On the existence of geodesics on globally hyperbolic spacetimes with a lightlike Killing vector field

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Geodesic connectedness

Geodesic connectedness problem:
a semi-Riemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_M)$ is geodesically connected if for any $p, q \in \mathcal{M}$ there exists a geodesic joining them

If $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ is a **Lorentzian manifold**:

* $\langle\cdot,\cdot\rangle_L$ *induces a scalar product of index* 1 *on* T_zL *for all* $z \in L$

- **physical interest:** 4–dimensional spacetimes are models of General Relativity
  - *a time–oriented connected Lorentzian manifold is a spacetime*
- **geometrical interest:** Hopf-Rinow theorem does not hold
- **analytical interest:** if $p, q \in \mathcal{L}$, γ is a geodesic joining them iff γ is a **critical point** of

$$f(\gamma) = \frac{1}{2}\int_0^1 \langle\dot\gamma, \dot\gamma\rangle_L \, \mathrm{d}s$$

  in the set of C^1 (or H^1) curves γ on $I = [0, 1]$ such that $\gamma(0) = p$, $\gamma(1) = q$

Overview (partial answers)

Bernal-Sánchez:
$\mathcal{L}$ is a **globally hyperbolic spacetime** if there exists a (smooth) spacelike **Cauchy hypersurface** S

- $S \subset \mathcal{L}$ *which is intersected* **exactly once** *by any inextendible timelike curve*
  $\zeta \in T_z\mathcal{L} \setminus \{0\}$ *is timelike* $\langle \zeta, \zeta \rangle_L < 0$, *lightlike* $\langle \zeta, \zeta \rangle_L = 0$, *spacelike* $\langle \zeta, \zeta \rangle_L > 0$, *causal* $\langle \zeta, \zeta \rangle_L \leq 0$

- **Avez-Seifert theorem**: on a globally hyperbolic spacetime, any couple of causally related points can be joined by a causal geodesic

- Globally hyperbolic spacetimes may be not geodesically connected

Overview

- Benci, Fortunato, Giannoni, Masiello, Piccione: **variational methods** and **critical point theory** apply to globally hyperbolic spacetimes, deal with geodesics of any causal character, can give multiplicity results

  - direct approach: by means of the theory for strongly indefinite functionals

  - **reduction**: $\langle \cdot, \cdot \rangle_L$ presents symmetries (**Killing vector fields**), for example **stationary spacetimes**

$\mathcal{L}$ is **stationary** if it admits a **timelike** Killing vector field

typical physical spacetimes, as Kerr's or Schwarzschild's

∗ *a vector field is Killing if the Lie derivative of* $\langle \cdot, \cdot \rangle_L$ *in the direction of* K *is* 0

∗ **Interplay between variational methods and causality theory**

Some preliminaries

The splitting theorem
$(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ *globally hyperbolic*, K *complete causal Killing vector field* on $\mathcal{L}$:
$\mathcal{L} = S \times \mathbb{R}$ and

$$\langle\zeta,\zeta'\rangle_L = \langle\xi,\xi'\rangle + \langle\delta(x),\xi\rangle\tau' + \langle\delta(x),\xi'\rangle\tau - \beta(x)\tau\tau'$$

for all $z = (x,t) \in \mathcal{L}$, $\zeta = (\xi,\tau)$, $\zeta' = (\xi',\tau') \in T_z\mathcal{L} = T_xS \times \mathbb{R}$, $(S, \langle\cdot,\cdot\rangle)$ Riemannian manifold, δ vector field on S, β non–negative function on S

- K **timelike** $\Rightarrow$ $\beta(x) > 0$ on S
- K **lightlike** $\Rightarrow$ $\beta \equiv 0$, δ non–vanishing

if $\beta(x) > 0$ on S, $\mathcal{L}$ is **standard stationary**
if $\delta \equiv 0$ $\mathcal{L}$ is (standard) **static**

Stationary extrinsic results

Connectedness if (S **is complete** and) **the metric coefficients satisfy**

- Giannoni-Masiello, Pisani:

$$\beta(x), |\delta(x)| \leq \lambda d^{\alpha}(x, \bar{x}) + k, \alpha \in [0, 1[$$

  $d(\cdot, \cdot)$ distance on S induced by $\langle \cdot, \cdot \rangle$, $\bar{x} \in S$, $\lambda, k \in \mathbb{R}$

- B-Candela-Flores-Sánchez:

$$\delta \equiv 0 \quad \text{and} \quad \alpha = 2$$

- B-Candela-Flores: $\beta(x), |\delta(x)|^2 \leq \lambda d^2(x, \bar{x}) + k$

Some remarks

If $\mathcal{L}$ is globally hyperbolic with a complete causal Killing vector field K, the **splitting** $\mathcal{L} = S \times \mathbb{R}$ is *neither unique nor canonically associated to* $\mathcal{L}$
Results should be independent of the chosen S, K

The bounds on β, δ have *no geometrical meaning*, except as optimal, sufficient (not necessary) conditions for global hyperbolicity (Sánchez)

Question: Does global hyperbolicity implies geodesic connectedness for stationary spacetimes?

Stationary intrinsic results

- $\mathcal{L}$ **stationary spacetime**: geodesics connecting $p, q \in \mathcal{L}$ are the **critical points** of f on $\Omega_K(p,q)$ defined by

$$\{z \in \Omega(p,q) :\ \exists\, C_z \in \mathbf{R} \text{ s.t. } \langle \dot{z}, K(z)\rangle_L \equiv C_z \text{ a.e. on } I\}$$

with $\Omega(p,q) = \{\gamma \in H^1(I,\mathcal{L}) : \gamma(0) = p, \gamma(1) = q\}$

- given $c \in \mathbb{R}$, the set $\Omega_K(p,q)$ is **c–precompact** for f if every sequence $(z_m)_m$ in $\Omega_K(p,q)$ such that $f(z_m) \leq c$ has a subsequence which converges weakly in $\Omega_K(p,q)$ (hence, uniformly in $\mathcal{L}$)

Theorem [Giannoni-Piccione] If $\Omega_K(p,q) \neq \emptyset$ and there exists $c > \inf f(\Omega_K(p,q))$ such that $\Omega_K(p,q)$ is c–precompact, then there exists at least one geodesic in $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ joining p to q

Candela-Flores-Sánchez result

- the restriction of f to $\Omega_K(p,q)$ is **pseudo–coercive** if $\Omega_K(p,q)$ is c–precompact for all $c \geq \inf f(\Omega_K(p,q))$
- pseudo–coercivity implies global hyperbolicity, but it is quite difficult to verify

Theorem [Candela-Flores-Sánchez] Let $\mathcal{L}$ be a **stationary** spacetime with a **complete timelike Killing** vector field. If $\mathcal{L}$ is **globally hyperbolic** with a **complete Cauchy** hypersurface, then it is geodesically connected

- K complete $\Rightarrow \Omega_K(p,q) \neq \emptyset$ for any p, $q \in \mathcal{L}$
- the **geometric assumptions** $\Rightarrow$ pseudo–coercivity

The limit case

Question:
If $\mathcal{L}$ is globally hyperbolic with a complete **lightlike Killing vector field** and a complete (smooth, spacelike) Cauchy hypersurface, is $\mathcal{L}$ geodesically connected?

Unluckily, the answer is negative

The existence of a complete lightlike Killing vector field and a complete Cauchy hypersurface *do not imply* geodesic conectedness: $\mathcal{L} = \mathbf{R}^3 \times \mathbf{R}$ equipped with a suitable Lorentzian metric

Motivation: the class of **generalized plane waves** (Candela-Flores-Sánchez)

Generalized plane waves

$(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ is a **generalized plane wave** if $\mathcal{L} = \mathcal{M} \times \mathbf{R}^2$ with $(\mathcal{M}, \langle\cdot,\cdot\rangle)$ Riemannian manifold and

$$\langle\cdot,\cdot\rangle_L \;=\; \langle\cdot,\cdot\rangle + 2dudv + \mathcal{H}(x,u)du^2,$$

$x \in \mathcal{M}$, $(u,v) \in \mathbf{R}^2$, $\mathcal{H} : \mathcal{M} \times \mathbb{R} \to \mathbb{R}$ smooth with $\mathcal{H} \not\equiv 0$

- a GPW becomes a **gravitational wave**: $\mathcal{M} = \mathbb{R}^2$ with the standard metric, $\mathcal{H}(x,u) \;=\; g_1(u)(x_1^2 - x_2^2) + 2g_2(u)x_1x_2$, $x = (x_1, x_2) \in \mathbb{R}^2$, for some g_1 and g_2 s.t. $g_1^2 + g_2^2 \not\equiv 0$
- the *extrinsic hypotheses* which imply global hyperbolicity give also geodesic connectedeness:
  $\mathcal{M}$ complete, $-\mathcal{H}$ spatially subquadratic $\Rightarrow$
  $\mathcal{L}$ geodesically connected (Candela-Flores-Sánchez) and globally hyperbolic (Flores-Sánchez)

Our result

Theorem [B-Candela-Flores] Let $(\mathcal{L}, \langle\cdot,\cdot\rangle_L)$ be globally hyperbolic with a complete **lightlike Killing vector field** K and a **complete** (smooth, spacelike) **Cauchy hypersurface** S.
Given $p, q \in \mathcal{L}$, the following statements are **equivalent**:

(i) p and q are geodesically connected in $\mathcal{L}$

(ii) p and q can be connected by a C^1 curve φ on $\mathcal{L}$ such that $\langle\dot{\varphi}, K(\varphi)\rangle_L$ has constant sign or is identically equal to 0

The limit argument

1. **perturbation of $\langle\cdot,\cdot\rangle_L$** by $\langle\cdot,\cdot\rangle_n$ *standard* stationary metrics

2. by a suitable use of the **Candela-Flores-Sánchez result**:
   $\langle\cdot,\cdot\rangle_n$ is geodesically connected for $n \geq n_0$

3. by (ii): **estimates** on the sequence of connecting geodesics

4. existence of a **limit connecting geodesic** for $\langle\cdot,\cdot\rangle_L$

Perturbation of $\langle\cdot,\cdot\rangle_L$

- for all $n \in \mathbf{N}$ **standard stationary spacetimes** $(\mathcal{L}_n, \langle\cdot,\cdot\rangle_n)$, where $\mathcal{L}_n = \mathcal{L}$ and

$$\langle\zeta,\zeta'\rangle_n \;=\; \langle\zeta,\zeta'\rangle_L - \frac{1}{n}\tau\tau'$$

  with $\langle\cdot,\cdot\rangle_L$ as in the **splitting theorem** ($\mathcal{L}$ globally hyperbolic)

- on standard stationary spacetimes:
  for $p = (x_p, t_p), q = (x_q, t_q) \in S \times \mathbf{R}$, **action functional** f defined on

$$\Omega(p,q) \;=\; \Omega(x_p, x_q; S) \times W(t_p, t_q)$$

  with

$$\Omega(x_p, x_q; S) \;=\; \left\{x \in H^1(I,S) :\; x(0) = x_p,\; x(1) = x_q\right\}$$

$$W(t_p, t_q) \;=\; \left\{t \in H^1(I,\mathbf{R}) :\; t(0) = t_p,\; t(1) = t_q\right\}$$

Connection of auxiliary standard spacetimes

The reduction argument:

- if $z = (x,t) \in \Omega_K(p,q)$ $(K = \partial_t)$

$$\dot{t} = \frac{\langle \delta(x), \dot{x} \rangle - C_z}{\beta(x)} \qquad \text{a.e. on } I$$

- **the restriction** of f to $\Omega_K(p,q)$ depends only on $\Delta_t = t_q - t_p$ and the component x of $z = (x,t) \in \Omega_K(p,q)$

Given $p = (x_p, t_p)$, $q = (x_q, t_q) \in (\mathcal{L} = S \times \mathbb{R}, \langle \cdot, \cdot \rangle_L)$ with $\Delta_t = t_q - t_p \geq 0$, p and q are **connected by a geodesic** $\gamma_n = (x_n, t_n)$ in $(\mathcal{L}_n, \langle \cdot, \cdot \rangle_n)$ $\forall n \geq n_0$

The key-point

$\mathcal{L}$ globally hyperbolic $\Rightarrow$
any past inextendible causal curve departing from $q = (x_q, t_q)$, $t_q \geq t_p$, must intersect $S \times \{t_p\}$

then:

- $J_n^-(q) \cap (S \times \{t_p\})$ is compact in $\mathcal{L}_n$ for all $n \geq n_0$
  - $J^-(q) \ = \ \{p \in \mathcal{L} : \ p \leq q\}$ *is the* **causal past of** q *in* $\mathcal{L}$
    a causal curve γ *is* future *or* past directed *depending on the time orientation of the cone determined by* $\dot{\gamma}$ *at each point;* p *is in the causal past of* q *(*$p < q$*) if there exists a future–directed causal curve from* p *to* q*;* $p \leq q$ *denotes either* $p < q$ *or* $p = q$
- on standard stationary spacetimes this implies that $\Omega_K(p, q)$ is c-precompact for some $c > \inf f(\Omega_K(p, q))$

A priori estimates

- by **Nash embedding theorem** (O. Müller) $\Rightarrow \Omega(x_p, x_q; S)$ is a **complete submanifold** of $H^1(I, \mathbb{R}^N)$
- as usual

$$\|y\|^2 = \|y\|_2^2 + \|\dot{y}\|_2^2 \quad \text{for all } y \in H^1(I, \mathbb{R}^N)$$

where $\|\cdot\|_2$ denotes the standard L^2–norm: $\|y\|_2^2 = \int_0^1 y^2 \, \mathrm{d}s$

$(\gamma_n = (x_n, t_n))_{n \geq n_0}$ sequence of curves connecting p to q, each γ_n geodesic in $\mathcal{L}_n$

- by condition *(ii)*: the sequence $(\|\dot{x}_n\|_2)_{n \geq n_0}$ **is bounded**
- by the geodesic equations satisfied by $\gamma_n = (x_n, t_n)$ on each $\mathcal{L}_n$: $\left(\|\dot{t}_n\|_2\right)_{n \geq n_0}$ **is bounded**

Limit geodesic

- there exists a subsequence $(\gamma_n)_n$ and γ

$$\gamma = (x, t) \in \Omega(p, q) = \Omega(x_p, x_q; S) \times W(t_p, t_q)$$

  such that $x_n \to x$ **strongly** in $\Omega(x_p, x_q; S)$ and $t_n \to t$ **strongly** in $W(t_p, t_q)$

- $\gamma_n = (x_n, t_n)$ geodesic of $(\mathcal{L}_n, \langle\cdot,\cdot\rangle_n) \Rightarrow$ $\gamma = (x, t)$ **must satisfy the geodesic equations for $\mathcal{L}$**
  - * a smooth curve $\gamma = (x, t)$ is a geodesic of $\mathcal{L}_n$ if and only if

$$\begin{cases} D_s\dot{x} - \dot{t}\,F(x)[\dot{x}] + \ddot{t}\,\delta(x) = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}s}\left(\dfrac{1}{n}\dot{t} - \langle\delta(x), \dot{x}\rangle\right) = 0 \end{cases}$$

    where $F(x)$ denotes the linear operator on T_xS associated to

$$\operatorname{curl}\delta(x)[\xi, \xi'] = \langle(\delta'(x))^T[\xi], \xi'\rangle - \langle\delta'(x)[\xi'], \xi\rangle$$

    for all $\xi, \xi' \in T_xS$

- **Counterexample** if the **lightlike Killing vector field** is **not complete**:
  removing from the Minkowski 2–space $\mathbb{L}^2$ the region $\{(x,t) : x \geq 0,\, t \geq 0\}$

- **Counterexample** if the **Cauchy hypersurface** is **not complete**:
  $\mathcal{L} = S \times \mathbb{R}$, $S = \mathbf{R}^2 \setminus \{(x_1, 0) : -1 \leq x_1 \leq 1\}$ equipped with a suitable Lorentzian metric

Some applications

- **Avez–Seifert result**:
  $\mathcal{L}$ as in the main theorem. Then, two points of $\mathcal{L}$ can be connected by a causal geodesic if and only if they are causally related

- **Geodesic connectedness for generalized plane waves**:

  Theorem: Any **globally hyperbolic GPW** with a **complete Cauchy hypersurface** is geodesically connected

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