

# Newton's equations in spaces of constant curvature

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# Goal: to present some results from

- F. Diacu. On the singularities of the curved  $n$ -body problem, *Trans. Amer. Math. Soc.* **363**, 4 (2011), 2249-2264.
- F. Diacu and E. Pérez-Chavela. Homographic solutions of the curved 3-body problem, *J. Differential Equations* **250** (2011), 340-366.
- F. Diacu. Polygonal homographic orbits of the curved  $n$ -body problem, *Trans. Amer. Math. Soc.* **364**, 5 (2012), 2783-2802.
- F. Diacu. *Relative equilibria in the curved  $N$ -body problem*, Atlantis Studies in Dynamical Systems, vol. I, Atlantis Press, 2012.
- F. Diacu. Relative equilibria in the 3-dimensional curved  $n$ -body problem, *Memoirs Amer. Math. Soc.* **228**, 1071 (2013), ISBN: 978-0-8218-9136-0.
- F. Diacu and S. Kordlou. Rotopulsating orbits of the curved  $N$ -body problem *J. Differential Equations* **255** (2013), 2709-2750.

# History of the problem

- 1830s Nikolai Lobachevsky and János Bolyai: 2-BP in  $\mathbf{H}^3$
- 1852 Lejeune Dirichlet: 2-BP in  $\mathbf{H}^3$
- 1860 Paul Joseph Serret: 2-BP in  $\mathbf{S}^2$
- 1870 Ernst Schering: 2-BP in  $\mathbf{H}^3$
- 1873 Rudolph Lipschitz: 2-BP in  $\mathbf{S}^3$
- 1885 Wilhelm Killing: 2-BP in  $\mathbf{H}^3$
- 1902 Heinrich Liebmann: 2-BP in  $\mathbf{S}^2$  and  $\mathbf{H}^2$  also proves an analogue of Bertrand's theorem
- 1940 Erwin Schrödinger: quantum 2-BP in  $\mathbf{H}^3$
- 1945 Leopold Infeld and Alfred Schild: quantum 2-BP in  $\mathbf{H}^3$
- 1990s Russian school of celestial mechanics
- 2005 José Cariñena, Manuel Rañada, Mariano Santander: 2-BP in  $\mathbf{S}^2$  and  $\mathbf{H}^2$

# Setting

The space in which the motion of the bodies takes place is:

$$\mathbb{M}_\kappa^3 = \{(w, x, y, z) \mid w^2 + x^2 + y^2 + \sigma z^2 = \kappa^{-1} (z > 0 \text{ if } \kappa < 0)\},$$

where  $\sigma$  is the *signum function*

$$\sigma = \begin{cases} +1, & \text{for } \kappa > 0 \\ -1, & \text{for } \kappa < 0 \end{cases}$$

Notice that

$$\mathbb{M}_1^3 = \mathbb{S}^3 \quad \text{and} \quad \mathbb{M}_{-1}^3 = \mathbb{H}^3$$

# Notations

Consider  $m_1, \dots, m_n > 0$  in  $\mathbb{R}^4$  for  $\kappa > 0$  and  $\mathbb{R}^{3,1}$  (Minkowski space) for  $\kappa < 0$ , with positions given by

$$\mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = \overline{1, n}$$

$\mathbf{q} = (\mathbf{q}_1, \dots, \mathbf{q}_n)$  is the configuration of the system

$\nabla_{\mathbf{q}_i} := (\partial_{w_i}, \partial_{x_i}, \partial_{y_i}, \sigma \partial_{z_i})$ ,  $\nabla := (\nabla_{\mathbf{q}_1}, \dots, \nabla_{\mathbf{q}_n})$  is the gradient

For  $\mathbf{a} := (a_w, a_x, a_y, a_z)$ ,  $\mathbf{b} := (b_w, b_x, b_y, b_z)$ ,

$$\mathbf{a} \cdot \mathbf{b} := (a_w b_w + a_x b_x + a_y b_y + \sigma a_z b_z)$$

is the inner product

# Potential

For  $\kappa \neq 0$ , the force function is

$$U_\kappa(\mathbf{q}) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j |\kappa|^{1/2} \kappa \mathbf{q}_i \cdot \mathbf{q}_j}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i)(\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{1/2}}$$

$-U_\kappa$  is the potential (a homogeneous function of degree 0).

Euler's formula for homogeneous functions:

$$\mathbf{q}_i \cdot \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = 0, \quad i = \overline{1, n}.$$

# Equations of motion

Using variational methods (constrained Lagrangian dynamics), we obtain the equations of motion:

$$m_i \ddot{\mathbf{q}}_i = \nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) - m_i \kappa (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i,$$
$$\mathbf{q}_i \cdot \mathbf{q}_i = \kappa^{-1}, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad \kappa \neq 0, \quad i = \overline{1, n}$$

$$\nabla_{\mathbf{q}_i} U_\kappa(\mathbf{q}) = \sum_{\substack{j=1 \\ j \neq i}}^n \frac{m_i m_j |\kappa|^{3/2} (\kappa \mathbf{q}_j \cdot \mathbf{q}_j) [(\kappa \mathbf{q}_i \cdot \mathbf{q}_i) \mathbf{q}_j - (\kappa \mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[\sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_i) (\kappa \mathbf{q}_j \cdot \mathbf{q}_j) - \sigma(\kappa \mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}},$$

$$i = \overline{1, n}$$

# Elimination of $\kappa$

Coordinate and time-rescaling transformations

$$\mathbf{q}_i = |\kappa|^{-1/2} \mathbf{r}_i, \quad i = \overline{1, n} \quad \text{and} \quad \tau = |\kappa|^{3/4} t$$

lead to the equations of motion

$$\mathbf{r}_i'' = \sum_{j=1, j \neq i}^n \frac{m_j [\mathbf{r}_j - \sigma(\mathbf{r}_i \cdot \mathbf{r}_j) \mathbf{r}_i]}{[\sigma - \sigma(\mathbf{r}_i \cdot \mathbf{r}_j)^2]^{3/2}} - \sigma(\mathbf{r}_i' \cdot \mathbf{r}_i') \mathbf{r}_i, \quad i = \overline{1, n},$$

where

$$' = \frac{d}{d\tau}, \quad \mathbf{r}_i \cdot \mathbf{r}_i = |\kappa| \mathbf{q}_i \cdot \mathbf{q}_i = |\kappa| \kappa^{-1} = \sigma$$



# The positive case and the negative case

Equations of motion in  $\mathbb{S}^3$ :

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^n \frac{m_j [\mathbf{q}_j - (\mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[1 - (\mathbf{q}_i \cdot \mathbf{q}_j)^2]^{3/2}} - (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q}_i = 1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

Equations of motion in  $\mathbb{H}^3$ :

$$\ddot{\mathbf{q}}_i = \sum_{j=1, j \neq i}^n \frac{m_j [\mathbf{q}_j + (\mathbf{q}_i \cdot \mathbf{q}_j) \mathbf{q}_i]}{[(\mathbf{q}_i \cdot \mathbf{q}_j)^2 - 1]^{3/2}} + (\dot{\mathbf{q}}_i \cdot \dot{\mathbf{q}}_i) \mathbf{q}_i,$$

$$\mathbf{q}_i \cdot \mathbf{q}_i = -1, \quad \mathbf{q}_i \cdot \dot{\mathbf{q}}_i = 0, \quad i = \overline{1, n}$$

# Hamiltonian form

$\mathbf{p} := (\mathbf{p}_1, \dots, \mathbf{p}_n)$ ,  $\mathbf{p}_i := m_i \dot{\mathbf{q}}_i$ ,  $i = \overline{1, n}$ , momenta

$T(\mathbf{q}, \mathbf{p}) = \frac{1}{2} \sum_{i=1}^n m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) (\sigma \mathbf{q}_i \cdot \mathbf{q}_i)$ , kinetic energy

$H(\mathbf{q}, \mathbf{p}) = T(\mathbf{q}, \mathbf{p}) - U(\mathbf{q})$ , Hamiltonian function

$$\begin{cases} \dot{\mathbf{q}}_i = \nabla_{\mathbf{p}_i} H(\mathbf{q}, \mathbf{p}) = m_i^{-1} \mathbf{p}_i, \\ \dot{\mathbf{p}}_i = -\nabla_{\mathbf{q}_i} H(\mathbf{q}, \mathbf{p}) = \nabla_{\mathbf{q}_i} U(\mathbf{q}) - \sigma m_i^{-1} (\mathbf{p}_i \cdot \mathbf{p}_i) \mathbf{q}_i, \\ \mathbf{q}_i \cdot \mathbf{q}_i = \sigma, \quad \mathbf{q}_i \cdot \mathbf{p}_i = 0, \quad i = \overline{1, n} \end{cases}$$

# The wedge product

Consider the basis

$$\mathbf{e}_w = (1, 0, 0, 0), \mathbf{e}_x = (0, 1, 0, 0), \mathbf{e}_y = (0, 0, 1, 0), \mathbf{e}_z = (0, 0, 0, 1)$$

The wedge product of

$\mathbf{u} = (u_w, u_x, u_y, u_z), \mathbf{v} = (v_w, v_x, v_y, v_z) \in \mathbb{R}^4$  is defined as

$$\begin{aligned} \mathbf{u} \wedge \mathbf{v} := & (u_w v_x - u_x v_w) \mathbf{e}_w \wedge \mathbf{e}_x + (u_w v_y - u_y v_w) \mathbf{e}_w \wedge \mathbf{e}_y + \\ & (u_w v_z - u_z v_w) \mathbf{e}_w \wedge \mathbf{e}_z + (u_x v_y - u_y v_x) \mathbf{e}_x \wedge \mathbf{e}_y + \\ & (u_x v_z - u_z v_x) \mathbf{e}_x \wedge \mathbf{e}_z + (u_y v_z - u_z v_y) \mathbf{e}_y \wedge \mathbf{e}_z, \end{aligned}$$

where  $\mathbf{e}_w \wedge \mathbf{e}_x, \mathbf{e}_w \wedge \mathbf{e}_y, \mathbf{e}_w \wedge \mathbf{e}_z, \mathbf{e}_x \wedge \mathbf{e}_y, \mathbf{e}_x \wedge \mathbf{e}_z, \mathbf{e}_y \wedge \mathbf{e}_z$  represent the bivectors that form a canonical basis of the exterior Grassmann algebra over  $\mathbb{R}^4$

# Integrals of the total angular momentum

$$\sum_{i=1}^n m_i \mathbf{q}_i \wedge \dot{\mathbf{q}}_i = \mathbf{c},$$

where  $\mathbf{c} =$

$c_{wx} \mathbf{e}_w \wedge \mathbf{e}_x + c_{wy} \mathbf{e}_w \wedge \mathbf{e}_y + c_{wz} \mathbf{e}_w \wedge \mathbf{e}_z + c_{xy} \mathbf{e}_x \wedge \mathbf{e}_y + c_{xz} \mathbf{e}_x \wedge \mathbf{e}_z + c_{yz} \mathbf{e}_y \wedge \mathbf{e}_z,$   
with the coefficients  $c_{wx}, c_{wy}, c_{wz}, c_{xy}, c_{xz}, c_{yz} \in \mathbb{R}$

– on components, 6 integrals:

$$\begin{aligned} \sum_{i=1}^n m_i (w_i \dot{x}_i - \dot{w}_i x_i) &= c_{wx}, & \sum_{i=1}^n m_i (w_i \dot{y}_i - \dot{w}_i y_i) &= c_{wy}, \\ \sum_{i=1}^n m_i (w_i \dot{z}_i - \dot{w}_i z_i) &= c_{wz}, & \sum_{i=1}^n m_i (x_i \dot{y}_i - \dot{x}_i y_i) &= c_{xy}, \\ \sum_{i=1}^n m_i (x_i \dot{z}_i - \dot{x}_i z_i) &= c_{xz}, & \sum_{i=1}^n m_i (y_i \dot{z}_i - \dot{y}_i z_i) &= c_{yz} \end{aligned}$$

# Isometries in $\mathbb{S}^3$

In some suitable basis, rotations can be written as

$$A = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}, \theta, \phi \in [0, 2\pi)$$

- simple rotations (elliptic): lead to new solutions
- double rotations (elliptic-elliptic): lead to new solutions

# Isometries in $\mathbb{H}^3$

In some suitable basis, rotations can be written as

$$B = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & \cosh \phi & \sinh \phi \\ 0 & 0 & \sinh \phi & \cosh \phi \end{pmatrix}, \theta \in [0, 2\pi), \phi \in \mathbb{R},$$

- simple rotations (elliptic): lead to new solutions
- simple rotations (hyperbolic): lead to new solutions
- double rotations (elliptic-hyperbolic): lead to new solutions

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\xi & \xi \\ 0 & \xi & 1 - \xi^2/2 & \xi^2/2 \\ 0 & \xi & -\xi^2/2 & 1 + \xi^2/2 \end{pmatrix}, \xi \in \mathbb{R}.$$

- simple rotations (parabolic): lead to no solutions

# Relative equilibria (RE) in $\mathbb{S}^3$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i = \overline{1, n},$$

$$[\text{positive elliptic}] : \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \text{ (constant)} \\ z_i(t) = z_i \text{ (constant)}, \end{cases}$$

$$\text{with } w_i^2 + x_i^2 = r_i^2, \quad r_i^2 + y_i^2 + z_i^2 = 1, \quad i = \overline{1, n}$$

$$[\text{positive elliptic-elliptic}] : \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \rho_i \cos(\beta t + b_i) \\ z_i(t) = \rho_i \sin(\beta t + b_i), \end{cases}$$

$$\text{with } w_i^2 + x_i^2 = r_i^2, \quad y_i^2 + z_i^2 = \rho_i^2, \quad r_i^2 + \rho_i^2 = 1, \quad i = \overline{1, n}$$

# Relative equilibria (RE) in $\mathbb{H}^3$

$$\text{[negative elliptic]} : \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = y_i \text{ (constant)} \\ z_i(t) = z_i \text{ (constant)}, \end{cases}$$

with  $w_i^2 + x_i^2 = r_i^2$ ,  $r_i^2 + y_i^2 - z_i^2 = -1$ ,  $i = \overline{1, n}$

$$\text{[negative hyperbolic]} : \begin{cases} w_i(t) = w_i \text{ (constant)} \\ x_i(t) = x_i \text{ (constant)} \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with  $y_i^2 - z_i^2 = -\eta_i^2$ ,  $w_i^2 + x_i^2 - \eta_i^2 = -1$ ,  $i = \overline{1, n}$

$$\text{[negative elliptic-hyperbolic]} : \begin{cases} w_i(t) = r_i \cos(\alpha t + a_i) \\ x_i(t) = r_i \sin(\alpha t + a_i) \\ y_i(t) = \eta_i \sinh(\beta t + b_i) \\ z_i(t) = \eta_i \cosh(\beta t + b_i), \end{cases}$$

with  $w_i^2 + x_i^2 = r_i^2$ ,  $y_i^2 - z_i^2 = -\eta_i^2$ , so  $r_i^2 - \eta_i^2 = -1$ ,  $i = \overline{1, n}$



# Fixed points (FP) in $\mathbb{S}^3$

- equilateral triangle on a great circle of a great sphere (equal masses, 3BP)
- any scalene acute triangle on a great circle of a great sphere (non-equal masses, 3BP)
- regular tetrahedron in a great sphere (equal masses, 4BP)
- two equilateral triangles, each on complementary great circles (equal masses, 6 BP):

$w_1 = 1,$	$x_1 = 0,$	$y_1 = 0,$	$z_1 = 0,$
$w_2 = -1/2,$	$x_2 = \sqrt{3}/2,$	$y_2 = 0,$	$z_2 = 0,$
$w_3 = -1/2,$	$x_3 = -\sqrt{3}/2,$	$y_3 = 0,$	$z_3 = 0,$
$w_4 = 0,$	$x_4 = 0,$	$y_4 = 1,$	$z_4 = 0,$
$w_5 = 0,$	$x_5 = 0,$	$y_5 = -1/2,$	$z_5 = \sqrt{3}/2,$
$w_6 = 0,$	$x_6 = 0,$	$y_6 = -1/2,$	$z_6 = -\sqrt{3}/2,$

- two, not necessarily congruent, scalene acute triangles, each on one of two complementary great circles (non-equal masses, 6 BP)

# Complementary circles in $\mathbb{S}^3$

## Definition 1

Two great circles,  $C_1$  and  $C_2$ , of two different great spheres of  $\mathbb{S}^3$  are called *complementary* if there is a coordinate system  $wxyz$  such that

$$C_1 = \mathbf{S}_{wx}^1 = \{(0, 0, y, z) \mid y^2 + z^2 = 1\},$$
$$C_2 = \mathbf{S}_{yz}^1 = \{(w, x, 0, 0) \mid w^2 + x^2 = 1\}.$$

Complementary circles form a Hopf link in a Hopf fibration,

$$h: \mathbb{S}^3 \rightarrow \mathbb{S}^2, \quad h(w, x, y, z) = (w^2 + x^2 - y^2 - z^2, 2(wz + xy), 2(xz - wy)),$$

which takes circles of  $\mathbb{S}^3$  to points of  $\mathbb{S}^2$ . Using the stereographic projection, it can be shown that the circles  $C_1$  and  $C_2$  are linked.

# Distance between complementary circles

Since, in  $\mathbb{S}^3$ , the distance between two points,  $\mathbf{a}$  and  $\mathbf{b}$ , is

$$d(\mathbf{a}, \mathbf{b}) = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}),$$

it follows that if  $\mathbf{a} \in C_1$  and  $\mathbf{b} \in C_2$ , then

$$d(\mathbf{a}, \mathbf{b}) = \pi/2 = \text{constant}$$

Therefore if the body  $m_1$  is on  $C_1$  and the body  $m_2$  is on  $C_2$ , the magnitude of the attraction between them is the same, no matter where each of them lies on the respective circle

# Clifford tori in $\mathbb{S}^3$

A remarkable family of surfaces in  $\mathbb{R}^4$  are the Clifford tori

$$\mathbf{T}_{r\rho}^2 = \{(r \cos \theta, r \sin \theta, \rho \cos \phi, \rho \sin \phi) \mid r^2 + \rho^2 = 1, 0 \leq \theta, \phi < 2\pi\},$$

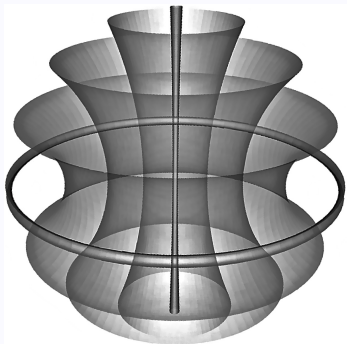
which lie in  $\mathbb{S}^3$ . Indeed, the Euclidean distance from the origin of the coordinate system to any point of a Clifford torus is

$$(r^2 \cos^2 \theta + r^2 \sin^2 \theta + \rho^2 \cos^2 \phi + \rho^2 \sin^2 \phi)^{1/2} = (r^2 + \rho^2)^{1/2} = 1$$

Unlike the standard torus, the Clifford torus is a flat surface, which divides  $\mathbb{S}^3$  into two solid tori, for which it forms the boundary

# Heegaard splitting of $\mathbb{S}^3$

The Clifford torus with  $r = \rho = 1/\sqrt{2}$  provides the standard genus 1 splitting of  $\mathbb{S}^3$ , a case in which the two solid tori are congruent.



A 3D projection of a 4D foliation of  $\mathbb{S}^3$  into Clifford tori

# A Lagrangian RE

$$w_1(t) = r \cos \omega t,$$

$$y_1(t) = y \text{ (constant),}$$

$$w_2(t) = r \cos(\omega t + 2\pi/3),$$

$$y_2(t) = y \text{ (constant),}$$

$$w_3(t) = r \cos(\omega t + 4\pi/3),$$

$$y_3(t) = y \text{ (constant),}$$

$$x_1(t) = r \sin \omega t,$$

$$z_1(t) = z \text{ (constant),}$$

$$x_2(t) = r \sin(\omega t + 2\pi/3),$$

$$z_2(t) = z \text{ (constant),}$$

$$x_3(t) = r \sin(\omega t + 4\pi/3),$$

$$z_3(t) = z \text{ (constant).}$$

Given  $m := m_1 = m_2 = m_3 > 0$ ,  $r \in (0, 1)$ , and  $y, z$  with  $r^2 + y^2 + z^2 = 1$ , we can always find two frequencies,

$$\alpha^+ = \frac{2}{r} \sqrt{\frac{2m}{\sqrt{3}r(4 - 3r^2)^{3/2}}} \quad \text{and} \quad \alpha^- = -\frac{2}{r} \sqrt{\frac{2m}{\sqrt{3}r(4 - 3r^2)^{3/2}}};$$

$$c_{wx} = 3m\omega \neq 0 \quad \text{and} \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = c_{yz} = 0.$$

# Stability of Lagrangian RE in $\mathbb{S}^2$

Regina Martínez and Carles Simó: On  $\mathbb{S}^2$ , the Lagrangian RE with masses  $m_1 = m_2 = m_3 = 1$  are linearly stable for  $r \in (r_1, r_2) \cup (r_3, 1)$ , where  $r = \sqrt{1 - z^2}$ ,

$$r_1 = 0.55778526844099498188467226566148375,$$

$$r_2 = 0.68145469725865414807206661241888645,$$

$$r_3 = 0.92893280143637470996280353121615412,$$

truncated to 35 decimal digits.

# Regular tetrahedron RE

Place the bodies  $m_1 = m_2 = m_3 = m_4$  at the vertices of a regular tetrahedron. Then  $m_1$  and  $m_2$  move on the Clifford torus with  $r = 0$  and  $\rho = 1$ , which is the only Clifford torus in the class of a given foliation of  $\mathbb{S}^3$  that is also a great circle of  $\mathbb{S}^3$ . The bodies of mass  $m_3$  and  $m_4$  move on the Clifford torus with  $r = \frac{\sqrt{6}}{3}$  and  $\rho = \frac{\sqrt{3}}{3}$ :

$$w_1 = 0, \quad x_1 = 0, \quad y_1 = \cos(\alpha t + \pi/2), \quad z_1 = \sin(\alpha t + \pi/2),$$

$$w_2 = 0, \quad x_2 = 0, \quad y_2 = \cos(\alpha t + b_2), \quad z_2 = \sin(\alpha t + b_2),$$

with  $\sin b_2 = -\frac{1}{3}$  and  $\cos b_2 = \frac{2\sqrt{2}}{3}$ ,



# Example of RE moving on Clifford tori

$$w_3 = \frac{\sqrt{6}}{3} \cos(\alpha t + 3\pi/2), \quad x_3 = \frac{\sqrt{6}}{3} \sin(\alpha t + 3\pi/2),$$

$$y_3 = \frac{\sqrt{3}}{3} \cos(\alpha t + b_3), \quad z_3 = \frac{\sqrt{3}}{3} \sin(\alpha t + b_3),$$

with  $\cos b_3 = -\frac{\sqrt{6}}{3}$  and  $\sin b_3 = -\frac{\sqrt{3}}{3}$ , and

$$w_4 = \frac{\sqrt{6}}{3} \cos(\alpha t + \pi/2), \quad x_4 = \frac{\sqrt{6}}{3} \sin(\alpha t + \pi/2),$$

$$y_4 = \frac{\sqrt{3}}{3} \cos(\alpha t + b_4), \quad z_4 = \frac{\sqrt{3}}{3} \sin(\alpha t + b_4),$$

with  $\cos b_4 = -\frac{\sqrt{6}}{3}$  and  $\sin b_4 = -\frac{\sqrt{3}}{3}$ . Notice that  $b_3 = b_4$ .

# RE with fixed bodies

This is a solution of the 6-body problem with two equilateral triangles, one inscribed in a great circle of a great sphere and the other inscribed in a complementary great circle of another great sphere. The first triangle rotates uniformly, while the second triangle is fixed:

$$m_1 = m_2 = m_3 = m_4 = m_5 = m_6 =: m,$$

$$\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3, \mathbf{q}_4, \mathbf{q}_5, \mathbf{q}_6), \quad \mathbf{q}_i = (w_i, x_i, y_i, z_i), \quad i \in \{1, 2, 3, 4, 5, 6\},$$

$$\begin{array}{llll} w_1 = \cos \alpha t, & x_1 = \sin \alpha t, & y_1 = 0, & z_1 = 0, \\ w_2 = \cos(\alpha t + a), & x_2 = \sin(\alpha t + a), & y_2 = 0, & z_2 = 0, \\ w_3 = \cos(\alpha t + b), & x_3 = \sin(\alpha t + b), & y_3 = 0, & z_3 = 0, \\ w_4 = 0, & x_4 = 0, & y_4 = 1, & z_4 = 0, \\ w_5 = 0, & x_5 = 0, & y_5 = -\frac{1}{2}, & z_5 = \frac{\sqrt{3}}{2}, \\ w_6 = 0, & x_6 = 0, & y_6 = -\frac{1}{2}, & z_6 = -\frac{\sqrt{3}}{2}, \end{array}$$

where  $a = 2\pi/3$  and  $b = 4\pi/3$ .

# RE on complementary circles

In general, the orbit described below is quasiperiodic:

$$w_1 = \cos \alpha t,$$

$$y_1 = 0,$$

$$w_2 = \cos(\alpha t + 2\pi/3),$$

$$y_2 = 0,$$

$$w_3 = \cos(\alpha t + 4\pi/3),$$

$$y_3 = 0,$$

$$w_4 = 0,$$

$$y_4 = \cos \beta t,$$

$$w_5 = 0,$$

$$y_5 = \cos(\beta t + 2\pi/3),$$

$$w_6 = 0,$$

$$y_6 = \cos(\beta t + 4\pi/3),$$

$$x_1 = \sin \alpha t,$$

$$z_1 = 0,$$

$$x_2 = \sin(\alpha t + 2\pi/3),$$

$$z_2 = 0,$$

$$x_3 = \sin(\alpha t + 4\pi/3),$$

$$z_3 = 0,$$

$$x_4 = 0,$$

$$z_4 = \sin \beta t,$$

$$x_5 = 0,$$

$$z_5 = \sin(\beta t + 2\pi/3),$$

$$x_6 = 0,$$

$$z_6 = \sin(\beta t + 4\pi/3).$$

$$c_{wx} = 3m\alpha \neq 0, \quad c_{yz} = 3m\beta \neq 0, \quad c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$

# Eulerian RE in $\mathbb{H}^3$

The motion described below takes place on a hyperbolic 2-sphere, and is not periodic:

$$\begin{aligned}w_1 = 0, \quad x_1 = 0, & & y_1 = \sinh \beta t, & \quad z_1 = \cosh \beta t, \\w_2 = 0, \quad x_2 = x \text{ (constant)}, & & y_2 = \eta \sinh \beta t, & \quad z_2 = \eta \cosh \beta t, \\w_3 = 0, \quad x_3 = -x \text{ (constant)}, & & y_3 = \eta \sinh \beta t, & \quad z_3 = \eta \cosh \beta t,\end{aligned}$$

Given  $m := m_1 = m_2 = m_3 > 0, x > 0, \eta > 0$  with  $x^2 - \eta^2 = -1$ , there exist two non-zero frequencies,

$$\beta^+ = \frac{1}{2\eta} \sqrt{\frac{1 + 4\eta^2}{\eta(\eta^2 - 1)^{3/2}}} \quad \text{and} \quad \beta^- = -\frac{1}{2\eta} \sqrt{\frac{1 + 4\eta^2}{\eta(\eta^2 - 1)^{3/2}}};$$

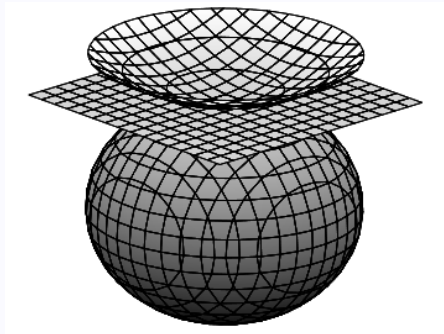
$$c_{wx} = c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0, \quad c_{yz} = m\beta(1 - 2\eta^2)$$

# RE on hyperbolic cylinders

The motion described below takes place on a hyperbolic cylinder, and is not periodic:

$$\begin{aligned}w_1 &= 0, & x_1 &= 0, & y_1 &= \sinh \beta t, & z_1 &= \cosh \beta t, \\w_2 &= r \cos \alpha t, & x_2 &= r \sin \alpha t, & y_2 &= \eta \sinh \beta t, & z_2 &= \eta \cosh \beta t, \\w_3 &= -r \cos \alpha t, & x_3 &= -r \sin \alpha t, & y_3 &= \eta \sinh \beta t, & z_3 &= \eta \cosh \beta t.\end{aligned}$$

$$c_{wx} = 2m\alpha r^2, c_{yz} = -1 - 2\beta\eta^2, c_{wy} = c_{wz} = c_{xy} = c_{xz} = 0$$



The passage from  $\mathbb{S}_{\kappa}^3$  up to  $\mathbb{R}^3$  and from  $\mathbb{H}_{\kappa}^3$  down to  $\mathbb{R}^3$  when  $\kappa \rightarrow 0$ .

Consider a coordinate system having the origin at the North Pole of the spheres  $\mathbb{S}_{\kappa}^3$ , i.e., the position of the body  $m_i$  is given by

$$\mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \quad i = \overline{1, n}.$$

# Extension of the equations to $\kappa = 0$

$$V_\kappa(\mathbf{r}) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j \left(1 - \frac{\kappa r_{ij}^2}{2}\right)}{r_{ij} \left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{1/2}}$$

$$\ddot{\mathbf{r}}_i = \sum_{j=1, j \neq i}^n \frac{m_j \left[ \mathbf{r}_j - \left(1 - \frac{\kappa r_{ij}^2}{2}\right) \mathbf{r}_i + \frac{r_{ij}^2 \mathbf{R}}{2} \right]}{r_{ij}^3 \left(1 - \frac{\kappa r_{ij}^2}{4}\right)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) (\kappa \mathbf{r}_i + \mathbf{R}), \quad i = \overline{1, n},$$

with

$$\kappa \mathbf{r}_i \cdot \mathbf{r}_i + 2|\kappa|^{1/2} \omega_i = 0, \quad \kappa \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i + |\kappa|^{1/2} \dot{\omega}_i = 0, \quad i = \overline{1, n},$$

where

$$\mathbf{R} = (0, 0, 0, \sigma |\kappa|^{1/2}), \quad \mathbf{r}_i = (x_i, y_i, z_i, \omega_i), \quad i = \overline{1, n},$$

$$r_{ij} := [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 + \sigma(\omega_i - \omega_j)^2]^{1/2}.$$

# The explicit equations

$$\left\{ \begin{aligned} \ddot{x}_i &= \sum_{j=1, j \neq i}^n \frac{m_j \left[ x_j - \left( 1 - \frac{\kappa r_{ij}^2}{2} \right) x_i \right]}{r_{ij}^3 \left( 1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) x_i \\ \ddot{y}_i &= \sum_{j=1, j \neq i}^n \frac{m_j \left[ y_j - \left( 1 - \frac{\kappa r_{ij}^2}{2} \right) y_i \right]}{r_{ij}^3 \left( 1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) y_i \\ \ddot{z}_i &= \sum_{j=1, j \neq i}^n \frac{m_j \left[ z_j - \left( 1 - \frac{\kappa r_{ij}^2}{2} \right) z_i \right]}{r_{ij}^3 \left( 1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - \kappa (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) z_i \\ \ddot{\omega}_i &= \sum_{j=1, j \neq i}^n \frac{m_j \left[ \omega_j - \left( 1 - \frac{\kappa r_{ij}^2}{2} \right) \omega_i + \frac{\sigma |\kappa|^{1/2} r_{ij}^2}{2} \right]}{r_{ij}^3 \left( 1 - \frac{\kappa r_{ij}^2}{4} \right)^{3/2}} - (\dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i) [\kappa \omega_i + \sigma |\kappa|^{1/2}], \end{aligned} \right.$$

$$\kappa(x_i^2 + y_i^2 + z_i^2 + \sigma \omega_i^2) + 2|\kappa|^{1/2} \omega_i = 0,$$

$$\kappa(x_i \dot{x}_i + y_i \dot{y}_i + z_i \dot{z}_i + \sigma \omega_i \dot{\omega}_i) + |\kappa|^{1/2} \dot{\omega}_i = 0, \quad i = \overline{1, n}.$$



# Newtonian equations

For  $\kappa = 0$  we recover the Newtonian equations:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1, j \neq i}^n \frac{m_i m_j (\mathbf{r}_j - \mathbf{r}_i)}{r_{ij}^3}, \quad i = \overline{1, n},$$

with  $\mathbf{r}_i = (x_i, y_i, z_i, 0)$ ,  $i = \overline{1, n}$ .

# Bifurcation of the first integrals

- Integral of energy:  
for all  $\kappa \in \mathbb{R}$ : 1 integral (no bifurcation)
- Integrals of the centre of mass:  
 $\kappa = 0$ : 3 integrals  
 $\kappa \neq 0$ : 0 integrals
- Integrals of the linear momentum:  
 $\kappa = 0$ : 3 integrals  
 $\kappa \neq 0$ : 0 integrals
- Integrals of the total angular momentum:  
 $\kappa = 0$ : 3 integrals  
 $\kappa \neq 0$ : 6 integrals

Muchas gracias!