# Polar Actions on Symmetric Spaces

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## Real hyperbolic plane (I)

- $SL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1,\ a, b, c, d \in \mathbb{R} \right\}$
- $H = \{z \in \mathbb{C} \mid \Im(z) > 0\} \subset \mathbb{C}$ upper half plane
- $SL_2(\mathbb{R})$ acts on $H$ by

$$z \mapsto \frac{az + b}{cz + d}$$

- isometric action with respect to $ds^2 = \frac{dzd\bar{z}}{\Im(z)^2} = \frac{dx^2+dy^2}{y^2}$
- $(H, ds^2) =$ upper half plane model for real hyperbolic plane
- $H = G/K$ homogeneous space with

$$G = SL_2(\mathbb{R})\ ,\ \ K = \left\{ \begin{pmatrix} \cos(s) & \sin(s) \\ -\sin(s) & \cos(s) \end{pmatrix} \middle| \, s \in \mathbb{R} \right\} = SO_2$$

## Real hyperbolic plane (II)

- There is a unique decomposition

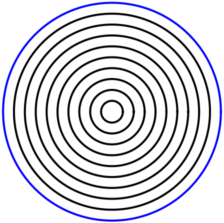
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} \exp(s) & 0 \\ 0 & \exp(-s) \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$$

- Iwasawa decomposition $SL_2(\mathbb{R}) \stackrel{\text{diff}}{=} KAN$
- $K \cong SO_2$ compact, $A \cong \mathbb{R}$ abelian, $N \cong \mathbb{R}$ nilpotent
- $H = G/K \stackrel{\text{iso}}{=} AN$ solvable Lie group with left-invariant Riemannian metric

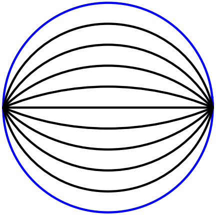
## Real hyperbolic plane (III)

Orbit structures (Poincaré model)

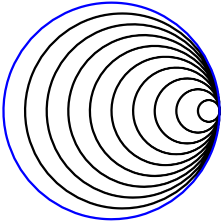
$K$-orbits



$A$-orbits



$N$-orbits



# Polar representations

- $H$ compact connected Lie group acting on
  $V$ real vector space with $H$-invariant inner product
- $\pi : H \to O(V)$ representation
- $v + (T_v(H \cdot v))^\perp = \Sigma_v \subset V$ cross-section of action at $v \in V$
- $\Sigma_v$ minimal $\iff$ $\dim H \cdot v$ maximal

Definition. $\pi : H \to O(V)$ **polar** if all orbits intersect a minimal cross-section orthogonally

Examples:

- standard representation $\pi : SO_2 \to O(\mathbb{R}^2)$ is polar
- $M = G/K$ Riemannian symmetric space, $o \in M$ with
  $K \cdot o = o$, isotropy representation $\pi : K \to O(T_o M)$ is polar

**THEOREM.** (**Dadok:** Trans. Amer. Math. Soc. 288 (1985), 125–137) *A representation on $\mathbb{R}^n$ is polar if and only if it is orbit equivalent to the isotropy representation of a Riemannian symmetric space $M^n = G/K$*

## Polar actions

$M$ connected Riemannian manifold, $H \subset I(M)$ connected closed subgroup

Definition. The action of $H$ on $M$ is **polar** if there exists a connected closed submanifold $\Sigma$ of $M$ such that

- $\forall p \in M : \Sigma \cap H \cdot p \neq \emptyset$
- $\forall p \in \Sigma : T_p\Sigma \subset (T_p(H \cdot p))^{\perp}$

Such a submanifold $\Sigma$ is called a **section** of the action

Fact. Sections are totally geodesic submanifolds

Definitions. A polar action is

- **hyperpolar** if $\Sigma$ is flat
- **of cohomogeneity one** if $\dim(\Sigma) = 1$

Problem. Classification of polar actions on Riemannian symmetric spaces of noncompact type

## Current state of affairs

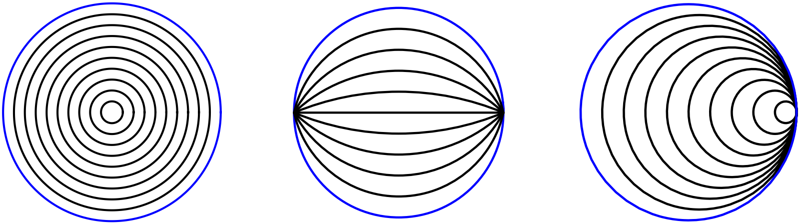
| | $\nexists$ singular orbit | $\exists$ singular orbit |
|---|---|---|
| cohom 1 | explicit classification | general construction; explicit classification for some $M$ with rank 1 or 2 |
| hyperpolar | explicit classification | ? |
| polar | examples | ? |

Joint work with

- José Carlos Díaz-Ramos (Santiago de Compostela)
- Miguel Domínguez-Vázquez (Rio de Janeiro)
- Hiroshi Tamaru (Hiroshima)

## Real hyperbolic plane

Every polar action on $\mathbb{R}H^2$ is orbit equivalent to one of these:



## Complex hyperbolic plane

- $N$ horosphere in $\mathbb{C}H^2$ (3-dim Heisenberg group)
- $S$ ruled real hypersurface in $\mathbb{C}H^2$ generated by a horocycle in $\mathbb{R}H^2 \subset \mathbb{C}H^2$
- $N \cap S = \mathbb{E}^2$ is a Euclidean plane embedded in $N$ as a minimal surface and in $\mathbb{C}H^2$ as a real surface with nonzero constant mean curvature

**THEOREM.** (**Berndt, Díaz-Ramos:** Ann. Global Anal. Geom. 43 (2013), 99–106): *Every polar action on $\mathbb{C}H^2$ is orbit equivalent to the action of the invariance group of one of the following geometric objects in $\mathbb{C}H^2$:*

- *Hyperpolar:* $pt$, $\mathbb{C}H^1$, $\mathbb{R}H^2$, $N$, $S$
- *Polar (and not hyperpolar):* $pt \subset \mathbb{C}H^1$ *(full flag)*, $\mathbb{R}H^1$, *horocycle in* $\mathbb{C}H^1$, $\mathbb{E}^2$

## The setup

- $M = G/K$ Riemannian symmetric space of noncompact type
  $G = I^o(M)$ noncompact semisimple real Lie group
  $K$ maximal compact subgroup of $G$
  $o \in M$ with $K \cdot o = o$
- $H$ connected closed subgroup of $G$ acting polarly on $M$

Problem: Classify $H$ up to orbit equivalence.

Start with case of cohomogeneity one

**FACT:**

1. The orbits of $H$ form a Riemannian foliation of $M$, or
2. There is exactly one singular orbit $W(= H \cdot o)$.

## The "conceptual" classification for cohomogeneity one

**THEOREM.** (**Berndt, Tamaru:** J. Reine Angew. Math. 683 (2013), 129–159):
*Let $M$ be a connected irreducible Riemannian symmetric space of noncompact type and assume that $H$ acts on $M$ with cohomogeneity one. Then one of the following statements holds:*

1. *The orbits of $H$ form a Riemannian foliation of $M$;*
2. *There is a totally geodesic singular orbit;*
3. *The action of $H$ is orbit equivalent to the canonical extension of a cohomogeneity one action on a boundary component of $M$;*
4. *The action of $H$ is orbit equivalent to one which is obtained by the nilpotent construction method.*

## The case of foliations (I)

- $G = KAN$ Iwasawa decomposition
- $M = G/K = AN$ solvable Lie group with left-invariant Riemannian metric
- Observation: Each codimension one subgroup of $AN$ induces a cohomogeneity one action on $M$ whose orbits form a Riemannian foliation on $M$
- There are two types of foliations arising in this way

## The case of foliations (II)

Type I:

- $\mathfrak{a} \oplus \mathfrak{n}$ Lie algebra of $AN$, $\ell \subset \mathfrak{a}$ one-dimensional subspace
- $\mathfrak{s}_\ell = (\mathfrak{a} \ominus \ell) \oplus \mathfrak{n}$ codimension one subalgebra
- $S_\ell$ acts on $M$ with cohomogeneity one, induces foliation $\mathcal{F}_\ell$
- All orbits of $\mathcal{F}_\ell$ are isometrically congruent to each other
- Special case: Horosphere foliations

Type II:

- $\mathfrak{n} = \bigoplus_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$
- $\alpha_1, \ldots, \alpha_r$ simple roots, $\ell \subset \mathfrak{g}_{\alpha_i}$ one-dimensional subspace
- $\mathfrak{s}_i = \mathfrak{a} \oplus (\mathfrak{n} \ominus \ell)$ codimension one subalgebra
- $S_i$ acts on $M$ with cohomogeneity one, induces foliation $\mathcal{F}_i$
- $\mathcal{F}_i$ has exactly one minimal orbit

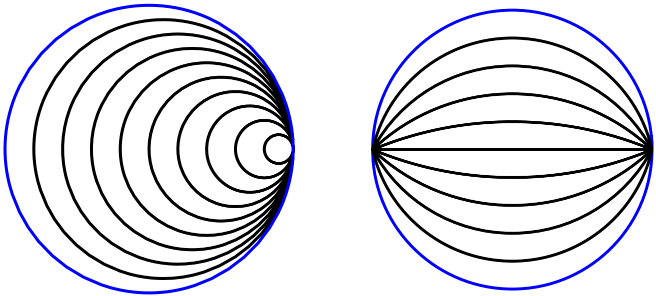
## The case of foliations (III)

- $M$ connected irreducible Riemannian symmetric space of noncompact type
- $\mathcal{M}_F$ = set of all homogeneous codimension one foliations on $M$ up to isometric congruence
- $r$ = rank of $M$
- $\mathrm{Aut}(\Delta) \in \{I, \mathbb{Z}_2, \mathfrak{S}_3\}$ automorphism group of the Dynkin diagram $\Delta$ associated to $M$

**THEOREM.** (**Berndt, Tamaru:** J. Differential Geom. 63 (2003), 1–40) *Every homogeneous codimension one foliation $\mathcal{F}$ of $M$ is congruent to one of the foliations $\mathcal{F}_\ell$ or $\mathcal{F}_i$ and*

$$\boxed{\mathcal{M}_F \cong (\mathbb{R}P^{r-1} \cup \{1, \ldots, r\})/\mathrm{Aut}(\Delta)}$$

## The two foliations on hyperbolic spaces ($r = 1$)



- horosphere foliation
- foliation with exactly one minimal leaf $S$
  - $\underline{M = \mathbb{R}H^n}$: $S = \mathbb{R}H^{n-1}$ totally geodesic
  - $\underline{M = \mathbb{C}H^n}$: $S =$ ruled real hypersurface associated to a horocycle in a totally geodesic $\mathbb{R}H^2 \subset \mathbb{C}H^n$

## Duality and triality

**$\mathcal{M}_F$ depends only on the rank and on possible duality or triality principles on the symmetric space**

Example: $r = 8$, $\mathrm{Aut}(\Delta) = I$

$$\boxed{\mathcal{M}_F = \mathbb{R}P^7 \cup \{1, \ldots, 8\}}$$

for the symmetric spaces

$$SO_{17}(\mathbb{C})/SO_{17}\ ,\ Sp_8(\mathbb{R})/U_8\ ,\ Sp_8(\mathbb{C})/Sp_8\ ,\ SO^*_{32}/U_{16}\ ,\ SO^*_{34}/U_{17}$$

$$E_8^8/SO_{16}\ ,\ E_8(\mathbb{C})/E_8$$

$$SO^o_{8,8+k}/SO_8 SO_{8+k}\ (k \geq 1)$$

$$SU_{8,8+k}/S(U_8 U_{8+k})\ (k \geq 0)\ ,\ Sp_{8,8+k}/Sp_8 Sp_{8+k}\ (k \geq 0)$$

## The singular case - basic observations

$L$ connected proper maximal subgroup of $G$ with $H \subset L$

**THEOREM.** (**Mostow:** Ann. of Math. (2) 74 (1961), 503–517)

$$\mathfrak{l} \textit{ is reductive or parabolic}$$

- $\mathfrak{l}$ reductive $\Longrightarrow L \cdot o \subsetneq M \Longrightarrow H, L$ orbit equivalent
- $\mathfrak{l}$ parabolic $\Longrightarrow L \cdot o = M$

# The reductive case (I)

**THEOREM.** (**Karpelevich:** Dokl. Akad Nauk SSSR (N.S.) 93 (1953), 401–404) *$L$ has a totally geodesic orbit $W \subsetneq M$*

Thus: $H$ has a totally geodesic orbit $W$

$W$ **reflective**

$\Updownarrow$

geodesic reflection of $M$ in $W$ is an isometry

$\Updownarrow$

$\exists$ totally geodesic submanifold $W^\perp$ of $M$
with $o \in W^\perp$ and $T_o W^\perp = \nu_o W$

**Leung** (J. Differential Geom. 14 (1979), 167–177):
Classification of reflective submanifolds in irreducible simply connected Riemannian symmetric spaces

## The reductive case (II)

**THEOREM.** (**Berndt, Tamaru:** Tohoku Math. J. (2) 56 (2004), 163–177) *$W$ is a totally geodesic singular orbit of a cohomogeneity one action on $M$ $\Longleftrightarrow$*

- *$W$ reflective and $\operatorname{rank} W^{\perp} = 1$ (**Leung 1979**), or*
- *$W$ is one of the following totally geodesic non-reflective submanifolds:*

| $W$ | $M$ | $\dim W$ | $\dim M$ |
|---|---|---|---|
| $\mathbb{C}H^2$ | $G_2^2/SO_4$ | 4 | 8 |
| $SL_3(\mathbb{R})/SO_3$ | $G_2^2/SO_4$ | 5 | 8 |
| $G_2^2/SO_4$ | $SO^o_{3,4}/SO_3SO_4$ | 8 | 12 |
| $SL_3(\mathbb{C})/SU_3$ | $G_2(\mathbb{C})/G_2$ | 8 | 14 |
| $G_2(\mathbb{C})/G_2$ | $SO_7(\mathbb{C})/SO_7$ | 14 | 21 |

## The reductive case for hyperbolic spaces

*$W$ is a totally geodesic singular orbit of a cohomogeneity one action on $\mathbb{F}H^n$ ($n \geq 2$)* $\Longleftrightarrow$

| $\mathbb{F}$ | $W$ | *Comments* |
|---|---|---|
| $\mathbb{R}$ | $\mathbb{R}H^k$ | $k \in \{0, \ldots, n-2\}$ |
| $\mathbb{C}$ | $\mathbb{C}H^k$ | $k \in \{0, \ldots, n-1\}$ |
| | $\mathbb{R}H^n$ | |
| $\mathbb{H}$ | $\mathbb{H}H^k$ | $k \in \{0, \ldots, n-1\}$ |
| | $\mathbb{C}H^n$ | |
| $\mathbb{O}$ | $\mathbb{O}H^k$ | $n = 2,\ k \in \{0, 1\}$ |
| | $\mathbb{H}H^2$ | |

## Parabolic subalgebras

**restricted root space decomposition:** $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$

$\Lambda = \{\alpha_1, \ldots, \alpha_r\}$ set of simple roots of $\Psi$, $\Psi = \Psi^+ \cup \Psi^-$

$\Phi \subset \Lambda$, $\Psi_\Phi = \Psi \cap \mathrm{span}\{\Phi\}$

$$\mathfrak{q}_\Phi = \underbrace{\mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi}_{\text{Chevalley}} = \underbrace{\mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi}_{\text{Langlands}} \quad \text{parabolic subalgebra}$$

$$\mathfrak{l}_\Phi = \mathfrak{g}_0 \oplus \left( \bigoplus_{\alpha\in\Psi_\Phi} \mathfrak{g}_\alpha \right) \quad \text{reductive subalgebra}$$

$$\mathfrak{n}_\Phi = \bigoplus_{\alpha\in\Psi^+\setminus\Psi^+_\Phi} \mathfrak{g}_\alpha \quad \text{nilpotent subalgebra}$$

$$\mathfrak{a}_\Phi = \bigcap_{\alpha\in\Phi} \mathrm{ker}(\alpha) \quad \text{abelian subalgebra}$$

$$\mathfrak{m}_\Phi = \mathfrak{l}_\Phi \ominus \mathfrak{a}_\Phi \quad \text{reductive subalgebra}$$

## Horospherical decomposition

$$\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi = \mathfrak{l}_\Phi \oplus \mathfrak{n}_\Phi \longleftrightarrow Q_\Phi = M_\Phi \times A_\Phi \times N_\Phi = L_\Phi \times N_\Phi$$

$$\underbrace{M \overset{\text{diff}}{=} B_\Phi \times \mathbb{E}^{r-|\Phi|} \times N_\Phi}_{\text{horospherical decomposition}} = F_\Phi \times N_\Phi$$

$$\begin{aligned}
B_\Phi &= M_\Phi \cdot o \text{ semisimple symmetric space with } \mathsf{rk}(B_\Phi) = |\Phi|, \\
&\quad \text{totally geodesic in } M, \textbf{ boundary component} \text{ of } M \\
\mathbb{E}^{r-|\Phi|} &= A_\Phi = A_\Phi \cdot o \text{ Euclidean space, totally geodesic in } M \\
F_\Phi &= L_\Phi \cdot o = B_\Phi \times \mathbb{E}^{r-|\Phi|} \text{ totally geodesic in } M
\end{aligned}$$

- $N_\Phi$ acts polarly on $M$ with section $F_\Phi$
- $N_\Phi$ acts hyperpolarly on $M \Longleftrightarrow \Phi = \emptyset$

## Canonical extension

Basic example:
Extension of $SO_2$-action on $\mathbb{R}^2$ to $(SO_2 \times \mathbb{R})$-action on $\mathbb{R}^3$

- $H_\Phi \subset I^o(B_\Phi) \subset M_\Phi$ acting on $B_\Phi$ with cohomogeneity one
- $\mathfrak{h} = \mathfrak{h}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ subalgebra of $\mathfrak{q}_\Phi$

$H$ acts on $M$ with cohomogeneity one

**Rank reduction** - Such a cohomogeneity one action can be constructed by a CANONICAL EXTENSION OF A COHOMOGENEITY ONE ACTION ON A BOUNDARY COMPONENT

Canonical extension also works for hyperpolar and polar actions

## Nilpotent construction

- $\Lambda = \{\alpha_1, \ldots, \alpha_r\}$, $\{H^1, \ldots, H^r\}$ dual basis of $\Lambda$ in $\mathfrak{a}$
- $\Phi_j = \Lambda \setminus \{\alpha_j\}$: Put $\mathfrak{q}_j = \mathfrak{q}_{\Phi_j}$, $\mathfrak{n}_j = \mathfrak{n}_{\Phi_j}$, etcetera
- $\mathfrak{n}_j^\nu = \bigoplus_{\alpha \in \Sigma^+ \setminus \Sigma_j^+, \alpha(H^j) = \nu} \mathfrak{g}_\alpha$
- $\mathfrak{n}_j = \bigoplus_{\nu > 0} \mathfrak{n}_j^\nu$ gradation generated by $\mathfrak{n}_j^1$

Assume that

- $\mathfrak{v} \subset \mathfrak{n}_j^1$; define $\mathfrak{n}_{j,\mathfrak{v}} = \mathfrak{n}_j \ominus \mathfrak{v}$ subalgebra of $\mathfrak{n}_j$
- $N^o_{L_j}(\mathfrak{n}_{j,\mathfrak{v}}) = \theta N^o_{L_j}(\mathfrak{v})$ acts transitively on $F_j = B_j \times \mathbb{E}$
- $N^o_{L_j \cap K}(\mathfrak{v})$ acts transitively on the unit sphere in $\mathfrak{v}$ if $\dim \mathfrak{v} \geq 2$

Then

$$H_{j,\mathfrak{v}} = N^o_{L_j}(\mathfrak{n}_{j,\mathfrak{v}}) N_{j,\mathfrak{v}} \text{ acts on } M \text{ with cohomogeneity one}$$

## Nilpotent construction for hyperbolic spaces

| $M$ | $G$ | $K$ | $K_0$ | $\mathfrak{g}_\alpha$ | $\mathfrak{n}$ |
|---|---|---|---|---|---|
| $\mathbb{R}H^n$ | $SO^o_{1,n}$ | $SO_n$ | $SO_{n-1}$ | $\mathbb{R}^{n-1}$ | $\mathbb{R}^{n-1}$ |
| $\mathbb{C}H^n$ | $SU_{1,n}$ | $U_n$ | $U_{n-1}$ | $\mathbb{C}^{n-1}$ | $\mathbb{C}^{n-1} \oplus \mathbb{R}$ |
| $\mathbb{H}H^n$ | $Sp_{1,n}$ | $Sp_1 Sp_n$ | $Sp_1 Sp_{n-1}$ | $\mathbb{H}^{n-1}$ | $\mathbb{H}^{n-1} \oplus \mathbb{R}^3$ |
| $\mathbb{O}H^2$ | $F_4^{-20}$ | $Spin_9$ | $Spin_7$ | $\mathbb{O}$ | $\mathbb{O} \oplus \mathbb{R}^7$ |

- $\Lambda = \{\alpha\}$, $\Phi = \emptyset$, $\mathfrak{l}_\Phi = \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$, $\mathfrak{n}_\Phi = \mathfrak{n} = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{2\alpha}$
- $\mathfrak{q}_\Phi = \mathfrak{g}_0 \oplus \mathfrak{n} = \mathfrak{k}_0 \oplus \mathfrak{a} \oplus \mathfrak{n}$ minimal parabolic subalgebra

PROBLEM: Find all $k$-dimensional ($k \geq 2$) linear subspaces $\mathfrak{v}$ of $\mathfrak{g}_\alpha$ for which there exists a subgroup of $K_0$ acting transitively on the unit sphere in $\mathfrak{v}$

## Nilpotent construction for hyperbolic spaces

| $M$ | $G$ | $K$ | $K_0$ | $\mathfrak{g}_\alpha$ | $\mathfrak{n}$ |
|---|---|---|---|---|---|
| $\mathbb{R}H^n$ | $SO^o_{1,n}$ | $SO_n$ | $SO_{n-1}$ | $\mathbb{R}^{n-1}$ | $\mathbb{R}^{n-1}$ |
| $\mathbb{C}H^n$ | $SU_{1,n}$ | $U_n$ | $U_{n-1}$ | $\mathbb{C}^{n-1}$ | $\mathbb{C}^{n-1} \oplus \mathbb{R}$ |
| $\mathbb{H}H^n$ | $Sp_{1,n}$ | $Sp_1Sp_n$ | $Sp_1Sp_{n-1}$ | $\mathbb{H}^{n-1}$ | $\mathbb{H}^{n-1} \oplus \mathbb{R}^3$ |
| $\mathbb{O}H^2$ | $F_4^{-20}$ | $Spin_9$ | $Spin_7$ | $\mathbb{O}$ | $\mathbb{O} \oplus \mathbb{R}^7$ |

**Berndt, Tamaru** (Trans. Amer. Math. Soc. 359 (2007), 3425–3438):

- $\mathbb{R}$: any linear subspace $\mathfrak{v} \subset \mathbb{R}^{n-1}$
- $\mathbb{C}$: any linear subspace $\mathfrak{v} \subset \mathbb{C}^{n-1}$ with constant Kähler angle
- $\mathbb{O}$: any linear subspace $\mathfrak{v} \subset \mathbb{O}$ of dimension $k \in \{2, 3, 4, 6, 7\}$
- $\mathbb{H}$: some linear subspaces $\mathfrak{v} \subset \mathbb{H}^{n-1}$ with constant quaternionic Kähler angle (no complete classification)

## Explicit classifications for cohomogeneity one

We have a complete classification of cohomogeneity one actions (up to orbit equivalence) on the following Riemannian symmetric spaces of noncompact type and rank 1 or 2:

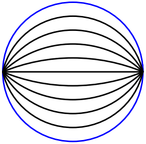
- **Berndt, Tamaru**: $\mathbb{C}H^n$, $\mathbb{H}H^2$, $\mathbb{O}H^2$, $SL_3(\mathbb{R})/SO_3$, $G_2^2/SO_4$
- **Berndt, Domínguez-Vázquez** (Transf. Groups, to appear): $SL_3(\mathbb{C})/SU_3$, $G_2(\mathbb{C})/G_2$, $SO^o_{2,2+k}/SO_2SO_{2+k}$

Some remaining problems are

- classification of cohomogeneity one actions on $\mathbb{H}H^n$ ($n \geq 3$),
- better understanding of the construction of cohomogeneity one actions of nilpotent type

## Examples of hyperpolar foliations

- $V$ linear subspace of $\mathbb{E}^m$
  $\Longrightarrow \mathcal{F}_V^m = \{p + V \mid p \in \mathbb{E}^m\}$ homogeneous hyperpolar foliation of $\mathbb{E}^m$
- $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, $M = G/K = \mathbb{F}H^n$
  $\Longrightarrow \mathcal{F}_{\mathbb{F}}^n$ homogeneous codimension one foliation of $\mathbb{F}H^n$ with unique minimal leaf



- $\mathcal{F}_{\mathbb{F}_1}^{n_1} \times \cdots \times \mathcal{F}_{\mathbb{F}_k}^{n_k} \times \mathcal{F}_V^m$ homogeneous hyperpolar foliation of $\mathbb{F}_1 H^{n_1} \times \cdot \times \mathbb{F}_k H^{n_k} \times \mathbb{E}^m$

## Examples of hyperpolar foliations (II)

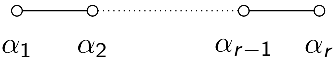
- $M = G/K$ symmetric space of noncompact type
- $\Phi$ orthogonal set of simple roots, $k = |\Phi|$
- $\mathfrak{q}_\Phi = \mathfrak{m}_\Phi \oplus \mathfrak{a}_\Phi \oplus \mathfrak{n}_\Phi$ Langlands decomposition of parabolic subalgebra $\mathfrak{q}_\Phi$ of $\mathfrak{g}$
- $F_\Phi \cong \underbrace{\mathbb{F}_1 H^{n_1} \times \cdots \times \mathbb{F}_k H^{n_k}}_{M_\Phi \cdot o} \times \underbrace{\mathbb{E}^{r-k}}_{A_\Phi \cdot o}$
- $\mathcal{F}^{n_1}_{\mathbb{F}_1} \times \cdots \times \mathcal{F}^{n_k}_{\mathbb{F}_k} \times \mathcal{F}^{r-k}_V$ homogeneous hyperpolar foliation of $F_\Phi$
- $\mathcal{F}_{\Phi,V} = \mathcal{F}^{n_1}_{\mathbb{F}_1} \times \cdots \times \mathcal{F}^{n_k}_{\mathbb{F}_k} \times \mathcal{F}^{r-k}_V \times N_\Phi$ homogeneous hyperpolar foliation of $M = F_\Phi \times N_\Phi$
- $\mathcal{F}_{\emptyset,\{0\}}$ horocycle foliation of $M$

## Classification of homogeneous hyperpolar foliations

**THEOREM.** (**Berndt, Díaz-Ramos, Tamaru:** J. Differential Geom. 86 (2010), 191–235)
*Let $M$ be a Riemannian symmetric space of noncompact type. Every homogeneous hyperpolar foliation on $M$ is isometrically congruent to $\mathcal{F}_{\Phi,V}$ for some orthogonal set $\Phi$ of simple roots and some linear subspace $V \subset \mathbb{E}^{r-|\Phi|}$.*

## The symmetric space $SL_{r+1}(\mathbb{R})/SO_{r+1}$

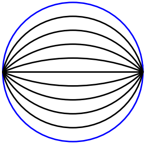
- Dynkin diagram




- $\Phi \subset \Lambda = \{\alpha_1, \ldots, \alpha_r\}$ orthogonal, $k = |\Phi|$
- horospherical decomposition:
  $SL_{r+1}(\mathbb{R})/SO_{r+1} \cong \underbrace{\mathbb{R}H^2 \times \ldots \times \mathbb{R}H^2}_{k \text{ factors}} \times \mathbb{E}^{r-k} \times N_\Phi$
- $N_\Phi$ corresponds to the set of all upper block diagonal matrices with certain $2 \times 2$ and $1 \times 1$ diagonal blocks, diagonal entries are 1

## The symmetric space $SL_{r+1}(\mathbb{R})/SO_{r+1}$ (II)

- horospherical decomposition:
$$SL_{r+1}(\mathbb{R})/SO_{r+1} \cong \underbrace{\mathbb{R}H^2 \times \ldots \times \mathbb{R}H^2}_{k \text{ factors}} \times \mathbb{E}^{r-k} \times N_\Phi$$
- On each $\mathbb{R}H^2$ select the foliation



- On $\mathbb{E}^{r-k}$ select a foliation by parallel affine subspaces
- On $N_\Phi$ select the foliation with one leaf $N_\Phi$

**The product foliation is hyperpolar, and every homogeneous hyperpolar foliation of $SL_{r+1}(\mathbb{R})/SO_{r+1}$ arises in this way**