

Then we can define the Weyl system:

$$\widehat{W}(z) = \widehat{U}(z_1)\widehat{V}(z_2)\exp\{-\imath\omega((z_1,0),(0,z_2))/2\hbar\} \tag{89}$$

We will denote z_1 and z_2 with $(q,0)$ and $(0,p)$ respectively, with q and p n-dimensional vectors ($n = \dim S_1 = \dim S_2$). Correspondingly, we will denote $\widehat{U}(z_1)$ and $\widehat{V}(z_2)$ as $\widehat{U}(q)$ and $\widehat{V}(p)$ respectively.

Von Neumann's theorem [14] states then that there exists a unitary map $T : \mathcal{H} \to L^2(\mathbb{R}^n, d\mu)$ such that:

$$\left(T\widehat{U}(q)T^{-1}\psi\right)(x) = \psi(x+q) \tag{90}$$

$$\left(T\widehat{V}(p)T^{-1}\psi\right)(x) = e^{\imath x\cdot p}\psi(x) \tag{91}$$

This theorem proves that all the representations of the Weyl commutation relations are unitarily equivalent to the Schroedinger representation, and hence are unitarily equivalent among themselves.

3.2. Linear transformations

Let us now examine what happens to a Weyl system under the action of a linear transformation that preserves the symplectic structure, i.e. linear maps: $T : \mathcal{S} \to \mathcal{S}$ such that: $\omega(Tz, Tz') = \omega(z, z')$, $\forall z, z' \in \mathcal{S}$. Clearly, this induces a map:

$$\widehat{W}_T : \mathcal{S} \to \mathcal{H} \tag{92}$$

$$\widehat{W}_T(z) \equiv \widehat{W}(Tz) \tag{93}$$

such that:

$$\widehat{W}_T(z+z') = \widehat{W}_T(z)\widehat{W}_T(z')\exp\{-\imath\omega(z,z')/2\hbar\} \tag{94}$$

since

$$\begin{aligned}\widehat{W}(T(z+z')) &= \widehat{W}(Tz)\widehat{W}(Tz')\exp\{-\imath\omega(Tz,Tz')/2\hbar\} = \\ &= \widehat{W}(Tz)\widehat{W}(Tz')\exp\{-\imath\omega(z,z')/2\hbar\}\end{aligned} \tag{95}$$

Hence $\widehat{W}_T$ is also a Weyl system, which, by von Neumann's theorem, it is unitarily equivalent to $\widehat{W}$.

Example: Fourier transform. Consider, in $\mathbb{R}^2$, the map:

$$(q,p) \to (-p,q) \tag{96}$$

for which

$$\widehat{U}(q) = \widehat{W}((q,0)) \mapsto \widehat{W}((0,-p)) = \widehat{V}(-p) \tag{97}$$

$$\widehat{V}(p) = \widehat{W}((0,p)) \mapsto \widehat{W}((q,0)) = \widehat{U}(q) \tag{98}$$

It is a simple exercise to show that this is exactly what the Fourier transform does. Indeed, setting

$$\begin{aligned} \mathcal{F} &: L^2(\mathbb{R}) \to L^2(\mathbb{R}) \\ &\psi(x) \mapsto \widetilde{\psi}(p) \end{aligned}$$

with

$$\psi(x) = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \widetilde{\psi}(p) e^{\imath px} \tag{99}$$

one finds easily that:

$$\begin{aligned} \left(\widetilde{e^{\imath x\widehat{P}}\psi}\right)(p) = e^{\imath xp}\widetilde{\psi}(p) \quad &\Rightarrow \quad \left(\widehat{P}\widetilde{\psi}\right)(p) = p\widetilde{\psi}(p) \\ \left(e^{\imath p_0\widehat{Q}}\psi\right)(p) = \widetilde{\psi}(p-p_0) \quad &\Rightarrow \quad \left(\widehat{Q}\widetilde{\psi}\right)(p) = \imath d\widetilde{\psi}(p)/dp \end{aligned} \tag{100}$$

and we can conclude that:

$$\mathcal{F}^\dagger \widehat{Q} \mathcal{F} = -\widehat{P} \, , \; \mathcal{F}^\dagger \widehat{P} \mathcal{F} = \widehat{Q} \tag{101}$$

□

As $\widehat{W_T}$ is unitarily equivalent to $\widehat{W}$, to the map T there is associated an automorphism of the group $\mathcal{U}(\mathcal{H})$ of unitary operators. As every automorphism of $\mathcal{U}(\mathcal{H})$ is inner, there is a unitary operator $\widehat{U}_T$ such that:

$$\widehat{W_T}(z) = \widehat{U}_T^\dagger \left(\widehat{W}(z)\right) \widehat{U}_T \tag{102}$$

More generally, we can consider a one-parameter group $\{T_\lambda\}_{\lambda\in\mathbb{R}}$ of linear symplectic transformations. Calling Γ the linear vector field that is the infinitesimal generator of the group, the condition:

$$\omega(T_\lambda z, T_\lambda z') = \omega(z, z') \quad , \quad \forall z, z' \in S \, , \, \forall \lambda \in \mathbb{R} \tag{103}$$

becomes

$$L_\Gamma \omega = 0 \tag{104}$$

with L_Γ the Lie derivative. There exists then (globally on a vector space) a function g such that:

$$i_\Gamma \omega = dg \tag{105}$$

and, for linear transformations, g will be a quadratic function of the coordinates.
According to what has just been said, the family $\{T_\lambda\}$ defines a (strongly continuous) one-parameter group $\{U_\lambda\}_{\lambda\in\mathbb{R}}$ of unitary operators such that:

$$\widehat{W}(z(\lambda)) = \widehat{U}_\lambda^\dagger \widehat{W}(z) \widehat{U}_\lambda \tag{106}$$

where $z(\lambda) = T_\lambda(z)$. By Stone's theorem, then:

$$\widehat{U}_\lambda = \exp\left\{-i\lambda\widehat{G}/\hbar\right\} \tag{107}$$

with $\widehat{G}$ self-adjoint. The self-adjoint operator $\widehat{G}$ is the quantum counterpart of the quadratic function g. In this way we have achieved a way to quantize all the quadratic functions: given G, we can define via Eq.(105) the associated Hamiltonian vector field. This in turns defines a one-parameter group of (linear) symplectic transformations, and the corresponding Weyl system allows us to find the (self-adjoint) quantum operator to be associated with g.

Let's consider now a general linear transformation $T \in GL(2n, \mathbb{R})$, not necessarily a symplectic one. Denoting with ω_0 the standard Darboux symplectic form, we can define then a new symplectic structure ω_T via:

$$\omega_T(z, z') \equiv \omega_0(Tz, Tz') \tag{108}$$

and a new Weyl system for $(\mathcal{S}, \omega_T)$ (the proof is left to the reader):

$$\widehat{W}_T(z) \equiv \widehat{W}(Tz) \tag{109}$$

such that

$$\widehat{W}_T(z + z') = \widehat{W}_T(z)\,\widehat{W}_T(z')\exp\{-\imath\omega_T(z, z')/2\hbar\} \tag{110}$$

Mimicking the analysis that has been done previously, we conclude that:

$$\widehat{W}_T(\lambda z) = \exp\{i\lambda\widehat{G}(z)\} \tag{111}$$

with

$$\left[\widehat{G}(z), \widehat{G}(z')\right] = -\imath\hbar\omega_T(z, z') \tag{112}$$

Now we are in a position to consider Weyl systems for a vector space with an arbitrary and translationally invariant symplectic structure ω. By Darboux' theorem, there exists always an invertible linear transformation T that maps ω_0 in ω. Then, the sequence of transformations:

$$T : (\mathcal{S}, \omega) \to (\mathcal{S}, \omega_0)\ ,\ W : (\mathcal{S}, \omega_0) \to \mathcal{U}(\mathcal{H}) \tag{113}$$

defines a Weyl system for $(\mathcal{S}, \omega)$ by setting: $W \circ T = W_T$ or, more explicitly:

$$\widehat{W}_T(z) \equiv \widehat{W}(Tz) \tag{114}$$

A conspicuous example of a one-parameter group of symplectic transformations is provided by the time evolution of a Hamiltonian system. Some simple examples are reported below.

Example: free particle evolution. In this case, the one-parameter group is given by: $(q, p) \to (q + tp/m, p)$ and is represented by the matrix:

$$\left|\begin{array}{c} q(t) \\ p(t) \end{array}\right| = F(t)\left|\begin{array}{c} q \\ p \end{array}\right|\ ,\ F(t) = \left|\begin{array}{cc} 1 & t/m \\ 0 & 1 \end{array}\right|\ ,\ F(t)\,F(t') = F(t + t') \tag{115}$$

Then:

$$\widehat{W}_t(q,p) = \widehat{W}(q(t),p(t)) = \exp\left\{(\imath/\hbar)\left[q(t)\widehat{P} + p(t)\widehat{Q}\right]\right\}$$
$$\equiv \exp\left\{(\imath/\hbar)\left[q\widehat{P}_t + p\widehat{Q}_t\right]\right\} \tag{116}$$

where

$$\widehat{P}_t = \widehat{P} \ , \ \ \widehat{Q}_t = \widehat{Q} + t\widehat{P}/m \tag{117}$$

There exists therefore a one-parameter family $\left\{\widehat{F}(t)\right\}_{t\in\mathbb{R}}$ of unitary operators such that:

$$\exp\left\{\imath p\widehat{Q}_t/\hbar\right\} = \widehat{F}^{\dagger}(t)\exp\left\{\imath p\widehat{Q}/\hbar\right\}\widehat{F}(t) \tag{118}$$

$$\exp\left\{\imath q\widehat{P}_t/\hbar\right\} = \widehat{F}^{\dagger}(t)\exp\left\{\imath q\widehat{P}/\hbar\right\}\widehat{F}(t) \tag{119}$$

Setting then:

$$\widehat{F}(t) = \exp\left\{-\imath\widehat{H}t/\hbar\right\} \tag{120}$$

using Eq.(117) and expanding for small q, p and t, one finds the commutation relations:

$$\left[\widehat{P},\widehat{H}\right] = 0, \ \left[\widehat{Q},\widehat{H}\right] = \frac{\imath\hbar}{m}\widehat{P} \tag{121}$$

As the generators of linear and homogeneous canonical transformations are quadratic functions, it is natural to look for a quantum operator $\widehat{H}$ that is also a quadratic function:

$$\widehat{H} = a\widehat{P}^2 + b\widehat{Q}^2 + c\left(\widehat{P}\widehat{Q} + \widehat{Q}\widehat{P}\right) \tag{122}$$

Then the solution of the previous commutation relations is precisely:

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + \lambda\widehat{\mathbb{I}} \tag{123}$$

where $\widehat{\mathbb{I}}$ is the identity operator and λ and arbitrary real constant. Apart from this, the quantum operator associated with the time evolution is the standard quantum Hamiltonian for a free particle of mass m. $\square$

Example: harmonic oscillator evolution. From the classical equations of motion

$$q(t) = q\cos\omega t + p\frac{\sin\omega t}{m\omega} \tag{124}$$
$$p(t) = p\cos\omega t - qm\omega\sin\omega t$$

one finds that $F(t)$ is given by:

$$F(t) = \begin{vmatrix} \cos\omega t & \frac{\sin\omega t}{m\omega} \\ -m\omega\sin\omega t & \cos\omega t \end{vmatrix} \tag{125}$$

Proceeding just as in the previous example, we obtains:

$$\widehat{W}_t(q,p) = \exp\left\{(\imath/\hbar)\left[q\widehat{P}_t + p\widehat{Q}_t\right]\right\} \tag{126}$$

with, now:

$$\widehat{Q}_t = \widehat{Q}\cos\omega t + \widehat{P}\frac{\sin\omega t}{m\omega} \tag{127}$$

$$\widehat{P}_t = \widehat{P}\cos\omega t - \widehat{Q}m\omega\sin\omega t \tag{128}$$

Defining again: $\widehat{F}(t) = \exp\left\{-\imath\widehat{H}t/\hbar\right\}$ and working out the commutation relations:

$$\left[\widehat{Q}, \widehat{H}\right] = \frac{\imath\hbar}{m}\widehat{P} \quad , \quad \left[\widehat{P}, \widehat{H}\right] = -\imath\hbar m\omega^2\widehat{Q} \tag{129}$$

one gets the Hamiltonian:

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + \frac{1}{2}m\omega^2\widehat{Q}^2 + \lambda\widehat{\mathbb{I}} \tag{130}$$

which, again up to an additive multiple of the identity, is the standard quantum Hamiltonian for the harmonic oscillator. □

Other examples may be found in [5].

3.3. Quantum Mechanics on phase space

For simplicity, we will work in $S \approx \mathbb{R}^2$, but generalizations to higher dimensions are easy to work out.

3.3.1. More on the Weyl map

As a preliminary remark, let us observe that we have the identity ($f \in L^2(\mathbb{R}^2)$):

$$\int \frac{d\xi d\eta dq' dp'}{(2\pi\hbar)^2} f(q', p')\, e^{-\imath\omega_0((q',p'),(\xi,\eta))/\hbar} e^{\imath(\xi p + \eta q)/\hbar} = f(q, -p) \tag{131}$$

This can also be rewritten as:

$$\int \frac{d\xi d\eta}{2\pi\hbar}\left[\frac{1}{\hbar}\mathcal{F}_s(f)\left(\frac{\eta}{\hbar}, \frac{\xi}{\hbar}\right)\right] e^{\imath(\xi p + \eta q)/\hbar} = f(q, -p) \tag{132}$$

where $\mathcal{F}_s(f)$ is the symplectic Fourier transform [9]:

$$\mathcal{F}_s(f)(\eta, \xi) = \int \frac{dqdp}{2\pi} f(q, p)\, e^{-\imath\omega_0((q,p),(\xi,\eta))} \tag{133}$$

where, as usual, $\omega_0((q,p),(\xi,\eta)) = q\eta - p\xi$.
Allowing also for distribution-valued transforms, we have, in particular:

$$\mathcal{F}_s(q)(\eta, \xi) = 2\pi\imath\,\delta'(\eta)\,\delta(\xi) \tag{134}$$

and

$$\mathcal{F}_s(p)(\eta, \xi) = -2\pi\imath\,\delta(\eta)\,\delta'(\xi) \tag{135}$$

The Weyl map amounts to the replacement in Eq.(132):

$$\exp\left\{\imath\left(\xi p+\eta q\right)/\hbar\right\}\mapsto\exp\left\{\imath\left(\xi\widehat{P}+\eta\widehat{Q}\right)/\hbar\right\}\equiv\widehat{W}\left(\xi,\eta\right) \tag{136}$$

whereby one obtains the map:

$$\Omega:\mathcal{F}\left(\mathbb{R}^{2}\right)\rightarrow\mathcal{O}p\left(\mathcal{H}\right) \tag{137}$$

defined by:

$$\begin{aligned}\Omega\left(f\right)&\equiv\int\frac{d\xi d\eta}{2\pi\hbar}\left[\frac{1}{\hbar}\mathcal{F}_{s}\left(f\right)\left(\frac{\eta}{\hbar},\frac{\xi}{\hbar}\right)\right]\widehat{W}\left(\xi,\eta\right)=\\&=\int\frac{d\xi d\eta}{2\pi}\mathcal{F}_{s}\left(f\right)\left(\eta,\xi\right)\widehat{W}\left(\hbar\xi,\hbar\eta\right)\end{aligned} \tag{138}$$

It is simple to show that, if f is real, then:

$$\left[\mathcal{F}_{s}\left(f\right)\left(\eta,\xi\right)\right]^{*}=\mathcal{F}_{s}\left(f\right)\left(-\eta,-\xi\right) \tag{139}$$

and this proves that $\Omega\left(f\right)$ is at least a symmetric operator.

Using then

$$\left(\widehat{W}\left(\xi,\eta\right)\psi\right)\left(x\right)=\exp\left\{\imath\eta\left[x+\xi/2\right]/\hbar\right\}\psi\left(x+\xi\right) \tag{140}$$

we obtain:

$$\left(\Omega\left(f\right)\psi\right)\left(x\right)=\int\frac{d\xi d\eta}{2\pi}\mathcal{F}_{s}\left(f\right)\left(\eta,\xi\right)\exp\left[\imath\eta\left(x+\hbar\xi/2\right)\right]\psi\left(x+\hbar\xi\right) \tag{141}$$

In particular, using (134) and (135), one has:

$$\left(\Omega\left(q\right)\psi\right)\left(x\right)=x\psi\left(x\right)\ ,\ \left(\Omega\left(p\right)\psi\right)\left(x\right)=\imath\hbar\frac{d\psi}{dx} \tag{142}$$

or, in other words:

$$\Omega\left(q\right)=\widehat{Q}\ \ ,\ \ \Omega\left(p\right)=-\widehat{P} \tag{143}$$

More generally, for arbitrary integers n and m:

$$\mathcal{F}_{s}\left(q^{n}p^{m}\right)\left(\eta,\xi\right)=2\pi\left(-1\right)^{m}\imath^{n+m}\,\delta^{(n)}\left(\eta\right)\delta^{(m)}\left(\xi\right) \tag{144}$$

which implies

$$\begin{aligned}\left(\Omega\left(q^{n}p^{m}\right)\psi\right)\left(x\right)&=\left(\imath\frac{d}{d\xi}\right)^{m}\left[\left(x+\hbar\xi/2\right)^{n}\psi\left(x+\hbar\xi\right)\right]|_{\xi=0}\\&=\frac{1}{2^{n}}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)x^{k}\left(\imath\hbar\frac{d}{dx}\right)^{m}\left[x^{n-k}\psi\left(x\right)\right]\end{aligned} \tag{145}$$

Hence:

$$\Omega\left(q^{n}p^{m}\right)=\frac{1}{2^{n}}\sum_{k=0}^{n}\left(\begin{array}{c}n\\k\end{array}\right)\left[\Omega\left(q\right)\right]^{k}\cdot\left[\Omega\left(p\right)\right]^{m}\cdot\left[\Omega\left(q\right)\right]^{n-k} \tag{146}$$

In particular, for $n=m=1$:

$$\Omega\left(qp\right)=\frac{1}{2}\left[\Omega\left(q\right)\cdot\Omega\left(p\right)+\Omega\left(p\right)\cdot\Omega\left(q\right)\right] \tag{147}$$

Notice that $\Omega(qp) = \Omega(pq)$ but $\Omega(qp) \neq \Omega(q) \cdot \Omega(p)$. Indeed, in general:

$$\Omega(fg) \neq \frac{1}{2}(\Omega(f) \cdot \Omega(g) + \Omega(g) \cdot \Omega(f)) \tag{148}$$

as it can be shown on examples and seen already from Eq.(146) when $m, n \neq 1$, i.e. the "Weyl symmetrization procedure" (148) holds only in very special cases.

Finally we observe that from eq.(141) we obtain, for the matrix elements of the Weyl operator $\Omega(f)$:

$$\langle\phi|\Omega(f)|\psi\rangle = \int \frac{dx d\xi d\eta}{2\pi} \mathcal{F}_s(f)(\eta, \xi) e^{\imath\eta(x+\hbar\xi/2)} \phi^*(x) \psi(x + \hbar\xi) \tag{149}$$

In particular, in a plane-wave basis:

$$\langle k'|\Omega(f)|k\rangle = \int \frac{d\xi}{2\pi} \mathcal{F}_s(f)(k' - k, \xi) \exp\{\imath\hbar\xi(k + k')/2\} \tag{150}$$

or:

$$\langle K + k/2|\Omega(f)|K - k/2\rangle = \int \frac{d\xi}{2\pi} \mathcal{F}_s(f)(k, \xi) \exp\{\imath\hbar\xi K/2\} \tag{151}$$

Inserting then the explicit form of the symplectic Fourier transform, we find eventually:

$$\langle K + k/2|\Omega(f)|K - k/2\rangle = \int \frac{dq}{2\pi} f(q, -\hbar K) \exp\{-\imath kq\} \tag{152}$$

3.3.2. The Wigner map

The Weyl map can be inverted, i.e there exists a map, called the *Wigner map*:

$$\Omega^{-1} : \mathcal{O}p(\mathcal{H}) \to \mathcal{F}(\mathbb{R}^2) \tag{153}$$

such that:

$$\Omega^{-1}(\Omega(f)) = f \tag{154}$$

In general, given any operator $\widehat{O}$ such that $\mathrm{Tr}\left[\widehat{O}\widehat{W}(x, k)\right]$ exists*, the Wigner map is defined as:

$$\Omega^{-1}\left(\widehat{O}\right)(q, p) \equiv \int \frac{dx dk}{2\pi\hbar} \exp\{-\imath\omega_0((x, k), (q, p))/\hbar\} \,\mathrm{Tr}\left[\widehat{O}\widehat{W}^\dagger(x, k)\right] \tag{155}$$

Proof. In order to prove Eq.(154), we need the trace:

$$\mathrm{Tr}[\widehat{W}(x, k)\widehat{W}^\dagger(\xi, \eta)] = \int dh dh' \left\langle h \left| \widehat{W}(x, k) \right| h' \right\rangle \left\langle h' \left| \widehat{W}^\dagger(\xi, \eta) \right| h \right\rangle \tag{156}$$

Using Eq. (84) we obtain:

$$\mathrm{Tr}\left[\widehat{W}(x, k)\widehat{W}^\dagger(\xi, \eta)\right] = 2\pi\hbar\,\delta(x - \xi)\,\delta(k - \eta) \tag{157}$$

Inserting then (157) into (155), we obtain:

$$\Omega^{-1}(\Omega(f))(q, p) = \int \frac{d\xi d\eta}{2\pi} \mathcal{F}_s(\eta, \xi) \exp\{-\imath\omega((\xi, \eta), (q, p))\} = f(q, p) \tag{158}$$

* As W is a bounded operator, this will be granted, e.g., if A is trace-class.

■

It is useful to have an expression for the Wigner map directly in terms of the matrix elements of the operators. In terms of plane waves:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\int dke^{\imath qk}\left\langle -p/\hbar+k/2|\widehat{O}|-p/\hbar-k/2\right\rangle \tag{159}$$

Proof. Introducing a resolution of identity, one has:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\int\frac{dxd\pi dldm}{2\pi\hbar}\exp\left\{-\imath\left(xp-\pi q\right)/\hbar\right\}\left\langle l|\widehat{O}|m\right\rangle\left\langle m|\widehat{W}^{\dagger}\left(x,\pi\right)|l\right\rangle$$

or ($\pi=\hbar k$):

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\int\frac{dxdkdldm}{2\pi}\exp\left\{-\imath\left(xp/\hbar-kq\right)\right\}\left\langle l|\widehat{O}|m\right\rangle\left\langle m|\widehat{W}^{\dagger}\left(x,\hbar k\right)|l\right\rangle$$

Formula (159) is obtained by using

$$\left\langle m|\widehat{W}^{\dagger}\left(x,\hbar k\right)|l\right\rangle=\exp\left\{-\imath x\left(m+l\right)/2\right\}\delta\left(l-m-k\right) \tag{160}$$

■.

Also, it is easy to prove that

$$\Omega^{-1}\left(\widehat{W}\left(q',,p'\right)\right)(q,p)=\exp\left\{\imath\omega_0\left(\left(q,p\right),\left(q',p'\right)\right)/\hbar\right\} \tag{161}$$

Introducing now resolutions of the identity relative to the coordinates:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\int dkdxdx'e^{\imath qk}\left\langle -p/\hbar+k/2|\,x\right\rangle\left\langle x\,|\widehat{O}|\,x'\right\rangle\left\langle x'\,|-p/\hbar-k/2\right\rangle$$

the integration over k yields a delta-function, and eventually we obtain the celebrated Wigner formula:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\int d\xi e^{\imath p\xi/\hbar}\left\langle q+\xi/2|\widehat{O}\,|q-\xi/2\right\rangle \tag{162}$$

Finally, it is easy to check that the Wigner transform inverts to:

$$\left\langle x|\widehat{O}|x'\right\rangle=\int\frac{dp}{2\pi\hbar}\exp\left\{-\imath p\left(x-x'\right)/\hbar\right\}\Omega^{-1}\left(\widehat{O}\right)\left(\frac{x+x'}{2},p\right) \tag{163}$$

Examples.

(i) If $\widehat{O}=-\widehat{P}$, since $\widehat{P}\,|m\rangle=\hbar m\,|m\rangle$, we have:

$$\left\langle -p/\hbar+k/2|(-\widehat{P})|-p/\hbar-k/2\right\rangle=\left(p\hbar+k/2\right)\left\langle -p/\hbar+k/2|-p/\hbar-k/2\right\rangle=p\,\delta\left(k\right)$$

and we find, as expected:

$$\Omega^{-1}\left((-\widehat{P})\right)(q,p)=p$$

(ii) Setting: $\widehat{A}=\widehat{Q}$, we find at once:

$$\Omega^{-1}\left(\widehat{Q}\right)(q,p)=q$$

(iii) Consider now $\widehat{O}=|\phi\rangle\langle\psi|$, which is a prototype of a finite-rank operator. Then it is immediate to see that:

$$\Omega^{-1}\left(|\phi\rangle\langle\psi|\right)(q,p)=\int\limits_{-\infty}^{\infty}d\xi e^{\imath p\xi/\hbar}\phi\left(q+\xi/2\right)\psi\left(q-\xi/2\right)^{*} \qquad (164)$$

It is also easy to check formula (163):

$$\int\frac{dp}{2\pi\hbar}e^{\{-\imath p(x-x')/\hbar\}}\Omega^{-1}\left(\widehat{O}\right)\left(\frac{x+x'}{2},p\right)=\phi\left(x\right)\psi^{*}\left(x'\right)=\langle x|\phi\rangle\langle\psi|x'\rangle$$

(iv) Proceeding in a somewhat heuristic manner, let now $\widehat{O}$ be a self-adjoint operator with a completely discrete spectrum: $\widehat{O}|\phi_n\rangle=\lambda_n|\phi_n\rangle$, $\langle\phi_n|\phi_m\rangle=\delta_{nm}$, $\sum_n|\phi_n\rangle\langle\phi_n|=\mathbb{I}$. Then:

$$\Omega^{-1}\left(\widehat{O}\right)(q,p)=\sum_n\lambda_n\int d\xi e^{\imath p\xi/\hbar}\phi_n\left(q+\xi/2\right)\phi_n^{*}\left(q+\xi/2\right) \qquad (165)$$

□

The most interesting consequence of eq. (159) is the fact that the Weyl and Wigner maps establish a bijection [9] between Hilbert-Schmidt operators and square-integrable functions on phase space, which is also (strongly) bicontinuous. Indeed the following theorem holds:
f will be square-integrable if and only if $\Omega\left(f\right)$ is Hilbert-Schmidt. Similarly, $\Omega^{-1}\left(\widehat{A}\right)$ will be square-integrable if and only if $\widehat{A}$ is Hilbert-Schmidt.
Proof. Let's calculate the L^2 norm of $\Omega^{-1}\left(\widehat{A}\right)(q,p)$, i.e.

$$\begin{aligned}\left\|\Omega^{-1}\left(\widehat{A}\right)\right\|^{2} &= \int\tfrac{dqdp}{2\pi\hbar}\left|\Omega^{-1}\left(\widehat{A}\right)(q,p)\right|^{2}\\ &= \int\tfrac{dqdp}{2\pi\hbar}dkdk'e^{\imath(k'-k)q}\left\langle p/\hbar-k/2|\widehat{A}^{\dagger}|p/\hbar+k/2\right\rangle\left\langle p/\hbar+k'/2|\widehat{A}|p/\hbar-k'/2\right\rangle\end{aligned}$$

Performing the integration over q, which produces a delta-function, and shifting variables: $p\to p+\hbar k/2$, one gets:

$$\left\|\Omega^{-1}\left(\widehat{A}\right)\right\|^{2}=\int d\left(p/\hbar\right)dk\left\langle p/\hbar|\widehat{A}^{\dagger}|p/\hbar+k\right\rangle\left\langle p/\hbar+k|\widehat{A}|p/\hbar\right\rangle$$

The integration over k yields a resolution of the identity, and we end up with:

$$\left\|\Omega^{-1}\left(\widehat{A}\right)\right\|^{2}=\int d\left(p/\hbar\right)\left\langle p/\hbar|\widehat{A}^{\dagger}\widehat{A}|p/\hbar\right\rangle$$

Thus:

$$\left\|\Omega^{-1}\left(\widehat{A}\right)\right\|^{2}=\mathrm{Tr}\left[\widehat{A}^{\dagger}\widehat{A}\right] \qquad (166)$$

and, if: $\widehat{A} = \Omega(f)$:

$$\|f\|^2 = \mathrm{Tr}\left\{\Omega(f)^\dagger \Omega(f)\right\} \tag{167}$$

The condition of finiteness (positivity is obvious) of $\mathrm{Tr}\left[\widehat{A}^\dagger \widehat{A}\right]$ characterizes $\widehat{A}$ as a Hilbert-Schmidt operator. ■.

The fact that: $[\mathcal{F}_s(\eta,\xi)]^* = \mathcal{F}_s(-\eta,-\xi)$ and $\widehat{W}^\dagger(\xi,\eta) = \widehat{W}(-\xi,-\eta)$ allows to prove at once that the Weyl and Wigner maps preserve conjugation:

$$\Omega(f^*) = \Omega(f)^\dagger \quad , \quad \Omega^{-1}\left(\widehat{O}^\dagger\right) = \Omega^{-1}\left(\widehat{O}\right)^* \tag{168}$$

Therefore, in particular, if f is real, then, as already mentioned, $\Omega(f)$ will be a symmetric operator.

As a final remark, we observe that eq. (163) implies also:

$$\mathrm{Tr}_x\left[\widehat{O}\right] \equiv \int dx \langle x|O|x\rangle = \int \frac{dqdp}{2\pi\hbar}\Omega^{-1}\left(\widehat{O}\right)(q,p) \tag{169}$$

as well as:

$$\int \frac{dqdp}{2\pi\hbar} f(q,p) = \mathrm{Tr}\left[\Omega(f)\right] \tag{170}$$

and this defines formally a *trace operation* on phase space*:

$$\mathrm{Tr}[f] \equiv \int \frac{dqdp}{2\pi\hbar} f(q,p) \tag{171}$$

Example: the 1D Harmonic oscillator. We go back to the Hamiltonian

$$\widehat{H} = \frac{\widehat{P}^2}{2m} + \frac{1}{2}m\omega^2\widehat{Q}^2 \tag{172}$$

which has eigenvalues: $E_n = \hbar\omega(n+1/2)$ $(n \geq 0)$ and eigenfunctions:

$$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} \exp\left(-\zeta^2/2\right) H_n(\zeta) \tag{173}$$

where ζ is the dimensionless variable $\zeta = x\sqrt{m\omega/\hbar}$ and the H_n's are the Hermite polynomials [4] . We want to evaluate here the Wigner function associated with the so called Boltzmann factor $\widehat{A} = \exp\left(-\beta\widehat{H}\right)$. From:

$$\left\langle x\left|e^{-\beta\widehat{H}}\right|x'\right\rangle = \sum_{n=0}^{\infty} e^{-\beta E_n}\psi_n^*(x)\,\psi_n(x') \tag{174}$$

inserting the explicit form (173) of the eigenfunctions and manipulating the expression (we refer to [5] for details), one finds that the matrix element (174) can be expressed as:

$$\left\langle x\left|e^{-\beta\widehat{H}}\right|x'\right\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}e^{-\left(\zeta^2+\zeta'^2\right)/2}\sqrt{\frac{z}{1-z^2}}\exp\left[\frac{2z\zeta\zeta' - z^2\left(\zeta^2+\zeta'^2\right)}{1-z^2}\right]$$

* Of course, all these results will make sense when all the quantities in the previous equations are finite.

which yield the Wigner function:

$$\Omega^{-1}\left(e^{-\beta\widehat{H}}\right)(q,p)=\frac{1}{\cosh\left(\beta\hbar\omega/2\right)}\exp\left\{-\tanh\left(\beta\hbar\omega/2\right)\left[\frac{m\omega}{\hbar}q^{2}+\frac{p^{2}}{m\hbar\omega}\right]\right\}\quad(175)$$

Finally, using eq. (175), we find with some long but elementary algebra:

$$\mathrm{Tr}\left[\Omega^{-1}\left(e^{-\beta\widehat{H}}\right)\right]=\int\frac{dqdp}{2\pi\hbar}\Omega^{-1}\left(e^{-\beta\widehat{H}}\right)=\frac{1}{2\sinh\left(\beta\hbar\omega/2\right)}\quad(176)$$

which is the expected result for the canonical partition function of the 1D harmonic oscillator [4] (see also one of the examples in the section on coherent states).

3.3.3. The Moyal product and the Quantum→Classical Transition

The Wigner map allows for the definition of a new algebra structure on the space of functions $\mathcal{F}\left(\mathbb{R}^{2}\right)$, the *Moyal* " $*$ "-*product* that is defined as:

$$f*g\equiv\Omega^{-1}\left(\widehat{\Omega}\left(f\right)\cdot\widehat{\Omega}\left(g\right)\right)\quad(177)$$

This product is *associative*, it is *distributive* with respect to the sum, but it is *non-local* and *non-commutative* (since in general $\widehat{\Omega}\left(f\right)\cdot\widehat{\Omega}\left(g\right)\neq\widehat{\Omega}\left(g\right)\cdot\widehat{\Omega}\left(f\right)$). Explicitly:

$$\left(f*g\right)(q,p)=\int\frac{dxdk}{2\pi\hbar}\exp\left\{-\imath\omega_{0}\left(\left(x,k\right),\left(q,p\right)\right)/\hbar\right\}\mathrm{Tr}\left[\widehat{\Omega}\left(f\right)\cdot\widehat{\Omega}\left(g\right)\widehat{W}^{\dagger}\left(x,k\right)\right]\quad(178)$$

with

$$\begin{aligned}&\mathrm{Tr}\left[\widehat{\Omega}\left(f\right)\cdot\widehat{\Omega}\left(g\right)\widehat{W}^{\dagger}\left(x,k\right)\right]=\\&=\int\tfrac{d\xi d\eta d\xi' d\eta'}{(2\pi)^{2}}\mathcal{F}_{s}\left(f\right)\left(\eta,\xi\right)\mathcal{F}_{s}\left(g\right)\left(\eta',\xi'\right)\mathrm{Tr}\left[\widehat{W}\left(\hbar\xi,\hbar\eta\right)\widehat{W}\left(\hbar\xi',\hbar\eta'\right)\widehat{W}^{\dagger}\left(x,k\right)\right]\end{aligned}$$

Skipping the details of calculations (see [5]), it is possible to show that the last expression can be recast in the form:

$$\begin{aligned}\left(f*g\right)(q,p)&=4\int\frac{dadbdsdt}{(2\pi\hbar)^{2}}f\left(a,b\right)g\left(s,t\right)\exp\left\{-\frac{2\imath}{\hbar}\left[\left(a-q\right)\left(t-p\right)+\left(s-q\right)\left(p-b\right)\right]\right\}\\&=4\int\frac{dadbdsdt}{(2\pi\hbar)^{2}}f\left(a,b\right)g\left(s,t\right)\exp\left\{2\imath\omega_{0}\left(\left(q-a,p-b\right),\left(q-s,p-t\right)\right)/\hbar\right\}\end{aligned}$$

and this exhibits explicitly the non-locality of the Moyal product.
There are several equivalent ways of re-writing such an expression, such as*:

$$\begin{aligned}\left(f*g\right)(q,p)&=\sum_{n,m=0}^{\infty}\left(\frac{i\hbar}{2}\right)^{n+m}\frac{(-1)^{n}}{n!m!}\left\{\frac{\partial^{m+n}f\left(a,b\right)}{\partial a^{m}\partial b^{n}}\frac{\partial^{m+n}g\left(a,b\right)}{\partial a^{n}\partial b^{m}}\right\}|_{a=q,b=p}\\&=f\left(q,p\right)\exp\left\{\frac{i\hbar}{2}\left[\frac{\overleftarrow{\partial}}{\partial q}\frac{\overrightarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial p}\frac{\overrightarrow{\partial}}{\partial q}\right]\right\}g\left(q,p\right)\end{aligned}$$

* All the above expressions for the Moyal product apply of course to functions that are regular enough for the right-hand side of the defining equations to make sense. In particular, they will hold when f,g are Schwartz functions.

The latter form exhibits explicitly the Moyal product as a series expansion in powers of $\hbar$. To lowest order:

$$f * g = fg + \frac{i\hbar}{2}\{f,g\} + \mathcal{O}\left(\hbar^2\right) \tag{179}$$

where $\{.,.\}$ is the Poisson bracket. The Planck constant $\hbar$ acts then as a "deformation parameter" of the usual associative product structure on the algebra of functions, making the product non-commutative. Indeed, it can be seen, e.g., from the expansion of the exponential in eq.(179), that terms proportional to even powers of $\hbar$ are symmetric under the interchange $f \leftrightarrow g$, but terms proportional to odd powers are *anti*symmetric, and this makes the product non-commutative.

Examples.

(i) If $f \equiv 1$ *or* $g \equiv 1$, then:

$$(1 * g)(q,p) = g(q,p), (f * 1)(q,p) = f(q,p) \tag{180}$$

(ii) If $f = q$ and at least $g \in S^\infty(\mathbb{R}^2)$, then:

$$(q * g)(q,p) = 4\int \frac{dadbdsdt}{(2\pi\hbar)^2} a g(s,t) \exp\left\{\frac{2\imath}{\hbar}\left[(a-q)(t-p)+(s-q)(p-b)\right]\right\} =$$
$$= 4\int \frac{dadbdsdt}{(2\pi\hbar)^2} g(s,t)\left(q + \frac{\imath\hbar}{2}\frac{\partial}{\partial t}\right) \exp\left\{\frac{2\imath}{\hbar}\left[(a-q)(t-p)+(s-q)(p-b)\right]\right\}$$

Integrating by parts in the second integral and using the previous result:

$$(q * g)(q,p) = \left(q + \frac{\imath\hbar}{2}\frac{\partial}{\partial p}\right) g(q,p) \tag{181}$$

Similarly:

$$(g * q)(q,p) = \left(q - \frac{\imath\hbar}{2}\frac{\partial}{\partial p}\right) g(q,p) \tag{182}$$

(iii) In the same way, if $f = p$, we have:

$$(p * g)(q,p) = \left(p - \frac{\imath\hbar}{2}\frac{\partial}{\partial q}\right) g(q,p) \tag{183}$$

(iv) If $f = q$ and $g = p$ (or viceversa), then one gets:

$$(q * p)(q,p) = qp + \frac{\imath\hbar}{2}; (p * q)(q,p) = qp - \frac{\imath\hbar}{2} \tag{184}$$

(v) Notice that Eq.(181) implies:

$$\widehat{\Omega}(q) \cdot \widehat{\Omega}(g) = \widehat{\Omega}(qg) + \frac{\imath\hbar}{2}\widehat{\Omega}\left(\frac{\partial g}{\partial p}\right) \tag{185}$$

and similarly for the others. □

Using the Moyal product we can define the *Moyal Bracket* $\{.,.\}_*$ as:

$$\{.,.\}_* : \mathcal{F}\left(\mathbb{R}^2\right) \times \mathcal{F}\left(\mathbb{R}^2\right) \to \mathcal{F}\left(\mathbb{R}^2\right) \tag{186}$$

$$\{f,g\}_* \equiv \frac{1}{i\hbar}(f * g - g * f) = \{f,g\} + \mathcal{O}\left(\hbar^2\right) \tag{187}$$

where $\{.,.\}$ is the standard Poisson bracket. Notice that the difference between the Moyal and Poisson brackets is $\mathcal{O}(\hbar^2)$, and not $\mathcal{O}(\hbar)$, as one could expect, since the difference $f * g - g * f$ contains only odd powers of $\hbar$.
Being defined in terms of an associative product, the Moyal bracket fulfills all the properties of a Poisson bracket (linearity, anti-symmetry and the Jacobi identity), and defines a new Poisson structure on the (non-commutative) algebra of functions with the Moyal product. In particular, just as for the ordinary Poisson brackets, the Jacobi identity implies that $\{f,.\}_*$ is a derivation (with respect to the $*$-product) on the algebra of functions:

$$\{f, g * h\}_* = \{f, g\}_* * h + g * \{f, h\}_* \tag{188}$$

Writing down explicitly the second term in (187): $\{f, g\}_* = \{f, g\} + \hbar^2 \{f, g\}_2 + ...$, we obtain:

$$\{f, g\}_2 (q, p) = \frac{1}{24} \left\{ \frac{\partial^3 f}{\partial q^3} \frac{\partial^3 g}{\partial p^3} - 3 \frac{\partial^3 f}{\partial p \partial q^2} \frac{\partial^3 g}{\partial q \partial p^2} + 3 \frac{\partial^3 f}{\partial p^2 \partial q} \frac{\partial^3 g}{\partial q \partial q^2} - \frac{\partial^3 f}{\partial p^3} \frac{\partial^3 g}{\partial q^3} \right\}$$

Therefore, $\{f, g\}_*$ contains, besides first-order derivatives, third and higher-order derivatives, and, although it is a derivation on the algebra of functions with the " $*$ " product, it is not a vector field (while $\{f,.\}$ *is* a vector field). The reason for that is precisely that the Moyal bracket is non-local. It is only when f is at most a quadratic polynomial that $\{f,.\}_*$ becomes a derivation on the usual pointwise product. Indeed, if this is the case, the Moyal and Poisson brackets of f with other functions coincide.
Using the definitions of the Weyl and Wigner maps we have, in general:

$$\{f, g\}_* = i\Omega^{-1} \left(\widehat{\Omega}(f) \cdot \widehat{\Omega}(g) - \widehat{\Omega}(g) \cdot \widehat{\Omega}(f) \right) / \hbar \tag{189}$$

i.e.:

$$\left[\widehat{\Omega}(f), \widehat{\Omega}(g) \right] = -i\hbar \widehat{\Omega}(\{f, g\}_*) \tag{190}$$

Unless f and/or g are at most quadratic, $\{f, g\}_M \neq \{f, g\}$. Therefore, the commutator of the quantum operators associated with observables on phases space is not (modulo a multiplicative constant) the quantum operator associated with the Poisson bracket. Generically, it becomes so only to lowest order in $\hbar$, reproducing the so called Ehrenfest theorem [4].

As discussed in the pre-mini course (see also [5]), QM can and should be consistently described in the framework of the projective Hilbert space $P\mathcal{H}$. Once this is identified with the space of rank-one projectors, it is natural to discuss QM not for vectors in the Hilbert space but for the associate rank-one projectors. Given then the Weyl-Wigner maps, one would like to discuss QM in the context of the phase space, i.e. on $\mathcal{H} = L^2 (T^* Q)$. To do so, we proceed to reformulate the postulate of QM as follows.

Space of States. Denoting by simplicity as $f_{\widehat{O}} = \Omega^{-1}\left(\widehat{O}\right)$ the Wigner function associated with $\widehat{O}$, we see that:

$$\widehat{P}^2 = \widehat{P} \Leftrightarrow \Omega^{-1}\left(\widehat{P}^2\right) = \Omega^{-1}\left(\widehat{P} = \Omega^{-1}\left(\widehat{P}\right) * \Omega^{-1}\left(\widehat{P}\right)\right) \tag{191}$$

and

$$\widehat{P}^{\dagger}=\widehat{P}\ \Leftrightarrow\ \Omega^{-1}\left(\widehat{P}\right)\ \text{real} \tag{192}$$

Moreover:

$$\text{Tr}\left[\Omega^{-1}\left(\widehat{P}\right)\right]=\text{Tr}[\widehat{P}]=1 \tag{193}$$

Therefore projection operators are represented in phase space by real, uniformly-bounded functions satisfying:

$$f*f=f\ \ \text{and}\ \ \text{Tr}\left[f\right]<+\infty \tag{194}$$

iff the associated projector is of finite rank. Density states will be represented in turn by real, again uniformly-bounded, phase-space functions $f\left(q,p\right)$ satisfying: $\text{Tr}\left[f\right]=1$ and $\text{Tr}\left[f*f\right]\leq 1$.

Superposition rule. A special remark has to be added about the problem of superposition of states, which is at the origin of all interference phenomena characteristics of QM (see also a similar discussion about $P\mathcal{H}$ in the pre-course). If we denote as f_0 the Wigner function associated with a reference pure state and as f_1, f_2 those associated with two orthogonal (i.e.: $f_1 * f_2 = 0$) pure states $\rho_1=\left|\psi_1\right\rangle\left\langle\psi_1\right|, \rho_2=\left|\psi_2\right\rangle\left\langle\psi_2\right|$, then to the linear superposition $\left|\psi\right\rangle=c_1\left|\psi_1\right\rangle+c_2\left|\psi_2\right\rangle$, with $\left|c_1\right|^2+\left|c_2\right|^2=1$, there corresponds the Wigner function :

$$f=\sum_{i,j=1}^{2}c_i c_j^{*}\frac{f_i*f_0*f_j}{\sqrt{\text{Tr}\left[f_i*f_0*f_j*f_0\right]}} \tag{195}$$

where the phase-space trace has been defined in Eq.(171).

Observables. It is natural to pose eigenvalue problems not for vectors in the Hilbert space but for the associate rank-one projectors, which assume the form:

$$\widehat{O}\widehat{P}=\lambda\widehat{P}\ \ \text{with}\ \ \widehat{P}^{\dagger}=\widehat{P}\ ,\ \widehat{P}^{2}=\widehat{P}\ ,\ \text{Tr}\left[\widehat{P}\right]=1 \tag{196}$$

with $\widehat{O}$ an observable and $\lambda\in\mathbb{R}$ the corresponding eigenvalue*. The equivalent phase-space formulation, for $f\in L^2\left(T^*Q\right)$ will be then:

$$f_{\widehat{O}}*f=\lambda f\ ,\ f*f=f \tag{197}$$

for a real (and uniformly-bounded) function f. This will qualify f as the Wigner function associated with a projection operator: $f=\Omega^{-1}\left(\widehat{P}\right)$, with: $\text{Tr}[f]=1$ if it corresponds to a pure state.

Time evolution. Since an observable $\widehat{O}$ evolves in time as $\widehat{O}\left(t\right)=\widehat{U}^{\dagger}\left(t\right)\widehat{O}\widehat{U}\left(t\right)$ with $\widehat{U}\left(t\right)=\exp\left(-\imath t\widehat{H}/\hbar\right)$, we immediately see, from the very definition of the Moyal product that:

$$f(t)=f_{\widehat{O}(t)}=f_{\widehat{U}^{\dagger}(t)\cdot\widehat{O}\cdot\widehat{U}(t)}=f_{\widehat{U}^{\dagger}(t)}*f_{\widehat{O}}*f_{\widehat{U}(t)} \tag{198}$$

* To avoid unnecessary technical complications, we pose here the problem in the discrete spectrum. Also, the last condition in Eq.(196) can be relaxed in favor of P becoming then a not necessarily one-dimensional eigenprojector onto the subspace spanned by the eigenvalue λ.

Using the (formal) series expansion of the evolution operator, we can also write explicitly the evolution operator in phase space $f_{\widehat{U}(t)}$ as :

$$f_{\widehat{U}(t)} = \sum_{n=0}^{\infty} \frac{(-\imath t/\hbar)^n}{n!} \left(f_{\widehat{H}}\right)_*^n \equiv \exp_* \left(-\imath t f_{\widehat{H}}/\hbar\right) \tag{199}$$

where $(.)_*^n$ stands for an n-fold star-product. Now, to lowest order in t: $f_{\widehat{U}(t)} \approx 1 - (\imath t/\hbar)\, f_{\widehat{H}}$ etc., and we obtain easily the infinitesimal version of (198):

$$\frac{d}{dt} f(t) = \left\{f(t), f_{\widehat{H}}\right\}_M \ , \ f(0) = f \tag{200}$$

leading to:

$$f(t) = \exp_* \left(\imath t f_{\widehat{H}}/\hbar\right) * f * \exp_* \left(-\imath t f_{\widehat{H}}/\hbar\right) \tag{201}$$

This gives the phase-space description of quantum dynamics.

We finally notice that, as the classical ($\hbar \to 0$) limit of the Moyal bracket is the Poisson bracket, eq. (200) reduces to the description of the dynamics in terms of Poisson brackets.

4. Feynman Path Integral

The aim of this section is to give an introduction to Feynman's approach [8] to quantization, by means of the now-called path-integral technique. In his doctoral thesis [6] he presented a way to describe a quantum mechanical system starting from a classical one described in terms of a principle of least action, and not necessarily by Hamilton equations of motion. The idea was not to concentrate on the evolution of states and/or of operators, but to look directly at probabilities. Thus one tries to construct the kernel of the evolution operator:

$$\mathcal{K}(x, x'; t) = \langle x, t | x' \rangle = \langle x | e^{-\frac{\imath t H}{\hbar}} | x' \rangle \tag{202}$$

which gives the probability of finding the particle at point x at time t, given that it was at point x' at time 0 (we are supposing $t > 0$ and invariance under time-translations).

4.1. Path integral in the space of coordinates

In this subsection we will consider systems described by a Hamiltonian of the form:

$$H = T + V \ \ , \ \ T = \frac{p^2}{2m} \ , \ V = V(x) \tag{203}$$

for a particle with mass m moving in a potential $V(x)$.

In the following, we will use the so-called Trotter formula, that we now recall. For any two self-adjoint operators A, B on some Hilbert space $\mathcal{H}$, one can trivially write:

$$e^{\imath(A+B)} = [e^{\imath\epsilon(A+B)}]^M \ \ , \ \ t \equiv \epsilon M$$

Then, using the fact that

$$e^{\imath\epsilon(A+B)} = e^{\imath\epsilon A} e^{\imath\epsilon B} + O(\epsilon^2)$$

we get:

$$e^{\imath(A+B)} = \lim_{\epsilon\to 0}\left[e^{\imath\epsilon A}e^{\imath\epsilon B}\right]^M \tag{204}$$

where the limit means $\epsilon \to 0$ and $M \to \infty$ so that $M\epsilon = t$ is kept constant. This formula holds in the operator-norm sense if $\mathcal{H}$ is finite-dimensional. In the infinite-dimensional case, the limit has to be understood in the strong sense and applied to vectors in the appropriate domains.

For the Hamiltonian (203), supposing $T+V$ to be self-adjoint on a dense domain $\mathcal{D}(T)\cap\mathcal{D}(V)$, we can re-write the kernel in the following form:

$$\begin{aligned}\langle x|e^{-\frac{\imath tH}{\hbar}}|x'\rangle &= \lim_{\epsilon\to 0}\langle x|\left[e^{-i\epsilon\frac{T}{\hbar}}e^{-\imath\epsilon\frac{V}{\hbar}}\right]^M|x'\rangle = \\ &= \lim_{\epsilon\to 0}\int dx_1 ... \int dx_{M-1}\langle x|e^{-\imath\epsilon\frac{T}{\hbar}}e^{-\imath\epsilon\frac{V}{\hbar}}|x_{M-1}\rangle \times \\ &\times \langle x_{M-1}|e^{-\imath\epsilon\frac{T}{\hbar}}e^{-\imath\epsilon\frac{V}{\hbar}}x_{M-2}\rangle\cdots\langle x_1|e^{-\imath\epsilon\frac{T}{\hbar}}e^{-\imath\epsilon\frac{V}{\hbar}}|x'\rangle = \\ &= \lim_{\epsilon\to 0}\int dx_1 ... \int dx_{M-1}\prod_{n=1}^{M}\langle x_n|e^{-i\epsilon\frac{T}{\hbar}}e^{-\imath\epsilon\frac{V}{\hbar}}|x_{n-1}\rangle\end{aligned} \tag{205}$$

To obtain this expression we have, in the second line, inserted M resolutions of the identities in coordinate space ($\mathbb{I} = \int dx_j|x_j\rangle\langle x_j|$) and defined: $x_M = x$ e $x_0 = x'$.
We insert now a resolution of the identity in momentum space ($\mathbb{I} = \int dp_|p_j\rangle\langle p_j|$) in each factor of the integrand, to get:

$$\begin{aligned}\langle x_n|e^{-\imath\epsilon\frac{T}{\hbar}}e^{-\imath\epsilon\frac{V}{\hbar}}|x_n - 1\rangle &= \int dp_n\langle x_n|e^{-\imath\epsilon\frac{T}{\hbar}}|p_n\rangle\langle p_n|e^{-\imath\epsilon\frac{V}{\hbar}}|x_{n-1\rangle} = \\ &= \int dp_n e^{-\imath\frac{\epsilon p_n^2}{2m\hbar}}\langle x_n|p_n\rangle e^{-\imath\frac{\epsilon V(x_{n-1})}{\hbar}}\langle p_n|x_{n-1}\rangle = \\ &= \frac{1}{2\pi\hbar}\int dp_n e^{-\imath\frac{\epsilon p_n^2}{2m\hbar}}e^{-\imath\frac{\epsilon V(x_{n-1})}{\hbar}}e^{\imath\frac{p_n(x_n-x_{n-1})}{\hbar}}\end{aligned} \tag{206}$$

where we have used the fact that:

$$\langle x_j|p_n\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{\imath\frac{x_j p_n}{\hbar}} \tag{207}$$

Now we insert (206) in(205), finding:

$$\begin{aligned}\langle x|e^{-\imath\frac{tH}{\hbar}}|x'\rangle &= \lim_{\epsilon\to 0}\int dx_1...dx_{M-1}\frac{dp_1...dp_m}{(2\pi\hbar)^M}\prod_{n=1}^{M}\left[\exp\{\imath\frac{p_n(x_n-x_{n-1})}{\hbar}+\right. \\ &- \left.\imath\frac{\epsilon p_n^2}{2m\hbar} - \imath\frac{\epsilon V(x_{n-1})}{\hbar}\}\right]\end{aligned} \tag{208}$$

Thanks to the form of the Hamiltonian, we see that the integrals in the p_n's are of Gaussian type and thus they can be easily performed, to finally achieve:

$$\begin{aligned}\langle x|e^{-\imath\frac{tH}{\hbar}}|x'\rangle = \lim_{\epsilon\to 0}&\int dx_1...dx_{M-1}\left(\frac{m}{2\pi\imath\hbar\epsilon}\right)^{\frac{M}{2}} \times \\ &\times \exp\left\{\frac{\imath\epsilon}{\hbar}\sum_{n=1}^{M}\left[\frac{m}{2}\left(\frac{x_n-x_{n-1}}{\epsilon}\right)^2 - V(x_{n-1})\right]\right\}\end{aligned} \tag{209}$$

Let us remark that we have use no approximations to get this formula, which is mathematically sound as long as the problem of domains is taken into account, and the limit is taken after all integrals have been calculated.

Disregarding the last comment, we are tempted to bring the limit inside the integrals, noticing that:

$$\frac{x_n - x_{n-1}}{\epsilon} \to \dot{x}(t') \qquad V(x_{n-1}) \to V(x(t')) \qquad \epsilon \sum_{n=1}^{M} \to \int_0^t dt' \tag{210}$$

Thus, we can write:

$$\epsilon \sum_{n=1}^{M} \left[\frac{m}{2} \left(\frac{x_n - x_{n-1}}{\epsilon} \right)^2 - V(x_{n-1}) \right] \to \int_0^t dt' \left[\frac{m\dot{x}^2(t')}{2} - V(x(t')) \right]$$

where the last expression gives the classical action S. If, for compactness, we set:

$$\int_{x(0)=x'}^{x(t)=x} [\mathcal{D}x] = \lim_{\epsilon \to 0} \int dx_1 ... dx_n \left(\frac{m}{2\pi i \epsilon} \right)^{\frac{M}{2}}$$

we can eventually write eq. (209) as a function of the classical action $S =$ as:

$$\langle x | e^{-\imath \frac{tH}{\hbar}} | x' \rangle = \int_{x(0)=x'}^{x(t)=x} [\mathcal{D}x] e^{\frac{\imath}{\hbar} S} \tag{211}$$

This is the celebrated path-integral formula for the kernel, which is telling us that to get the quantum amplitude (202) we have to integrate over all possible paths $x(t)$, starting at x' and ending at x, the exponential of the classical action, evaluated on such trajectories.

Let us remark that formula (209) is compact and has an elegant interpretation, but it has (at least at this point of the discussion) only a formal meaning, since we have not specified which class of trajectories, i.e. space of functions, we work on and therefore what measure of integration we need. Notice also that they way we have obtained it suggests to interpret $x(t)$ as a trajectory obtained in configuration space by discretizing it with the set of points $\{x_j = x(t = j\epsilon)\}_j$. Thus, for the very way in which it is constructed, we expect it to have no nice properties such as continuity or differentiability. We will say something more about this in the following. For now, we use formula (209) in a symbolic way and refer to eq. (209) to do explicit calculations.

Example: the free particle. As a first example, let us choose $V(x) = 0$ and work in 1D*. In this case the kernel is given by:

$$\langle x | e^{-\imath \frac{tH}{\hbar}} | x' \rangle = \lim_{\epsilon \to 0} \int dx_1 ... dx_{M-1} \left(\frac{m}{2\pi \imath \hbar \epsilon} \right)^{\frac{M}{2}} \exp \left[\frac{\imath m}{2\hbar} \sum_{n=1}^{M} \frac{(x_n - x_{n-1})}{\epsilon} \right] \tag{212}$$

Using the identity:

$$\int dy \exp \left\{ \frac{\imath A}{2} \left[\frac{(x-y)^2}{\alpha} + \frac{(y-z)^2}{\alpha'} \right] \right\} = \exp \left\{ \frac{\imath A (x-z)^2}{2(\alpha + \alpha')} \right\} \left(\frac{2\pi \imath \alpha \alpha'}{A(\alpha + \alpha')} \right)^{\frac{1}{2}}$$

* The calculations can be easily generalized to arbitrary dimensions.

it is possible to calculate recursively all integrals, finding:

$$\langle x|e^{-\imath\frac{tH}{\hbar}}|x'\rangle = \lim_{\epsilon\to 0}\left(\frac{m}{2\pi\imath\hbar\epsilon}\right)^{\frac{M}{2}}\left(\frac{2\pi\imath\hbar}{m}\right)^{\frac{M-1}{2}}\left(\frac{\epsilon^M}{t}\right)^{\frac{1}{2}}\exp\left[\frac{\imath m(x-x')^2}{2\hbar t}\right] = \\ = \sqrt{\frac{m}{2\pi\imath\hbar t}}\exp\left[\frac{\imath m(x-x')^2}{2\hbar t}\right] \tag{213}$$

Let us recall that the classical action is given by:

$$S = \int_0^t dt'\frac{m}{2}\left(\frac{dx}{dt}\right)^2 \tag{214}$$

while integration of the equation of motion $d^2x(t)/dt^2 = 0$ yields the classical trajectory, satisfying the boundary conditions $x(0) = x', x(t) = x$:

$$x_{cl}(t') = \frac{t'(x-x')}{t} + x' \quad , \quad \dot{x}_{cl}(t') = \frac{x-x'}{t} \tag{215}$$

from which

$$S_{clas}(x,x') \equiv S|_{x_{cl}} = \int_0^t dt'\frac{m}{2}\left(\frac{x-x'}{t'}\right)^2 \tag{216}$$

Comparing with (213) we might see that:

$$\langle x|e^{-\imath\frac{tH}{\hbar}}|x'\rangle = F(t)\exp\left[\frac{\imath}{\hbar}S_{clas}(x,x')\right] \quad , \quad F(t) = \sqrt{\frac{m}{2\pi it\hbar}} \tag{217}$$

□

Example: the 1D harmonic oscillator. Similar calculations can be done for the hamiltonian of the 1D harmonic oscillator, to get:

$$K(x,x';t) = F_{ho}(t)\exp\left[\frac{\imath}{\hbar}S_{clas}(x,x')\right] \tag{218}$$

with

$$S_{cl}(x,x') = \frac{\omega m}{2\sin(\omega t)}\left[(x+x')^2\cos(\omega t) - 2xx'\right] \tag{219}$$

$$F_{ho} = \sqrt{\frac{m\omega}{2\pi\imath\hbar\sin(\omega t)}}. \tag{220}$$

□

Example: particle in a magnetic field. We briefly outline here one more example, since in this case the Hamiltonian cannot be written as in (203). We consider a particle of mass m and charge e moving in a magnetic potential $\vec{A}$, thus described by the Hamiltonian:

$$H = \frac{1}{2m}\left[\vec{p} - \frac{e\vec{A}}{c}\right]^2 \tag{221}$$

To proceed, we have first to write H in its normal form, by putting all terms containing coordinates to the right of those containing momenta:

$$: H :\equiv \frac{1}{2m}\left[\vec{p}^2 - \frac{2e}{c}\vec{p}\cdot\vec{A} - \frac{ie\hbar}{c}\vec{\nabla}\cdot\vec{A} + \frac{e^2\vec{A}^2}{c^2}\right] \tag{222}$$

Then we consider:

$$\begin{aligned}\int d^3q_n\langle q_n|e^{-\frac{\imath\epsilon}{\hbar}:H:}|q_{n-1}\rangle &\cong \int d^3p_n\langle q_n|p_n\rangle\langle p_n| : e^{-\frac{\epsilon}{\hbar}H(\hat{p},\hat{x})} : |q_n\rangle = \\ &= \int d^3p_n\langle q_n|p_n\rangle e^{-\imath\frac{\epsilon H}{\hbar}}\langle p_n|q_n\rangle = \\ &= \frac{1}{(2\pi\hbar)^3}\int d^3p_n e^{\frac{\imath}{\hbar}p(q_n-q_{n-1})}e^{-\frac{\imath\epsilon H}{\hbar}}\end{aligned} \tag{223}$$

where we have inserted the following normal ordering expression for the exponential function:

$$: e^{-\frac{\imath\epsilon H}{\hbar}} := e^{-\imath\frac{\epsilon:H:}{\hbar}} - \left(\frac{\epsilon}{\hbar}\right)^2\sum_{n=0}^{\infty}\frac{\left(-\frac{\imath\epsilon}{\hbar}\right)^2}{(n+2)!}\left[H^{n+2} - : H^{n+2} :\right] \tag{224}$$

and kept only terms up to $O(\epsilon)$, since we want to take the limit $\epsilon \to 0$.
Thus the kernel is given by:

$$\begin{aligned}K(x,x';t) = \lim_{\epsilon\to 0}\int &\frac{dq_1...dq_{M-1}dp_1...dp_M}{(2\pi\hbar)^{3M}} \times \\ &\times \exp\left\{\frac{\imath\epsilon}{\hbar}\sum_{n=1}^{M}\left[p_n\frac{q_n - q_{n-1}}{\epsilon} - H(p_n,q_n)\right]\right\}\end{aligned} \tag{225}$$

or, formally:

$$K(x,x';t) = \int_{x(0)=x',x(t)=x}[\mathcal{D}x][\mathcal{D}p]\exp\left\{\frac{\imath}{\hbar}\int_0^t dt'\left[p(t)\dot{q}(t) - H(p,q)\right]\right\}. \tag{226}$$

Let us notice, that the Hamiltonian (221) is invariant under the (local) gauge transformations:

$$\vec{A} \mapsto \vec{A} + \vec{\nabla}_q\Lambda(q)$$

$$\psi(q) \mapsto \tilde{\psi}(q) \equiv U(q)\psi(q) = \exp\left[\frac{ie}{\hbar c}\int_{q_0}^{q}dq'\vec{\nabla}\Lambda\right]\psi(q) = e^{\frac{\imath e}{\hbar c}[\Lambda(q)-\Lambda(q_0)]}\psi(q)$$

having fixed a fiducial point q_0. It is not difficult then to see that the kernel is not invariant under such a transformation. Indeed:

$$\begin{aligned}\langle\tilde{q}|e^{-\frac{\imath t\tilde{H}'}{\hbar}}|\tilde{q}'\rangle &= e^{-\frac{\imath e}{\hbar c}[\Lambda(x)-\Lambda(x_0)]}\langle q|e^{-\frac{\imath tH}{\hbar}}|q'\rangle e^{\frac{\imath e}{\hbar c}[\Lambda(q')-\Lambda(q_0)]} = \\ &= e^{-\frac{\imath e}{\hbar c}\Lambda(q)}\langle q|e^{-\frac{\imath tH}{\hbar}}|q'\rangle e^{\frac{\imath e}{\hbar c}\Lambda(q')}\end{aligned} \tag{227}$$

$$K(q,q';t)_{H'} = e^{-\frac{\imath e}{\hbar c}[\Lambda(q)-\Lambda(q')]}K(q,q';t)_H \tag{228}$$

□

4.1.1. Feynman integral in imaginary time and partition function

For a quantum system, the (canonical) partition function can be calculated as:

$$\mathcal{Z} \equiv \mathrm{Tr}\left[e^{-\beta H}\right] = \int dx \langle x|e^{-\beta H}|x\rangle \tag{229}$$

Therefore it is possible to proceed as in the previous section by taking into account the following two minor differences:
- in the expression we have $e^{-\beta H}$ instead of $e^{-\frac{\imath}{\hbar}tH}$;
- boundary conditions are now $x(t) = x(0) = x$, on which we have to integrate.
We will skip all details here (which we leave to the reader) and write only the final result:

$$\begin{aligned} \mathcal{Z} = \; & \lim_{\epsilon\to 0} \int_{x_0 = x_M} \left(\prod_{n=1}^{M} d^3x_n\right)\left(\frac{m}{2\pi\hbar\epsilon}\right)^{\frac{3M}{2}} \times \\ & \times \; \exp\left\{-\frac{\epsilon}{\hbar}\sum_{n=1}^{\infty}\left[\frac{(x_n - x_{n-1})^2}{\epsilon^2} + V(x_n)\right]\right\}. \end{aligned} \tag{230}$$

which can be written, formally, as:

$$\begin{aligned} \mathcal{Z} = & \int dx \int_{x(0)=x', x(\tau)=x} [\mathcal{D}x(\tau)] \exp\left\{-\frac{1}{\hbar}\int_0^{\tau} d\tau' H[x(\tau')]\right\} = \\ = & \int_{x(0)=x(\beta\hbar)} [\mathcal{D}x(\tau)] \exp\left\{-\frac{1}{\hbar}\int_0^{\beta\hbar} d\tau' H[x(\tau')]\right\}. \end{aligned} \tag{231}$$

Let us remark that the same result could have been obtained by performing a "Wick-rotation", i.e. the analytic continuation from the real variable t to the imaginary one $\tau = \imath t$. Indeed this would have changed the expressions of the derivatives appearing in the action S according to:

$$\frac{dx}{dt} = \imath\frac{dx}{d\tau} \;\; , \;\; \left(\frac{dx}{dt}\right)^2 = -\left(\frac{dx}{d\tau}\right)^2 .$$

Finally, it is important to notice that the measure that appears in (231) is the same one used in the context of stochastic processes, i.e. Wiener measure. In this case it is possible to give a rigorous mathematical treatment to make sense of the continuous version of Feynman path-integral [7].

4.2. Path integral with coherent states

In this section we will work out the path integral formulation of QM using coherent states. In the following we will concentrate on one single degree of freedom, either bosonic or fermionic, but the construction can be generalized to many body systems and quantum field theory. Indeed the technique of Feynman path integral with coherent states is one of the most exploited in these fields.

Let us take two coherent states, either bosonic or fermionic: $|\phi_i\rangle$ e $\langle\phi_f|$. We write the propagator in the following way:

$$\begin{aligned}\langle\phi_f|e^{-\imath\frac{tH}{\hbar}}|\phi_i\rangle &= \lim_{\epsilon\to 0}\langle\phi_f|(e^{-\imath\frac{\epsilon H}{\hbar}})^M|\phi_i\rangle = \\ &= \lim_{\epsilon\to 0}\int\left(\prod_{n=1}^{M-1}\frac{d\phi_n^* d\phi_n}{\mathcal{N}}\right)e^{-\sum_{n=1}^{M-1}\phi_n^*\phi_n}\langle\phi_f|e^{-\frac{i\epsilon H}{\hbar}}|\phi_{M-1}\rangle\times \\ &\times\langle\phi_{M-1}|e^{-\frac{\imath\epsilon H}{\hbar}}|\phi_{M-2}\rangle\ldots\langle\phi_1|e^{-\frac{\imath\epsilon H}{\hbar}}|\phi_i\rangle = \\ &= \lim_{\epsilon\to 0}\int\left(\prod_{n=1}^{M-1}\frac{d\phi_n^* d\phi_n}{\mathcal{N}}\right)e^{-\sum_{n=1}^{M-1}\phi_n^*\phi_n}\left(\prod_{n=1}^{M}\langle\phi_n|e^{-\imath\frac{\epsilon H}{\hbar}}|\phi_{n-1}\rangle\right)\end{aligned}$$

where we have set: $\langle\phi_f|\equiv\langle\phi_M|$, $|\phi_i\rangle\equiv|\phi_0\rangle$. To arrive at this expression we have divided the time interval t in M intervals of length ϵ, inserted the resolution of the identity written in terms of coherent states:

$$\mathbb{I} = \int\left(\prod_j\frac{d\phi_j^* d\phi_j}{\mathcal{N}}e^{-\phi_j^*\phi_j}|\phi_j\rangle\langle\phi_j|\right)$$

with

$$\phi = \begin{cases} z\in\mathbb{C} \\ \xi\in\mathbb{G}\end{cases} \qquad \mathcal{N} = \begin{cases} 2\pi i & \text{(B)} \\ 1 & \text{(F)}\end{cases}$$

and supposed that $H = H(a^\dagger, a)$ is written in its normal form so that:

$$\langle\phi_n|e^{-\imath\frac{\epsilon H}{\hbar}}|\phi_{n-1}\rangle = \langle\phi_n|1-\frac{\imath\epsilon}{\hbar}H(a^\dagger,a)+\cdots|\phi_{n-1}\rangle = e^{\phi_n^*\phi_{n-1}}e^{-\imath\frac{\epsilon}{\hbar}H(\phi_n^*,\phi_{n-1})}$$

where $H(\phi_n^*,\phi_{n-1})$ has been obtained form $H = H(a^\dagger, a)$ by means of the substitution: $a^\dagger\mapsto\phi_n^*$, $a\mapsto\phi_n$. Thus:

$$\begin{aligned}\langle\phi_f|e^{-\imath\frac{tH}{\hbar}}|\phi_i\rangle &= \lim_{\epsilon\to 0}\int\left(\prod_{n=1}^{M-1}\frac{d\phi_n^* d\phi_n}{\mathcal{N}}\right)e^{-\sum_{n=1}^{M-1}\phi_n^*\phi_n}\times \\ &\times e^{\sum_{n=1}^{M}\phi_n^*\phi_{n-1}}e^{-\imath\frac{\epsilon}{\hbar}\sum_{n=1}^{M}H(\phi_n^*\phi_{n-1})}. \end{aligned} \tag{232}$$

As before, we can interpret $\{\phi_n\}$ as a discretized trajectory, so that:

$$\phi_n^*\frac{\phi_n-\phi_{n-1}^*}{\epsilon}\to\phi_n^*(t')\frac{\partial}{\partial t'}\phi(t') \tag{233}$$

$$H(\phi_n^*,\phi_{n-1})\to H(\phi_n^*(t'),\phi_{n-1}(t')) \tag{234}$$

and

$$\begin{aligned}&-\sum_{n=1}^{M-1}\phi_n^*\phi_n+\sum_{n=1}^{M}\phi_n^*\phi_{n-1}-\frac{\imath\epsilon}{\hbar}\sum_{n=1}^{M}H(\phi_n^*,\phi_{n-1})\to \\ &\phi^*(t)\phi(t)+\frac{\imath}{\hbar}\int_0^t dt'\left[(i\hbar)\phi^*(t')\frac{\partial\phi(t')}{\partial t'}-H(\phi^*(t'),\phi(t'))\right]\end{aligned} \tag{235}$$

Thus, in a formal way we can write:

$$\langle\phi_f|e^{-\imath\frac{tH}{\hbar}}|\phi_i\rangle = \int_{\phi(0)=\phi_i,\phi(t)=\phi_f}[\mathcal{D}\phi^*\mathcal{D}\phi]e^{\phi^*(t)\phi(t)}e^{\frac{\imath}{\hbar}\int_0^t dt\left[i\hbar\phi^*\frac{\partial\phi}{\partial t'}-H(\phi^*,\phi)\right]} \tag{236}$$

where the integral in the exponential represents the so-called Schroedinger Lagrangian of the classical system, which contains a kinetic term which is linear in the first derivatives with respect to time.

We can now proceed to evaluate the (grancanonical) partition function of the system, by recalling that:

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta(\hat{H}-\mu\hat{N})}] = \int \frac{d\tilde{\phi}^* d\tilde{\phi}}{\mathcal{N}} e^{-\tilde{\phi}^*\tilde{\phi}} \langle \zeta\tilde{\phi}| e^{-\beta(\hat{H}-\mu\hat{N})} |\tilde{\phi}\rangle \tag{237}$$

with

$$\zeta = \begin{cases} +1 & \text{for boson} \\ -1 & \text{for fermion} \end{cases}$$

From (232), putting $\tau' = \imath t', \beta = it, \hbar = 1$, we have:

$$\langle \zeta\tilde{\phi}| e^{-\beta(\hat{H}-\mu\hat{N})} |\tilde{\phi}\rangle = \lim_{\epsilon\to 0} \int \prod_{n=1}^{M-1} \frac{d\phi_n^* d\phi_n}{\mathcal{N}} e^{-i\sum_{n=1}^{M-1}\phi_n^*\phi_n} e^{\sum_{n=1}^{M}\phi_n^*\phi_n} \times$$
$$\times \prod_{n=1}^{M} \exp\{-\epsilon[H(\phi_n^*, \phi_{n-1}) - \mu\phi_n^*\phi_{n-1}]\} \tag{238}$$

with the boundary conditions:

$$\begin{cases} \phi_0 = \tilde{\phi} \\ \phi_M^* = \zeta\tilde{\phi}^* = \zeta\phi_0^* \end{cases}, \qquad \begin{cases} \phi_M^* = \zeta\tilde{\phi}^* \\ \phi_M = \zeta\tilde{\phi}^* \end{cases}. \tag{239}$$

Thus we get:

$$\mathcal{Z} = \lim_{\epsilon\to 0} \int \left(\prod_{n=1}^{M} \frac{d\phi_n^* d\phi_n}{\mathcal{N}} \right) \exp\left\{ -\epsilon \sum_{n=1}^{M} \phi_n^* \frac{\phi_n - \phi_{n-1}}{\epsilon} \right\} \times$$
$$\times \exp\left\{ -\epsilon \sum_{n=1}^{M} [H(\phi_n^*, \phi_{n-1}) - \mu\phi_n^*\phi_{n-1}] \right\}. \tag{240}$$

Example: 1D bosonic/fermionic harmonic oscillator. We consider the Hamiltonian

$$H = \Omega a^\dagger a \tag{241}$$

so that $H(\phi_n^*, \phi_{n-1}) = \Omega\phi_n^*\phi_{n-1}$. The argument in the exponential of (240) is then given by:

$$\exp\left\{ -\sum_{i,j=1}^{M} \phi_i^* M_{ij} \phi_j \right\} \tag{242}$$

with the matrix $\mathbb{M} = [M_{ij}]$ of the form:

$$\mathbb{M} = \begin{vmatrix} 1 & 0 & 0 & ... & .... & -\zeta\Omega_0 \\ -\Omega_0 & 1 & 0 & ... & ... & 0 \\ 0 & -\Omega_0 & 1 & ... & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & -\Omega_0 & 1 \end{vmatrix}, \quad \Omega_0 \equiv 1 - \frac{\beta}{M}(\Omega - \mu) \tag{243}$$

The partition function is then given by an integral over (complex or grassmann) variables of gaussian type, which can be performed to get:

$$\mathcal{Z} = \lim_{\epsilon\to 0} (\mathtt{det}\,\mathbb{M})^{-\zeta} \tag{244}$$

where

$$\mathtt{det}\,\mathbb{M} = 1 + (-)^{M-1}(-\zeta\Omega_0)(-\Omega_0)^{M-1} = 1 - \zeta\Omega_0^M \tag{245}$$

hence

$$\mathcal{Z} = \lim_{\epsilon\to 0} \left[1 - \zeta\left(\frac{\beta(\Omega-\mu)}{M}\right)^M\right]^{-\zeta} = \left[1 - \zeta e^{-\beta(\Omega-\mu)}\right]^{-\zeta} \tag{246}$$

□

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