Trigroups and M-theory

John Huerta

arXiv:1409.4361

http://math.ucr.edu/~huerta

CAMGSD
Instituto Superior Técnico

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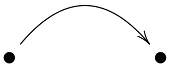
Prologue: higher gauge theory

Gauge theory tells us how particles transform as they move along a path:



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Gauge theory tells us how particles transform as they move along a path:

$$\bullet \longrightarrow \bullet \, .$$

A connection assigns a group element to each path:

$$\bullet \xrightarrow{\;g\;} \bullet \, , \qquad g \in G.$$

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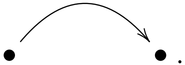
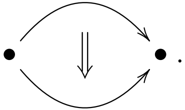
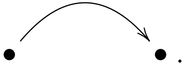
For example, a 2-connection tells us how to transform particles as they move along paths:



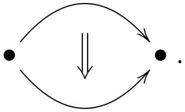
And strings as they sweep out surfaces:



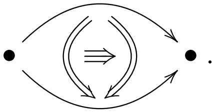
For 2d membranes, we need 3-connections, which tell us how to transform along paths:



surfaces:

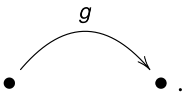


and volumes:

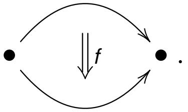


For 1-connections, we assign group elements to paths. For 3-connections, we need a new kind of algebraic gadget: a **3-group**.

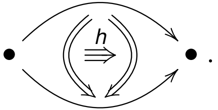
3-groups have objects:



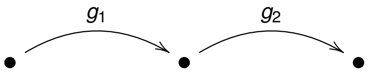
morphims going between objects:



and 2-morphisms going between morphisms:



In a 3-group, we can multiply objects:



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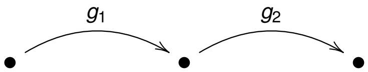

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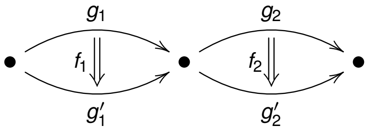

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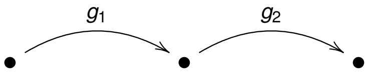

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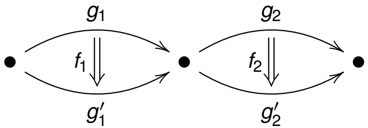
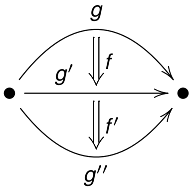
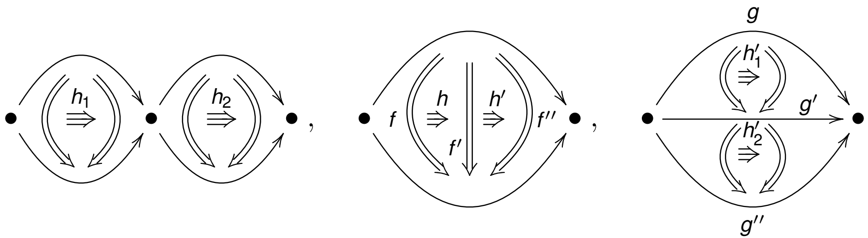


and also compose morphisms:



We can compose a pair of 2-morphisms in three distinct ways:



Various axioms must hold. The definition is unfortunately complex, but there is a simple example!

3-groups and cohomology

Group cohomology is a cohomology theory of groups going back to Eilenberg and Mac Lane.

Given a group G, a G-module H, we define a cochain complex:

$$C^p(G,H) = \{F\colon G^{p+1} \to H,\ \text{equivariant}\}$$

with a coboundary operator

$$d\colon C^p(G,H) \to C^{p+1}(G,H)$$

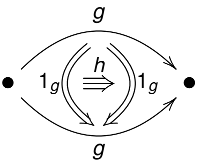
defined as follows:

$$dF(g_0, g_1, \ldots, g_{p+1}) = \sum_i (-1)^i F(g_0, g_1, \ldots, \hat{g}_i, \ldots, g_{p+1}).$$

Theorem

Given a 4-cocycle $\pi \colon G^5 \to H$ in group cohomology, there is a 3-group with:

- An object $\bullet \xrightarrow{g} \bullet$ for each element $g \in G$.
- For each object, only the identity morphism: $\bullet \underset{g}{\overset{g}{\rightrightarrows}} \bullet$ with 2-cell $\Downarrow 1_g$.

- A 2-morphism between identities 

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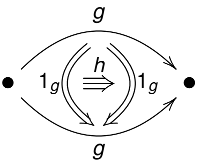
- A 2-morphism between identities 

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Call a 3-group of this form a **slim 3-group**.

This is trivial to internalize:

- Get a **slim Lie 3-group** when G and H are Lie groups and π is a smooth map.
- Get a **slim 3-supergroup** when G and H are ‘supergroups’ and π is a smooth map.

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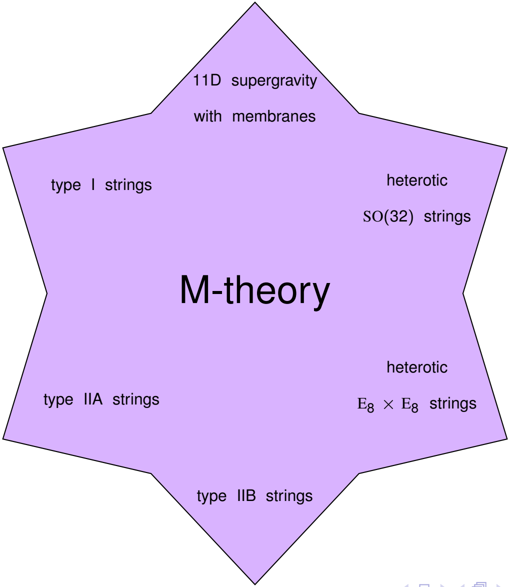
We will describe a 3-group related to M-theory by giving a 4-cocycle.


11D supergravity
with membranes
type I strings
heterotic
SO(32) strings
M-theory
heterotic
$E_8 \times E_8$ strings
type IIA strings
type IIB strings

M-theory is a conjectured theory of physics uniting the five 10D string theories and 11D supergravity coupled to membranes.

It includes the theory of a two-dimensional membrane in 11D. This is where our 4-cocycle appears.

Classically, this is the theory of maps from a 3-dimensional manifold to 11-dimensional superspacetime:

$$\phi \colon \Sigma^3 \to T.$$

- Σ^3 is the worldvolume of 2D membrane as it evolves in time.
- T is the super analogue of 11D Minkowski spacetime, $\mathbb{R}^{10,1}$. It is slightly noncommutative.

11D has a supersymmetric theory of the 2-brane, thanks to:

- T has a closed 4-form β on it.
- β is invariant under translations on T.
- β is invariant under Lorentz transformations, $\mathrm{Spin}(10,1)$.
- Hence, β is invariant under the full group of supersymmetries, the **super-Poincaré group**:

$$\mathrm{SISO}(T) = \mathrm{Spin}(10,1) \ltimes T.$$

We can use β to define the topological Wess–Zumino term in the 2-brane action. If ϕ maps into a contractible neighborhood $U \subseteq T$, this term is:

$$S_{WZ} = \int_{\Sigma^3} \phi^* \alpha,$$

where α is any 3-form on U such that $d\alpha = \beta$. This ought to be the holonomy of a 2-gerbe with curvature β on T.

So, β defines a de Rham 4-cocycle on T:

$$d\beta = 0.$$

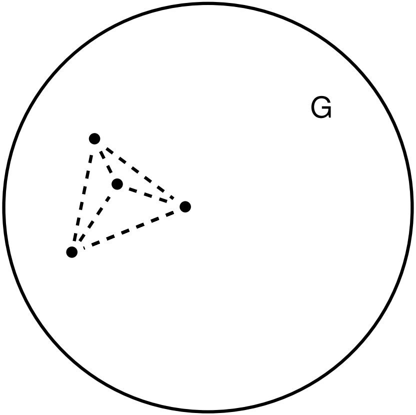
We will turn this into a group 4-cocycle:

$$\int \beta \colon T^5 \to \mathbb{R}$$

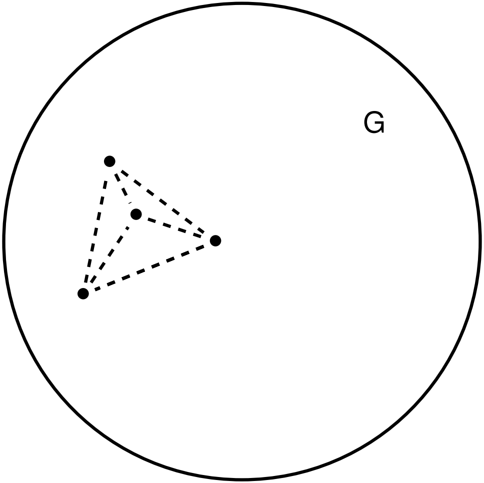
and hence obtain a 3-group.

Interlude on group cohomology

We can think of a $(p+1)$-tuple in G as the vertices of a p-simplex in G:



A cochain $F\colon G^{p+1} \to H$ just assigns H-values to these simplices:



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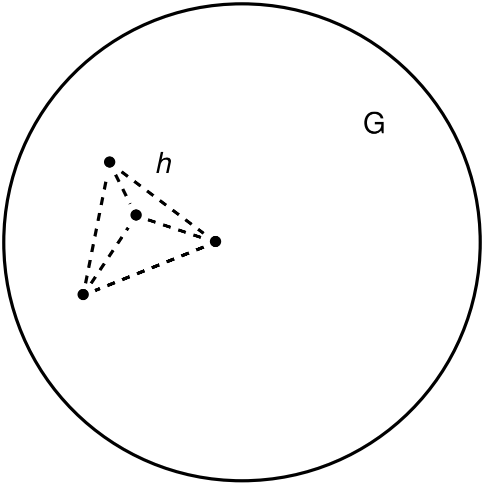
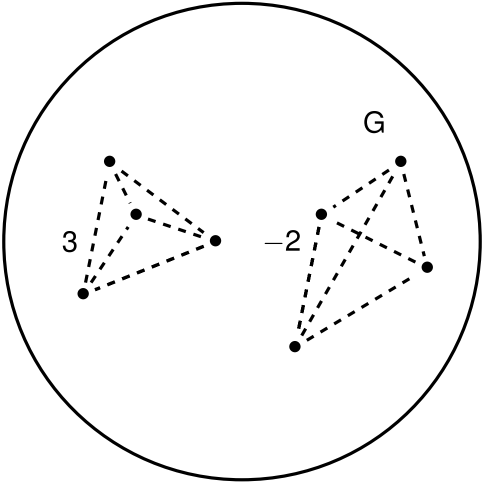


Trivially extending to the free $\mathbb{Z}$-module of simplices:



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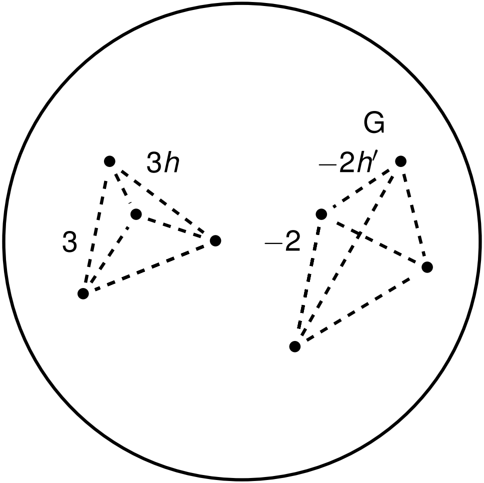

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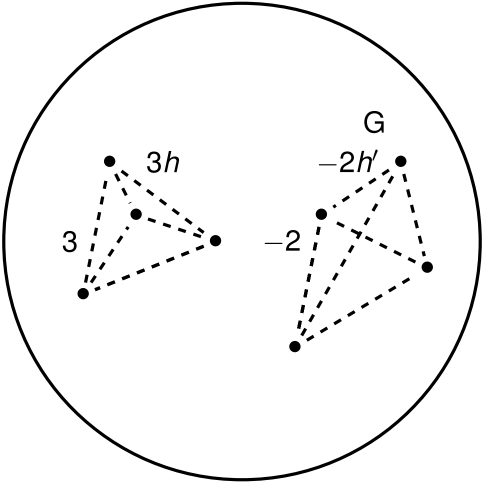

Equivariance just says F is a map of $\mathbb{Z}G$-modules.

Of course, these aren't geometric simplices—we have not said how to get a map

$$[g_0, \ldots, g_p] \colon \Delta^p \to G$$

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$$dF(\Delta) := F(\partial\Delta).$$

In particular, F is a ***p*-cocycle** if

$$F(\partial\Delta) = 0$$

for all combinatorial $(p+1)$-simplices Δ.

Integrating β

Remember our mission is to massage β, a closed 4-form on Minkowski superspacetime T, into a 4-cocycle:

$$\int \beta \colon T^5 \to \mathbb{R}.$$

The key is to take our pictures of simplices seriously.

- As a 4-form, integrating β over a 4-simplex yields a number.

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- As a 4-form, integrating β over a 4-simplex yields a number.
- If we had a *standard way* to flesh a combinatorial 4-simplex Δ out into a geometric 4-simplex:

  $$\overline{\Delta} \colon \Delta^4 \to T$$

  then we could define a 4-cochain:

  $$(\int \beta)(\Delta) = \int_{\overline{\Delta}} \beta.$$

After all, if T were merely $\mathbb{R}^{10,1}$, there is a standard way to flesh out a combinatorial p-simplex into a geometric one—just take the convex hull of the $(p+1)$-vertices.

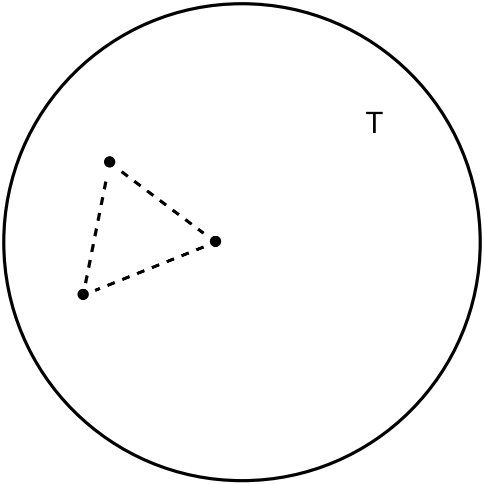
But T is not $\mathbb{R}^{10,1}$, and for our proposed 4-cochain:

$$(\textstyle\int \beta)(\Delta) = \int_{\overline{\Delta}} \beta$$

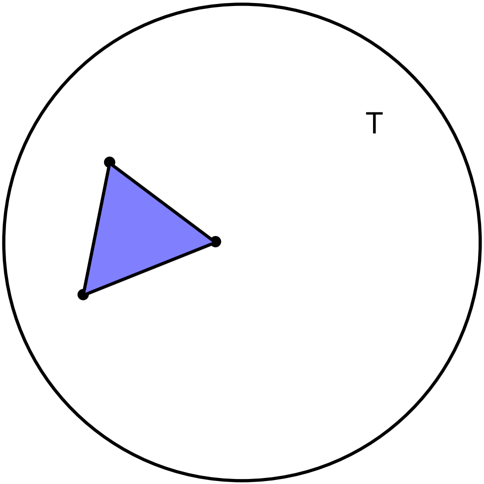
to be equivariant, we need our standard way of fleshing out combinatorial p-simplices to be equivariant:

$$\overline{t\Delta} = t\overline{\Delta}, \quad t \in T.$$

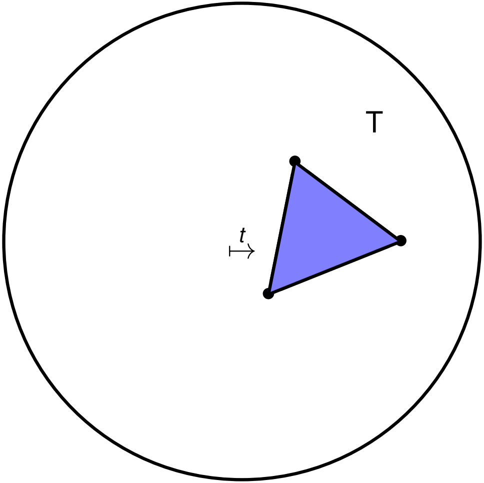
A priori, it need not be:



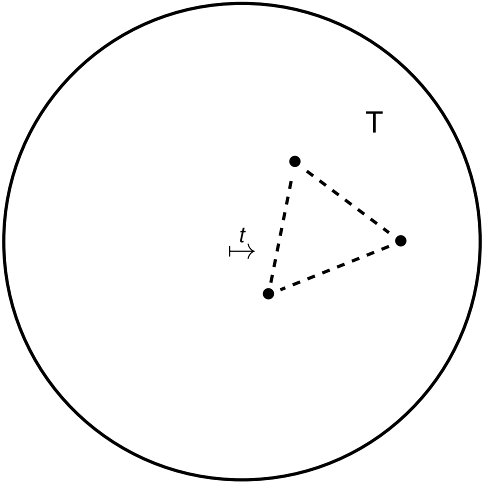
We can fill . . .



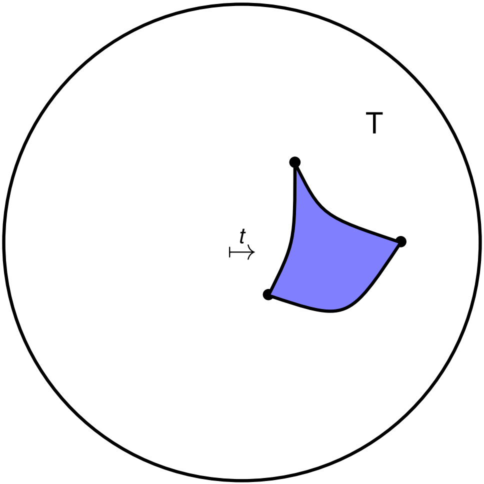
. . . and then translate.



Or we can translate . . .



…and then fill.



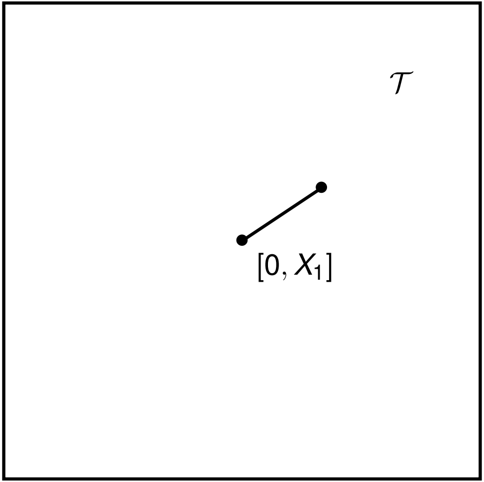
We say that T is equipped with a **left invariant notion of simplices** if we can fill equivariantly.

Only some supergroups can be equipped with this. The key to making it work is that T is **exponential**—the exponetial map is a diffeomorphism:

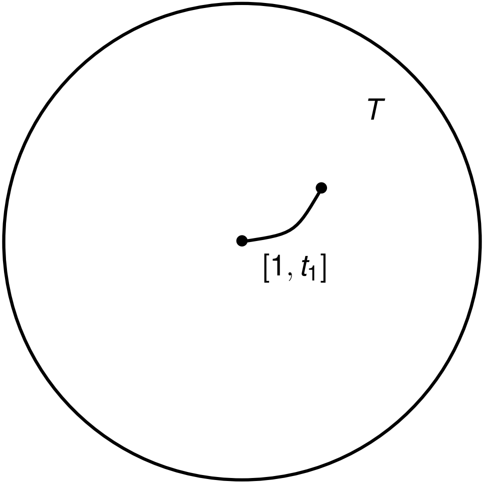
$$\exp\colon \mathcal{T} \to T.$$

Here, $\mathcal{T}$ is the Lie superalgebra of T.

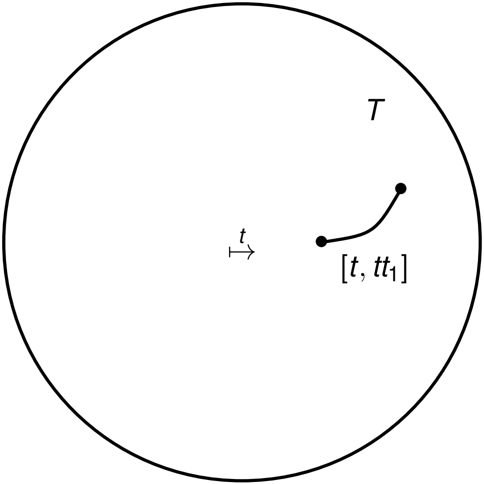
This gives us a natural notion of 1-simplex: take a path in the Lie algebra starting at 0 ...



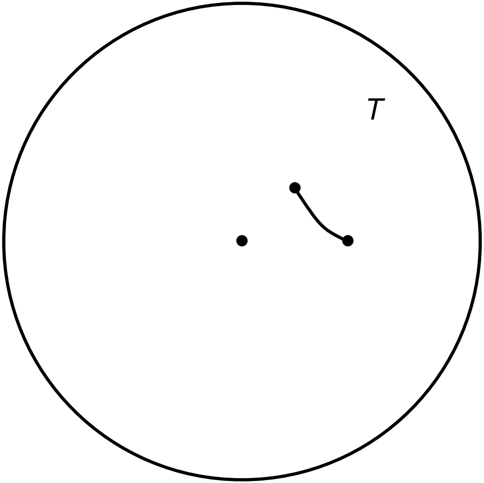
...exponentiate it to get a 1-simplex starting at 1 ...



...and translate it to get an arbitrary 1-simplex. Note equivariance is built in.



Build 2-simplices from 1-simplices . . . and induct for all higher simplices!



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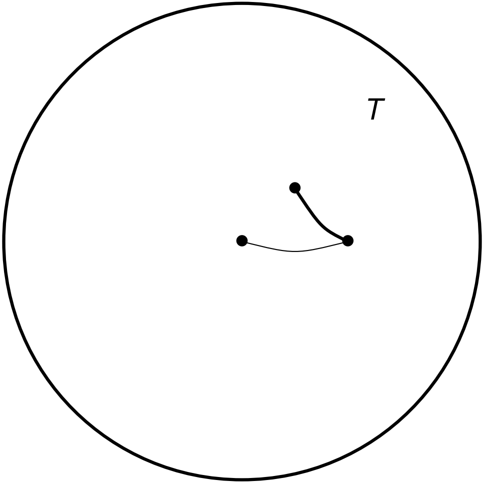

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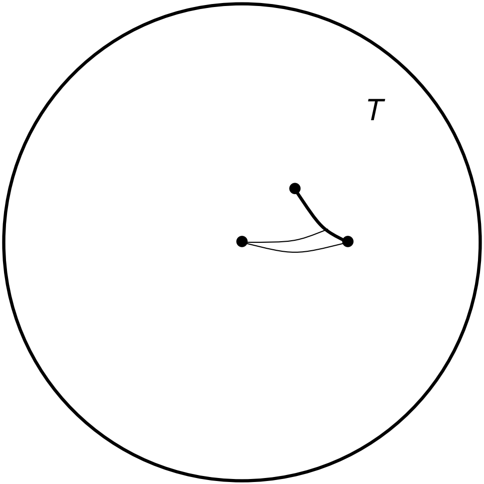

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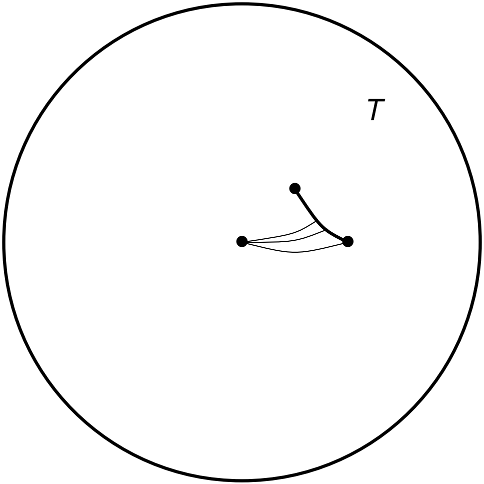

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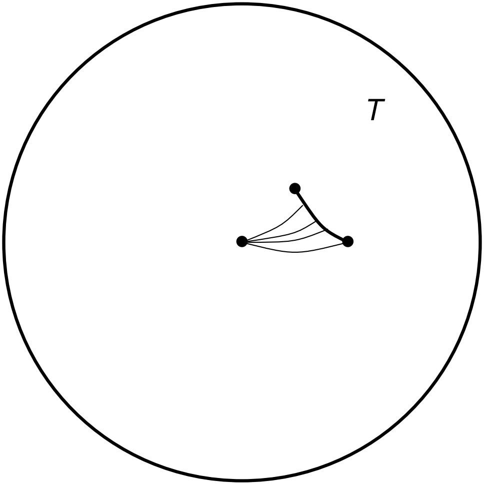

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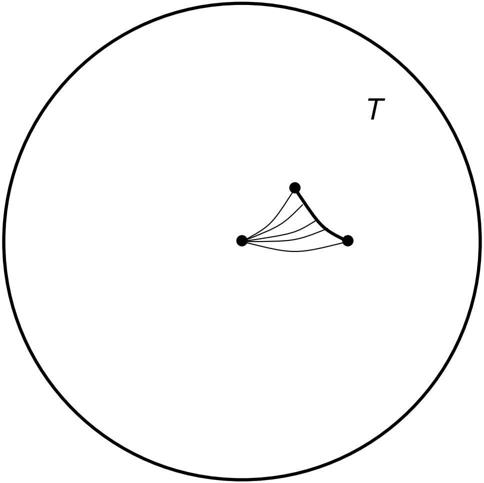

Now that we have our left invariant notion of simplices, we can turn β into a 4-cocycle simply be integrating:

$$(\int \beta)(\Delta) = \int_{\overline{\Delta}} \beta.$$

Checking $\int \beta$ is a 4-cocyle is easy:

$$(d \int \beta)(\Delta) \quad = \quad (\int \beta)(\partial \Delta)$$

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$$\begin{aligned}(d \textstyle\int \beta)(\Delta) &= (\textstyle\int \beta)(\partial\Delta) \\ &= \int_{\partial\overline{\Delta}} \beta \\ &= \int_{\overline{\Delta}} d\beta \\ &= 0\end{aligned}$$

Because β is closed, this follows from Stokes' theorem.

The 3-supergroup

Moreover, because β is invariant under the super-Poincaré group $\mathrm{SISO}(T) = \mathrm{Spin}(10,1) \ltimes T$, so is $\int \beta$. This allows us to extend it to a 4-cocycle on $\mathrm{SISO}(T)$.

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Thus, we get the **2-brane 3-supergroup 2-$\mathbf{Brane}(T)$**, with:

- $\mathrm{SISO}(T)$ as objects;
- trivial morphisms;
- $\mathbb{R}$ as 2-morphisms over any trivial morphism;
- 4-cocycle given by $\int \beta$.

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- It should play a role in anomaly cancellation for the 2-brane.
- Recent work of Schreiber, Sati and Fiorenza indicates that 2-$\mathrm{Brane}(T)$, viewed as a higher stack, may be the appropriate background for the "M5-brane"—the other, much more mysterious membrane in M-theory.