Geometric Hamiltonian formulation of Quantum Mechanics in complex projective spaces

IFWGP in Granada

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Outline

Classical Mechanics
- Hamiltonian formulation
- Classical states as measures

Quantum States and Frame fucntions
- Quantum states as measures
- Frame functions

Geometric Hamiltonian QM
- Projective space as phase space
- Geometric Hamiltonian QM (finite dimension)
- C*-algebra of classical-like observables

Extension to infinite dimension

Hamiltonian formulation of Classical Mechanics

Phase space

A classical system with n spatial degrees of freedom is described in a $2n$-**dimensional symplectic manifold** $(\mathcal{M}, \omega)$.

Physical state

A point $s = (q^1, ..., q^n, p_1, ..., p_n)$

Dynamics

A curve in $(a, b) \ni t \mapsto s(t) \in \mathcal{M}$ satisfying Hamilton equations:

$$\frac{ds}{dt} = X_H(s(t))$$

$H : \mathcal{M} \to \mathbb{R}$ is the *Hamiltonian function*.
X_H is the *Hamiltonian vector field*, given by: $\omega_s(X_H, \cdot) = dH_s(\cdot)$

Classical states as measures

Statistical description
$\Longrightarrow$ State of the system as a C^1-function $\rho = \rho(t, p, q)$.

Dynamics

$$\frac{\partial \rho}{\partial t} + \{\rho, H\}_{PB} = 0$$

Classical expecation values

Physical quantity $f : \mathcal{M} \to \mathbb{R}$
A state $\rho : \mathcal{M} \to [0, 1]$ (positive, normalized to 1)

$$\langle f \rangle_\rho = \int_{\mathcal{M}} f(s)\rho(t, s) d\mu(s)$$

State as a measure $\nu : \mathcal{B}(\mathcal{M}) \to [0, 1] \quad , \quad \nu(E) := \int_E \rho d\mu$

States as measures in Quantum Mechanics

Notations

$\mathcal{H}$: Complex Hilbert space

$\langle\ |\ \rangle$: Inner product

$\mathfrak{B}(\mathcal{H})$: Bounded operators on $\mathcal{H}$

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Quantum states as a *generalized probability measures*

Orthogonal projectors on $\mathcal{H}$:

$$\mathfrak{P}(\mathcal{H}) = \{P \in \mathfrak{B}(\mathcal{H}) : PP = P, P^* = P\}$$

A **quantum state** is a map $\mu : \mathfrak{P}(\mathcal{H}) \to [0, 1]$ such that:

i) $\mu(I) = 1$;

ii) If $\{P_i\}_{i\in\mathbb{N}} \subset \mathfrak{P}(\mathcal{H})$ with $P_iP_j = 0$ for $i \neq j$ then:

$$\mu\left(s - \sum_i P_i\right) = \sum_i \mu(P_i)$$

States as measures in Quantum Mechanics

Theorem [Gleason 1957]

For any state $\mu : \mathfrak{P}(\mathcal{H}) \to [0,1]$ with $\dim \mathcal{H} > 2$, $\exists! \sigma \in \mathfrak{B}(\mathcal{H})$ s.t.:
i) $\sigma \geq 0$;
ii) $\sigma \in \mathfrak{B}_1(\mathcal{H})$ (σ is trace-class) with $tr(\sigma) = 1$;
iii) $\mu(P) = tr(\sigma P)$ for every $P \in \mathfrak{P}(\mathcal{H})$

And $\sigma \in \mathfrak{B}(\mathcal{H})$ satisfying i) and ii) defines a state $\mu(P) = tr(\sigma P)$.

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Density matrices

$$\mathfrak{D}(\mathcal{H}) = \{\sigma \in \mathfrak{B}_1(\mathcal{H}) | \sigma \geq 0, tr(\sigma) = 1\}$$

-) $\mathfrak{D}(\mathcal{H})$ is convex in $\mathfrak{B}_1(\mathcal{H})$.
-) Extremal points (pure states) have form:
$\mathfrak{D}(\mathcal{H}) \ni |\psi\rangle\langle\psi|$ with $\psi \in \mathcal{H}$, $\| \psi \| = 1$.

Frame functions

Let $\mathcal{H}$ be separable and $\mathbb{S}(\mathcal{H}) = \{\psi \in \mathcal{H} : \| \psi \| = 1\}$.

$f : \mathbb{S}(\mathcal{H}) \to \mathbb{C}$ is called **frame funtion** if $\exists W_f \in \mathbb{C}$ (**weight**) s.t.:

$$\sum_{\psi \in N} f(\psi) = W_f \qquad \forall N \text{ orthonormal basis of } \mathcal{H}.$$

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Every quantum state (as probability measure) $\mu : \mathfrak{P}(\mathcal{H}) \to [0,1]$ defines a frame function:

$$f_\mu(\psi) := \mu(p_\psi) \qquad p_\psi = |\psi\rangle\langle\psi|$$

$$W_{f_\mu} = \sum_{\psi \in N} f_\mu(\psi) = \sum_{\psi \in N} \mu(p_\psi) = \mu\left(\sum_{\psi \in N} p_\psi\right) = \mu(I) = 1$$

Frame functions

Finite dimensional case: $2 < \dim \mathcal{H}_n = n < +\infty\,,\, \mathbb{S}(\mathcal{H}) = \mathbb{S}^{2n-1}$

$$\mathcal{L}^2(\mathbb{S}^{2n-1}, \nu_n) = \left\{ f : \mathbb{S}^{2n-1} \to \mathbb{C} \mid \int_{\mathbb{S}^{2n-1}} \overline{f(x)} f(x) d\nu_n(x) < +\infty \right\}$$

$\nu_n : \mathcal{B}(\mathbb{S}^{2n-1}) \to [0,1]$ is the unique regular Borel measure s.t.:
i) $\nu_n(\mathbb{S}^{2n-1}) = 1$;
ii) $\nu_n(UE) = \nu_n(E) \quad \forall U \in U(n), \forall E \in \mathcal{B}(\mathbb{S}^{2n-1})$.

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Theorem [V.Moretti, D.P. 2013]

Let $\mathcal{H}$ be a Hilbert space with $2 < \dim \mathcal{H}_n < +\infty$. For every frame function $f : \mathbb{S}^{2n-1} \to \mathbb{C}$ s.t. $f \in \mathcal{L}^2(\mathbb{S}^{2n-1}, \nu_n)$, $\exists!\ A \in \mathfrak{B}(\mathcal{H})$ s.t.:

$$f(\psi) = \langle \psi | A\psi \rangle \quad \forall \psi \in \mathbb{S}^{2n-1}$$

Projective space as phase space

Let $\mathcal{P}(\mathcal{H}_n) = \frac{U(n)}{U(n-1)U(1)}$ be the projective space of $\mathcal{H}_n$ and let f be a $\mathcal{L}^2$-frame function on $\mathbb{S}(\mathcal{H}_n)$:

$$f(\psi) = \langle\psi|A\psi\rangle = tr(A\,|\psi\rangle\langle\psi|) = tr(Ap_\psi)$$

$$\Longrightarrow f(p) = tr(A\,p),\ \forall p \in \mathcal{P}(\mathcal{H})\ ,\ \ f \in \mathcal{L}^2(\mathcal{P}(\mathcal{H}_n), \nu_n)$$

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Geometry of $\mathcal{P}(\mathcal{H})$

$\mathcal{P}(\mathcal{H}_n)$ is a real smooth $(2n-2)$-dimensional manifold.
Tangent vectors at $p \in \mathcal{P}(\mathcal{H}_n)$ has form:

$$v = -i[A_v, p] \in T_p\mathcal{P}(\mathcal{H}_n) \text{ for some } A_v \in i\mathfrak{u}(n),$$

where $\mathfrak{u}(n)$ is the Lie algebra of $U(n)$.

Projective space $\mathcal{P}(\mathcal{H})$ as phase space

Kähler structure on $\mathcal{P}(\mathcal{H})$

-) Symplectic form : $\omega_p(u,v) := -ik\, tr\,(p[A_u, A_v]) \qquad k > 0$

-) Fubini-Study metric:

$$g_p(u,v) = -ktr\,(p\,([A_u,p][A_v,p] + [A_v,p][A_u,p]))$$

-) Almost complex form:

$$j_p : T_p\mathcal{P}(\mathcal{H}_n) \ni v \mapsto i[v,p] \in T_p\mathcal{P}(\mathcal{H}_n)$$

$p \mapsto j_p$ is smooth, $j_p j_p = -id$ and $\omega_p(u,v) = g_p(u, j_p v)$.

$(\mathcal{P}(\mathcal{H}_n), \omega, g, j)$ is a Kähler manifold

Geometric Hamiltonian QM

The goal

Correspondence *quantum observables* – *classical-like observables*:

$$i\mathfrak{u}(n) \ni A \mapsto f_A : \mathcal{P}(\mathcal{H}) \to \mathbb{R},$$

such that **Schrödinger dynamics** given by a Hamiltonian operator H is equivalent to the **flow of the Hamilton vector field** X_{f_H}.

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A candidate

$$\mathcal{O} : A \mapsto f_A \quad , \quad f_A(p) := k\, tr(Ap) + c\, tr(A)$$

with $k > 0$ and $c \in \mathbb{R}$.

Geometric Hamiltonian QM

It is a *not bad choice*

With the correspondence $f_A(p) = k\,tr(Ap) + c\,tr(A)$, we have:

i) Let $t \mapsto p(t) \in \mathcal{P}(\mathcal{H}_n)$ a curve,

$$p = p(t) \text{ is a solution of } \dot{p}(t) = -i[H, p(t)] \quad \Longleftrightarrow$$

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ii) $\{f_A, f_B\} = f_{-i[A,B]}$

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Two questions at least

1st) A more general form for classical-like observables?
2nd) Can we fix k and c?

Geometric Hamiltonian QM

Comparison of expectation values

We are looking for:

$$\mathcal{O} : i\mathfrak{u}(n) \ni A \longmapsto f_A : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$$

$$\mathcal{S} : \mathfrak{D}(\mathcal{H}) \ni \sigma \longmapsto \rho_\sigma : \mathcal{P}(\mathcal{H}) \to [0, 1]$$

in order to obtain:

$$\langle A \rangle_\sigma = tr(A\sigma) = \int_{\mathcal{P}(\mathcal{H})} f_A \rho_\sigma d\nu_n \quad \text{and} \quad \int_{\mathcal{P}(\mathcal{H})} \rho_\sigma d\nu_n = 1$$

Geometric Hamiltonian QM

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Measure ν_n on the manifold $\mathcal{P}(\mathcal{H})$

ν_n is the Liouville measure induced by ω.

ν_n coincides to the Riemannian measure induced by g.

ν_n coincides (up to its normalization) with the unique normalized $U(n)$-invariant Borel measure on $\mathcal{P}(\mathcal{H})$.

Geometric Hamiltonian QM

Physical requirements on $\mathcal{O} : i\mathfrak{u}(n) \ni A \mapsto f_A$

O1) $\mathcal{O}$ is injective;
O2) $\mathcal{O}$ is $\mathbb{R}$-linear;
O3) If $H \in i\mathfrak{u}(n)$ then $\mathcal{O}(H) = f_H \in C^1(\mathcal{P}(\mathcal{H}))$ and X_{f_H} can be defined:

$$\dot{p}(t) = X_{f_H}(p(t)) \quad \Longleftrightarrow \quad \dot{p}(t) = -i[H, p]$$

O4) $U(n)$-covariance: $f_A(UpU^{-1}) = f_{U^{-1}AU}(p)$ for any $U \in U(n)$;

Geometric Hamiltonian QM

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Theorem [V.Moretti, D.P. 2014]

$\mathcal{O} : A \mapsto f_A$ satisfies O1) and O4) $\Longleftrightarrow f_A(p) = ktr(Ap) + ctr(A)$

with $c \in \mathbb{R}$ and $k + nc \neq 0$.

Geometric Hamiltonian QM

Physical requirements on $\mathcal{S} : \mathfrak{D}(\mathcal{H}) \ni \sigma \mapsto \rho_\sigma$

S1) $\rho_\sigma \geq 0$ for every $\sigma \in \mathfrak{D}(\mathcal{H})$;
S2) $\mathcal{S}$ is convex-linear;
S3) $\rho_\sigma \in \mathcal{L}^2(\mathcal{P}(\mathcal{H}), \nu_n)$ and

$$\int_{\mathcal{P}(\mathcal{H})} \rho_\sigma d\nu_n = 1;$$

S4) $\rho_\sigma(UpU^{-1}) = \rho_{U^{-1}\sigma U}(p)$
S5) If $A \in i\mathfrak{u}(n)$ and $f_A = \mathcal{O}(A)$ then:

$$tr(A\sigma) = \int_{\mathcal{P}(\mathcal{H})} f_A \rho_\sigma d\nu_n$$

Geometric Hamiltonian QM

Theorem [V.Moretti, D.P. 2014]

$\mathcal{S} : \sigma \mapsto \rho_\sigma$ satisfies S2) and S5) $\iff$ $\rho_\sigma(p) = k' tr(Ap) + c'$

$$\text{with} \quad k' = \frac{n(n+1)}{k}, \quad c' = \frac{k-(n+1)}{k}, \quad c = \frac{1-k}{n}.$$

So k is the only degree of freedom of the construction.

Geometric Hamiltonian QM

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All possible prescriptions to set up a Hamiltonian theory

From *quantum observables* to *classical-like observables*:

$$f_A(p) = ktr(Ap) - \frac{1-k}{n} tr(A)$$

From *denisty matrices* to *Liouville densities*:

$$\rho_\sigma(p) = \frac{n(n+1)}{k} tr(\sigma p) + \frac{k-(n+1)}{k}$$

C*-algebra of classical-like observables

$\mathcal{O} : i\mathfrak{u}(n) \ni A \mapsto f_A$ linear extension $\mathcal{O} : \mathfrak{B}(\mathcal{H}) \to \mathcal{F}^2(\mathcal{H})$

$$\mathcal{F}^2(\mathcal{H}) = \left\{ f \in \mathcal{L}^2(\mathcal{P}(\mathcal{H}), \nu_n) | f \text{ is a frame function} \right\}$$

$\mathcal{F}^2(\mathcal{H})$ as C*-algebra of observables

-) Involution: $A = \mathcal{O}(f)$, $A^* = \mathcal{O}(\overline{f})$;
-) $\star$ - product: $f \star g = \mathcal{O}\left(\mathcal{O}^{-1}(f)\mathcal{O}^{-1}(g)\right)$:

$$f \star g = \frac{i}{2}\{f, g\}_{PB} + \frac{1}{2}G(df, dg) + fg \qquad k = 1$$

-) Norm: $|||f||| = \| \mathcal{O}^{-1}(f) \|$

$$|||f||| = \frac{1}{k}\left\|f - \frac{1-k}{n}\int_{\mathcal{P}(\mathcal{H})} f \; d\nu_n\right\|_\infty \qquad k > 0$$

Geometry of quantum entanglement

Composite system described in $\mathcal{H}_1 \otimes \mathcal{H}_2$

The phase space is $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and not $\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$... But:

$\mathcal{P}(\mathcal{H}_1) \times \mathcal{P}(\mathcal{H}_2)$ is embedded in $\mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ by Segre embedding:

$$Seg(|\psi_1\rangle\langle\psi_1|, |\psi_2\rangle\langle\psi_2|) = |\psi_1 \otimes \psi_2\rangle\langle\psi_1 \otimes \psi_2|$$

and $Seg^* \left(\mathcal{F}^2(\mathcal{H}_1 \otimes \mathcal{H}_2)\right) = \mathcal{F}^2(\mathcal{H}_1) \otimes \mathcal{F}^2(\mathcal{H}_2)$.

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and $Seg^* \left(\mathcal{F}^2(\mathcal{H}_1 \otimes \mathcal{H}_2)\right) = \mathcal{F}^2(\mathcal{H}_1) \otimes \mathcal{F}^2(\mathcal{H}_2)$.

Measure of entanglement [D.P. 2014]

Let $\rho : \mathcal{P}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to [0,1]$ be a Liouville density.

$$\rho_1 := \int_{\mathcal{P}(\mathcal{H}_2)} Seg^* \rho \, d\nu_2 \qquad \rho_2 := \int_{\mathcal{P}(\mathcal{H}_1)} Seg^* \rho \, d\nu_1$$

$$E(\rho) = \int_{\mathcal{P}(\mathcal{H}_1)\times\mathcal{P}(\mathcal{H}_2)} |Seg^* \rho(p_1, p_2) - \rho_1(p_1)\rho_2(p_2)|^2 d\nu_1(p_1) d\nu_2(p_2)$$

Extension to infinite dimension

(work in progress with V.Moretti and S.Mazzucchi)

Finite dimension: Link with Gaussian measure

$$\pi : \mathcal{H}_n \setminus \{0\} \to \mathcal{P}(\mathcal{H}_n) \quad , \quad \mu_G(x) = \frac{e^{-\frac{\|x\|^2}{2}}}{(2\pi)^n} dx$$

For any bounded Borel function $f : \mathcal{P}(\mathcal{H}_n) \to \mathbb{C}$:

$$\int_{\mathcal{P}(\mathcal{H}_n)} f(p)\, d\nu_n(p) = \int_{\mathcal{H}_n} f(\pi x) \mu_G(x)$$

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Infinite dimension: Direct limit of projective spaces

$\{\mathcal{P}(V) | V \subset \mathcal{H}, \dim V < \infty\}$, $\mathcal{P}(V) \leq \mathcal{P}(W)$ if $V \subseteq W$,
$\Pi_V^W : \mathcal{P}(W) \to \mathcal{P}(V)$. Direct limit: $\mathcal{P}^\infty$.
We have $\mathcal{P}(\mathcal{H}) \subset \mathcal{P}^\infty$. $\pi_V : \mathcal{P}^\infty \to \mathcal{P}(V)$. $\exists \mu_\infty$ s.t.:

$$\nu_V = \pi_V \circ \mu_\infty \qquad \mu_\infty(\mathcal{P}(\mathcal{H})) = 0$$

Extension to infinite dimension

Finite dimension: $k = 1$

Observables: $f_A(p) = tr(Ap)$, $A \in i\mathfrak{u}(n)$;
States: $\rho_\sigma(p) = n(n+1)tr(\sigma p) - n$, $\sigma \in \mathfrak{D}(\mathcal{H})$

$$tr(A\sigma) = \int_{\mathcal{P}(\mathcal{H})} f_A(p) d\mu_\sigma(p) \qquad \mu_\sigma(p) = (n(n+1)tr(\sigma p) - n)\nu_n(p)$$

$$\mu_\sigma = \pi \circ \nu_\sigma \text{ where } \nu_\sigma(x) = \left(\frac{\langle x|\sigma x\rangle}{4} - \frac{1}{2}\right) \| x \|^2 \mu_G(x)$$

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Total variation of ν_σ

$$\| \nu_\sigma \| = \int_{\mathbb{R}} \left| \frac{|x_1|^2}{4} - \frac{1}{2} \right| |x_1|^2 \mu_G(x_1) + (2n-1) \int_{\mathbb{R}} \left| \frac{|x_1|^2}{4} - \frac{1}{2} \right| \mu_G(x_1)$$

$$\| \nu_\sigma \| \to \infty \quad \text{for} \quad n \to \infty$$

Application to Quantum Control

The bilinear model of *controllability quantum problem*:

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = \left[H_0 + \sum_{i=1}^{m} H_i u_i(t)\right]|\psi(t)\rangle.$$

Equivalence with the classical-like control system:

$$\dot{p}(t) = X_{f_{H_0}}(p(t)) + \sum_{i=1}^{m} X_{f_{H_i}}(p(t))u_i(t) \qquad p(t) = |\psi(t)\rangle\langle\psi(t)|$$

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A first result

Complete controllability of the quantum system is equivalent to **local accessibility** of the associated classical-like system.

Conclusions

- All possible prescriptions to translate standard QM in $\mathcal{H}_n$ in geometric Hamiltonian folrmulation on $\mathcal{P}(\mathcal{H}_n)$:

$$\mathcal{O} : i\mathfrak{u}(n) \ni A \longmapsto f_A : \mathcal{P}(\mathcal{H}) \to \mathbb{R}$$

$$\mathcal{S} : \mathfrak{D}(\mathcal{H}) \ni \sigma \longmapsto \rho_\sigma : \mathcal{P}(\mathcal{H}) \to [0, 1]$$

- Application of Hamiltonian formalism to describing compound quantum systems and measuring entanglement.
- Relationship between Liouville measure on $\mathcal{P}(\mathcal{H}_n)$ and standard Gaussian measure on $\mathcal{H}_n$. *Open issues of the generalization to infinite dimension*.
- Classical-like approach to Quantum Control Theory.

Thank you for your attention!