New examples of capillary surfaces in polyhedral regions

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Joint work with Rabah Souam

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Aim and references

The aim of this talk is to provide a large new family of **embedded capillary surfaces** inside polyhedral regions in the Euclidean space $\mathbb{R}^3$. We will also classify these examples.

- A. Alarcón and R. Souam, *The Minkowski problem, new constant curvature surfaces in* $\mathbb{R}^3$*, and some applications.* J. Reine Angew. Math., in press.
- A. Alarcón and R. Souam, *Capillary surfaces inside polyhedral regions.* Preprint 2014, arXiv:1401.6935.

Capillary surfaces

Consider a closed region $\mathcal{B}$ in $\mathbb{R}^3$.

Definition

A **capillary surface** in $\mathcal{B}$ is a compact H-surface with non-empty boundary, which is $\mathcal{C}^1$ up to the boundary and meeting the frontier $\partial\mathcal{B}$ of $\mathcal{B}$ at a **constant angle** $\theta \in [0, \pi]$ along its boundary.

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  $$\mathcal{E}(\Sigma) := \mathrm{Area}(\Sigma) - \cos\theta \, \mathrm{Area}(\mathrm{W}) \quad (\theta \in [0, \pi]).$$

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  The stationary surfaces of $\mathcal{E}$ for variations preserving the enclosed volume are precisely the capillary surfaces which make a constant angle θ with $\partial\mathcal{B}$.

- **Physical Interpretation:** Capillary surfaces model incompressible liquids inside a container in the absence of gravity; Σ represents the free surface of the liquid and W the region of the container wetted by the liquid.

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- When $\theta = \pi/2$, the surface is said to be a capillary surface with *free boundary* and there are many existence results for such surfaces. However no general existence results are available when $\theta \neq \pi/2$.

Theorem (AA, Souam 2014)

Let $\{p_1, \ldots, p_m\}$ *be a subset of* $\mathbb{S}^2$ *with cardinal number* $m \in \mathbb{N}$ *and let* $H > 0$ *be a positive real number. Let also* $(\theta_1, \ldots, \theta_m) \in (\frac{\pi}{2}, \pi)^m$ *be given numbers.*

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Denote by B_j the open disk in $\mathbb{S}^2$ of radius $\pi - \theta_j$ centered at $p_j, j = 1, \ldots, m$, and suppose the closed disks $\overline{B}_j$ are pairwise disjoint.
Consider real numbers $a_j > 0$, $j = 1, \ldots, m$, such that the balancing condition

$$\sum_{j=1}^{m} \left(a_j - \frac{\pi}{4H^2} \sin^2 \theta_j \right) p_j = \vec{0} \in \mathbb{R}^3 \tag{1}$$

is satisfied.

Capillary surfaces in polyhedral regions

Theorem (Continued)

Then there exists a polyhedral region $\mathcal{B}$ in $\mathbb{R}^3$ with frontier $\partial\mathcal{B}$ consisting of m planar regions $F_1, \ldots, F_m$, where F_j is orthogonal to p_j for all $j \in \{1, \ldots, m\}$, and there exists an **embedded capillary surface** *Σ in $\mathcal{B}$, with constant mean curvature H, satisfying:*

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- *$\gamma_j := \Sigma \cap F_j$ is a convex Jordan curve contained in the relative interior of F_j.*
- *Σ meets F_j at* **constant angle** *θ_j along γ_j for all $j \in \{1, \ldots, m\}$.*

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- *$\Sigma \setminus \partial\Sigma$ is positively curved and $\Sigma \cup (\cup_{j=1}^{m} D_j)$ is the* **boundary surface of a convex body** *in $\mathbb{R}^3$, where D_j is the convex connected component of $F_j \setminus \gamma_j$ for all $j \in \{1, \ldots, m\}$.*

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- $a_j = \mathrm{Area}(D_j) - \mathrm{Length}(\partial D_j)\frac{\sin\theta_j}{2H} + \pi\left(\frac{\sin\theta_j}{2H}\right)^2$ *for all* j*.*

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- *$a_j = \mathrm{Area}(D_j) - \mathrm{Length}(\partial D_j)\frac{\sin\theta_j}{2H} + \pi\left(\frac{\sin\theta_j}{2H}\right)^2$ for all j.*

Furthermore, the data $(\mathcal{B}, \Sigma)$ satisfying the previous conditions are **unique up to translations**.

Theorem (Continued)

Conversely let $\{p_1, \ldots, p_m\} \subset \mathbb{S}^2$*, let* $H > 0$*, let* $(\theta_1, \ldots, \theta_m) \in (\frac{\pi}{2}, \pi)^m$*, and let* $B_j \subset \mathbb{S}^2$*,* $j = 1, \ldots, m$*, be as above.*

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Assume there exists a polyhedral region $\mathcal{B}$ *in* $\mathbb{R}^3$ *with frontier* $\partial\mathcal{B}$ *consisting of* m *planar regions* $F_1, \ldots, F_m$*, where* F_j *is orthogonal to* p_j *for all* $j = 1, \ldots, m$*, and there exists a capillary surface* Σ *which is* $\mathcal{C}^1$ *up to the boundary, with constant mean curvature* H*, in* $\mathcal{B}$ *meeting the above description. Suppose in addition that* Σ *is not a part of an unduloid.*

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Then the numbers a_j*,* $j = 1, \ldots, m$*, defined above are all positive and satisfy the balancing condition* (1).

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- Why not unduloids?

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- The Gauss map of Σ is a **homeomorphism** onto $\mathbb{S}^2 \setminus \bigcup_{j=1}^{m} B_j$.
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- We allow the angle of contact to vary from one boundary component to the other.
  Physical interpretation: We allow the bounding faces of the polyhedral container to be composed of different (homogeneous) materials.
- If we take

$$a_j = \frac{\pi}{4H^2} \sin^2 \theta_j \quad \text{for all } j = 1, \ldots, m,$$

  then the balancing condition (1) trivially holds and the resulting surface is a piece of the round sphere $\mathbb{S}^2(1/H)$.

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  - The Gauss map of Σ_n is injective and converges to a homeomorphism onto $\mathbb{S}^2 \setminus \bigcup_{j=1}^m B_j$.
  - Σ_n is a piece of the boundary Ω_n of a strictly convex body.
  - $\Omega_n \setminus \Sigma_n = \bigcup_{j=1}^m O_{j,n}$, where $O_{j,n}$ is a disc converging to a planar disc D_j which is orthogonal to p_j and satisfies

$$a_j = \mathrm{Area}(D_j) - \mathrm{Length}(\partial D_j)\frac{\sin\theta_j}{2H} + \pi\left(\frac{\sin\theta_j}{2H}\right)^2, \quad j = 1, \ldots, m.$$

Existence. The Minkowski problem

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- Pogorelov 1952, Nirenberg 1953, Cheng-Yau 1976 Let $\kappa : \mathbb{S}^2 \to \mathbb{R}$ be a *smooth positive* function satisfying

$$\int_{\mathbb{S}^2} \frac{p}{\kappa(p)}\, dp = 0.$$

  Then there exists a unique, up to translations, smooth **embedding** $X : \mathbb{S}^2 \to \mathbb{R}^3$ such that
  - $X(\mathbb{S}^2)$ is a closed **strictly convex** surface.
  - κ is the curvature function of X.
  - The Gauss map of X is the identity map of $\mathbb{S}^2$.

Proof

For $q \in \mathbb{S}^2$, $r \in (0,1)$, and $s \in (0, 1-r)$, we denote by $B(q;r)$ the spherical cup of $\mathbb{S}^2$ given by

$$B(q;r) = \{p \in \mathbb{S}^2 \colon \sin \sphericalangle(p,q) < r,\ \cos \sphericalangle(p,q) > 0\},$$

where $\sphericalangle(p,q) \in [0,\pi]$ denotes the spherical angle between p and q.

We denote

$$A(q;r,s) = B(q;r+s) \setminus \overline{B}(q;s);$$

an open geodesic annulus.

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- Choose $(a_1, \ldots, a_m) \in (\mathbb{R}_+)^+$ satisfying (1) for $H = 1/2$, i.e.

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- For any large $n \in \mathbb{N}$, let $\kappa_n \colon \mathbb{S}^2 \to \mathbb{R}_+$ be a smooth function satisfying

$$\kappa_n\big|_{\mathbb{S}^2 \setminus \bigcup_{j=1}^m B(p_j; r_j + \frac{1}{n})} = 1,$$

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$$\int_{A(p_j; r_j, \frac{1}{n})} \frac{p}{\kappa_n(p)}\, dp = \pi \left(\frac{1}{n^2} + \frac{2 r_j}{n} \right) p_j, \quad \forall j = 1, \ldots, m.$$

Proof

- $\sum_{j=1}^{m}(a_j - \pi \sin^2 \theta_j) p_j = \vec{0}$.
- $\kappa_n = 1$ on $\Delta_n := \mathbb{S}^2 \setminus \bigcup_{j=1}^{m} B(p_j; r_j + \frac{1}{n})$.
- $\kappa_n = \frac{\pi}{a_j n^2}$ on $B(p_j; \frac{1}{n})$.
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- Then $\displaystyle\int_{\mathbb{S}^2} \frac{p}{\kappa_n(p)}\, dp = \sum_{j=1}^{m}(a_j - \pi r_j^2)p_j = \vec{0}$, and so the Minkowski problem can be solved for κ_n, giving a smooth embedding $X_n : \mathbb{S}^2 \to \mathbb{R}^3$ such that

1. $\mathscr{S}_n := X_n(\mathbb{S}^2)$ is a closed smooth strictly convex surface.
2. The Gauss map of X_n is the identity map of $\mathbb{S}^2$.
3. The curvature function of X_n agrees κ_n; in particular $X_n(\Delta_n)$ is a K-surface and $X_n(B(p_j; \frac{1}{n}))$ is *close* to flat.

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4. $\lim_{n\to\infty} \mathrm{Area}(X_n(B(p_j; \frac{1}{n}))) = a_j$ for all j.
5. $\lim_{n\to\infty} \mathrm{Area}(X_n(A(p_j; r_j, \frac{1}{n}))) = 0$ for all j.

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Claim

There exists $\xi > 0$ *(not depending on n) such that* $\mathbb{B}(\xi) \subset \mathscr{K}_n \subset \mathbb{B}(1/\xi)$ $\forall n$. *(The proof follows* Cheng-Yau 1976.*)*

6. Blaschke selection theorem $\Rightarrow \{\mathscr{K}_n\}_{n\in\mathbb{N}}$ converges in the Hausforff distance to a convex body $\mathscr{K}$.

Existence

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and $\mu(H) > 0$ for any hemisphere $H \subset \mathbb{S}^2$.
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- This gives the uniqueness and converse parts of the theorem.

New examples of capillary surfaces in polyhedral regions

Antonio Alarcón

Universidad de Granada

Joint work with Rabah Souam

Granada, September 2014