Universidade do Minho
Centro de Matemática

The classification of naturally reductive homogeneous spaces in small dimensions

Ana Cristina Ferreira
CMAT - Universidade do Minho

XXIII IFWGP - Granada,
3rd september 2014

Based on

I. Agricola, A.C. Ferreira, T. Friedrich,

The classification of naturally reductive homogeneous spaces in dimensions $n \leq 6$.

ArXiv: 1407.4936v1

Motivation

- É. Cartan 1926

  →classification of (Riemannian) symmetric spaces

  →Links algebraic theory of Lie groups and geometric notions such as isometry and curvature

- Homogeneous spaces

  →good examples in Riemannian geometry

However classification without further assumptions is impossible

Substancial work in special cases:

- very small dimensions
- positive curvature
- isotropy irreducible (examples of Einstein manifolds)

Our class: naturally reductive homogeneous spaces

Homogeneous space

- (M, g) plus $G \subseteq \mathrm{Iso}(M)$ s.t. G acts on M transitively and effectively.

  K stabilizer of a point $p \in M$, $M = G/K$.

→Assume M with good properties: connected, simply connected, complete.

Thm. [Ambrose-Singer, 1958]

(M, g) is homogeneous iff $\exists T \in TM \otimes \Lambda^2 M$ s. t.

→$(\nabla^g_X R^g)(Y, Z) = [T(X), R^g(Y, Z)] - R^g(T(X)(Y), Z) - R^g(Y, T(X)(Z))$

→$\nabla^g_X T(Y) = [T(X), T(Y)] - T(T(X)Y))$

∇ metric connection with torsion T

→$\nabla T = 0 = \nabla R$

Constructive proof!

Naturally reductive spaces

Dfn. $M = G/K$ is *naturally reductive* if $\mathfrak{k}$ admits a reductive complement $\mathfrak{m}$ in $\mathfrak{g}$ s. t.

$$\langle [X,Y]_{\mathfrak{m}}, Z\rangle + \langle Y, [X,Z]_{\mathfrak{m}}\rangle \;=\; 0 \qquad (*)$$

The PFB $G \to G/K$ induces a metric connection ∇ with torsion

$$g(T(X,Y),Z) := T(X,Y,Z) = -\langle [X,Y]_{\mathfrak{m}}, Z\rangle,$$

the so-called *canonical connection* of the nat. red. homog. space.

OBS: condition $(*)$ $\Leftrightarrow$ T is a 3-form.

Conversely, **Nomizu construction**

given (M,g,T) as in (AS) we can recover $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$:

$\rightarrow$ $\mathfrak{k}$ is the holonomy algebra, $\mathfrak{m}$ is identified with T_pM, for some p

$\rightarrow$ $[A+X, B+Y] = ([A,B]_{\mathfrak{k}} - R(X,Y)) + (AY - BX - T(X,Y))$

Examples

- all isotropy irreducible homogeneous manifolds
- any Lie group with a biinv. metric (b.t.w. flat) [Cartan-Schouten, 1926]
- construction/classification (under some assumptions) of left-invariant nat. red. metrics on compact Lie groups [D'Atri-Ziller, 1979]
- All 6-dim. homog. nearly Kähler mnfds. These are: $S^3 \times S^3$, $\mathbb{CP}^3$, the flag manifold $F^3 = \mathrm{U}(3)/\mathrm{U}(1)^3$, and $S^6 = G_2/\mathrm{SU}(3)$. [Butruille, 2005]
- spheres can carry several nat.red.structures, for example

$$S^{2n+1} = \mathrm{SO}(2n+2)/\mathrm{SO}(2n+1) = \mathrm{SU}(n+1)/\mathrm{SU}(n),$$

$$S^6 = G_2/\mathrm{SU}(3),\ S^7 = \mathrm{Spin}(7)/G_2,\ S^{15} = \mathrm{Spin}(9)/\mathrm{Spin}(7).$$

But if (M, g) not loc. isometric to sphere or Lie group then admits at most ONE nat. red. structure. [Olmos-Reggiani, 2012]

Known classifications

- In dimension 3 [Tricerri-Vanhecke, 1983]

  →space forms: $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$

  →one of the following (with left. inv. metric): $SU(2), \widetilde{SL}(2,\mathbb{R}), H^3$

- In dimension 4 [Kowalski-Vanhecke, 1983]

  →M is loc. $\mathbb{R} \times N^3$ with N^3 nat. red.

- In dimension 5 [Kowalski-Vanhecke, 1985]

  →$\mathrm{SU}(3)/SU(2)$ or $\mathrm{SU}(2,1)/\mathrm{SU}(2)$

  →H^5

  →$(K_1 \times K_2)/SO(2)$, where K_1 and K_2 are either $\mathrm{SU}(2)$, $\mathrm{SL}(2,\mathbb{R})$ or H^3

Our approach

- Look at the parallel torsion as the fundamental object.

- For "non-degenerate" torsion, the connection in (AS) is the characteristic connection for some known geometry (almost contact, almost Hermitian...).

An important tool

- $\sigma_T := \frac{1}{2}\sum_{i=1}^{n}(e_i \lrcorner T) \wedge (e_i \lrcorner T) \; = \overset{X,Y,Z}{\mathfrak{S}} \; g(T(X,Y), T(Z,V))$ $\quad$ $(= 0 \text{ if } n \leq 4)$

$\rightarrow$if non degenerate induces the geometric structure on M

* 1st Bianchi identity: $\overset{X,Y,Z}{\mathfrak{S}} \; \mathcal{R}(X,Y,Z,V) = \sigma_T(X,Y,Z,V)$
* $T^2 = -2\sigma_T + \|T\|^2$ in the Clifford algebra
* If $\nabla T = 0$: $dT = 2\sigma_T$ and $\nabla^g T = \frac{1}{2}\sigma_T$

Thm. [AFF]

M irreducible, $n \geq 5$, $\nabla T = 0$, $\sigma_T = 0$, then M is a simple compact Lie group (with biinv. metric) or its dual noncompact symmetric space.

Dimension 5

$(M^5, g, T \neq 0)$ Riemannian mnfd with parallel skew torsion

- $\exists$ a local frame s. t (for constants λ, $\varrho \in \mathbb{R}$)

$$T = -(\varrho e_{125} + \lambda e_{345}), \ \ *T = -(\varrho e_{34} + \lambda e_{12}), \ \ \sigma_T = \varrho\lambda e_{1234}, \ \ *\sigma_T = \varrho\lambda e_5$$

- **Case A:** $\sigma_T = 0$, M^5 is then loc. a product $N^3 \times N^2$ (if nat. red., N^2 has constant Gaussian curvature)

- **Case B:** $\sigma_T \neq 0$, two subcases:

  * Case B.1: $\lambda \neq \varrho$, $[\mathrm{Iso}(T) = \mathrm{SO}(2) \times \mathrm{SO}(2)]$

  * Case B.2: $\lambda = \varrho$, $[\mathrm{Iso}(T) = \mathrm{U}(2)]$

Dimension 5 - Induced contact structure

Case B: $\sigma_T \neq 0$

Dfn. A metric almost contact structure (φ, η) on (M^{2n+1}, g) is called
(N: Nijenhuis tensor, $F(X,Y) := g(X, \varphi Y)$)

- quasi-Sasakian if $N = 0$ and $dF = 0$
- α-Sasakian if $N = 0$ and $d\eta = \alpha F$ (Sasaki: $\alpha = 2$)

Thm. [AFF]

Let (M^5, g, T) be a Riemannian 5-mnfld with parallel skew torsion T such that $\sigma_T \neq 0$. Then M is a quasi-Sasakian manifold and ∇ is its characteristic connection.

The structure is α-Sasakian iff $\lambda = \varrho$ (case B.2), and it is Sasakian if $\lambda = \varrho = 2$.

Construction: $V := *\sigma_T \neq 0$ is a ∇-parallel Killing vector field of constant length
$\equiv$ contact direction $\eta = e_5$ (up to normalisation)
Check: $T = \eta \wedge d\eta$, define $F = -(e_{12} + e_{34})$, then prove that this works.

Dimension 5 - classification

Case B: $\sigma_T \neq 0$

For $\lambda = \varrho$ (case B.2), no classification for parallel skew torsion is possible (many non-homogeneous Sasakian mnfds are known). However:

Case B.1**:** $\lambda \neq \varrho$

Thm. [AFF]

Let (M^5, g, T) be Riemannian 5-mnfd with parallel skew torsion s. t. T has the normal form

$$T = -(\varrho e_{125} + \lambda e_{345}), \quad \varrho\lambda \neq 0 \text{ and } \varrho \neq \lambda.$$

Then $\nabla\mathcal{R} = 0$, i. e. M is loc. nat. red.

→Now we can apply the Nomizu construction to obtain the classification of Kowalski and Vanhecke.

Dimension 6

Normal forms in dim 6:

$$T_1 = ae_{123} + be_{456}, \quad T_2 = ae_{126} + be_{135} + ce_{234},$$
$$T_3 = ae_{135} + be_{146} + ce_{256} + de_{234}$$

$*\sigma_T$ is a 2-form. Induces an almost complex struture?
Can be seen as a skew endomorphism. Classify by its rank (=0,2,4,6)

Case A: $\sigma_T = 0$

Thm. [AFF]

A Riem. 6-mnfld with parallel skew torsion T s.t. $\sigma_T = 0$ splits into two 3-dimensional manifolds with parallel skew torsion.

Cor. [AFF]

Any 6-dim. nat. red. homog. space with $\sigma_T = 0$ [and $\ker T = 0$] is loc. isometric to a product of two 3-dimensional nat. red. homog. spaces.

Dimension 6

Case B: $\mathrm{rk}(*\sigma_T) = 2$

Thm. [AFF]

Let (M^6, g, T) be a 6-mnfd with parallel skew torsion s.t. $\mathrm{rk}(*\sigma_T) = 2$. Then $\nabla\mathcal{R} = 0$, i. e. M is nat. red.

Furthermore, M is a product $K_1 \times K_2$ of two Lie groups equiped with a left inv. metric and K_1 and K_2 are either $\mathrm{SU}(2), \mathrm{SL}(2, \mathbb{R})$ or H^3

Case C: $\mathrm{rk}(*\sigma_T) = 4$

Cannot occur.

Dimension 6

Case D: $\mathrm{rk}(*\sigma_T) = 6$

Thm. [AFF]

Let (M^6, g, T) be a 6-mnfd with parallel skew torsion s.t. $\mathrm{rk}(*\sigma_T) = 6$. Then:

Case D.1 (M^6, g) is nearly Kähler

Case D.2 (M^6, g) is nat. red. of type $\mathcal{W}_1 \oplus \mathcal{W}_3$ and is one of the following

→a two-step nilpotent Lie group with Lie algebra $\mathbb{R}^3 \times \mathbb{R}^3$ s.t.

$[(u_1, v_1), (u_2, v_2)] = (0, u_1 \times u_2)$

→$S^3 \times \mathbb{R}^3$, →$S^3 \ltimes \mathbb{R}^3$

→$S^3 \times S^3$, →$\mathrm{SL}(2, \mathbb{C})$